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Ripple Geometry : The Analytical Equation of a Dynamic Hyperbola

Definitions

1. Dynamic Circle

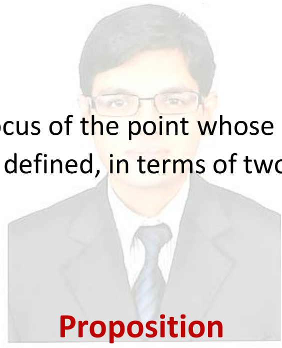
A dynamic circle is one whose radius is a function of time. In other words, it is a circle that is expanding (or equivalently, contracting) in a plane, at a uniform time rate.

2. Source

The center of uniform expansion of a dynamic circle, is called a source.

3. Inter-Source Interval

The interval of time spanning the instants at which two dynamic circles with equal expansion rates, emerge from their respective sources, is called the Inter-Source Interval.



4. Static Hyperbola

A hyperbola is classically defined as the locus of the point whose difference in the distances from two fixed points (foci), is a constant. A hyperbola so defined, in terms of two fixed points is referred to here as a Static Hyperbola.

Proposition

Dynamic Hyperbola

Two dynamic circles with equal expansion rates, non-coincident sources and distinct instants of emergence, come to intersect each other in a hyperbolic branch. A hyperbola so defined, in terms of two dynamic circles is referred to here as a Dynamic Hyperbola. A formal justification for this proposition is given in the proof of the theorem below.

Theorem

The locus of the intersection points of two dynamic circles is a dynamic hyperbola. Its analytical equation in the XY-plane is given by:

$$\frac{x^2}{\left(\frac{u.\Delta t_{AB}}{2}\right)^2} - \frac{y^2}{a^2 - \left(\frac{u.\Delta t_{AB}}{2}\right)^2} = 1$$

Where $(-a, 0)$ and $(a, 0)$ are the point locations of sources A and B respectively, Δt_{AB} is the Inter-Source Interval and u is the uniform rate of expansion of the dynamic circles.

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Proof

Consider two point Sources A and B located at positions $(-a, 0)$ and $(a, 0)$, respectively in a two dimensional XY-plane, with the Origin $O(0,0)$ lying mid-way between them. Say that a dynamic circle emerges from source A at an instant of time t_A and a similar dynamic circle from source B at a later instant t_B. Also assume that the speed of propagation u of both dynamic circles is equal and uniform in all directions. Then the equation of the dynamic circle emanating from source $A(-a, 0)$, at a given time $t > t_A$, can be written as:

$$(x + a)^2 + y^2 = R^2 \qquad \text{...(1)}$$

Similarly, the equation of the dynamic circle emanating from source $B(a, 0)$, at the instant $t > t_B$, can be written as:

$$(x - a)^2 + y^2 = r^2 \qquad \text{...(2)}$$

Where R and r are the instantaneous radii of the dynamic circles emanating from sources A and B, respectively. Note that, $R > r$ for $t_A < t_B$. Also, since the speed of propagation u of both dynamic circles is equal and uniform in all directions, we can write:

$$u = \frac{dR}{dt} = \frac{dr}{dt} \qquad \text{...(3)}$$

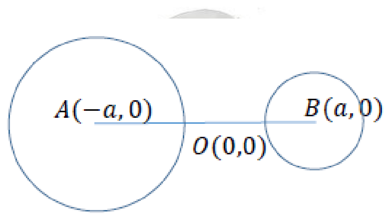

Figure 1: *Dynamic Circles emergent from sources A and B, in temporal succession*

Subtracting (2) from (1),

$$(x + a)^2 - (x - a)^2 = R^2 - r^2$$

On simplifying,

$$x = \frac{(R^2 - r^2)}{4a} \qquad \text{...(4)}$$

Squaring (4),

$$x^2 = \frac{(R^2 - r^2)^2}{16a^2} \qquad \text{....(5)}$$

Differentiating (5) with respect to time,

$$2x\frac{dx}{dt} = \frac{2(R^2 - r^2)(2R.\frac{dR}{dt} - 2r.\frac{dr}{dt})}{16a^2}$$

$$2x\frac{dx}{dt} = \frac{4u(R^2 - r^2)(R - r)}{16a^2} \qquad \text{(By (3))}$$

$$2x\frac{dx}{dt} = \frac{4u(R + r)(R - r)^2}{16a^2} \qquad \text{....(6)}$$

Substituting (4) in (1),

$$y^2 = R^2 - (x + a)^2$$

$$= R^2 - \left(\frac{(R^2 - r^2)}{4a} + a\right)^2$$

$$= (R + \left(\frac{(R^2 - r^2)}{4a} + a\right))(R - \left(\frac{(R^2 - r^2)}{4a} + a\right))$$

$$= \frac{(R^2 - r^2 + 4a^2 + 4aR).(-R^2 + r^2 - 4a^2 + 4aR)}{16a^2}$$

$$= -\frac{(R^4 + r^4 + 16a^4 - 2R^2r^2 - 8a^2R^2 - 8a^2r^2)}{16a^2}$$

$$= -\frac{\left[(R^2 + r^2 - 4a^2)^2 - 4R^2r^2\right]}{16a^2}$$

$$= -\frac{\left[((R-r)^2 + 2Rr - 4a^2)^2 - 4R^2r^2\right]}{16a^2}$$

$$= -\frac{[((R-r)^2 + 2Rr - 4a^2) + 2Rr][((R-r)^2 + 2Rr - 4a^2) - 2Rr]}{16a^2}$$

$$= -\frac{((R-r)^2 + 4Rr - 4a^2)((R-r)^2 - 4a^2)}{16a^2}$$

$$y^2 = -\frac{((R+r)^2 - 4a^2)((R-r)^2 - 4a^2)}{16a^2} \qquad ...(7)$$

From (7), it is clear that in order for $y \in \mathbb{R}$**,** either one of the following two conditions must hold true:

(i) $R + r > 2a$ and $R - r < 2a$, or
(ii) $R + r < 2a$ and $R - r > 2a$

In order that the two dynamic circles intersect each other to trace out the locus of some curve, (it will be later shown that the curve is a branch of a hyperbola with vertex V lying somewhere on the line AB joining the point sources A and B), it is necessary that condition (i) holds true. Condition (ii) would geometrically imply that the dynamic circles intersect nowhere in the XY-plane and is therefore rejected. So provided condition (i) holds true, we can write:

$$y = \pm\sqrt{-\frac{((R+r)^2 - 4a^2)((R-r)^2 - 4a^2)}{16a^2}} \quad \in \mathbb{R} \qquad ...(8)$$

Differentiating (7) with respect to time,

$$2y.\frac{dy}{dt} = -\frac{\left[((R+r)^2 - 4a^2).2(R-r)\left(\frac{dR}{dt} - \frac{dr}{dt}\right) + ((R-r)^2 - 4a^2).2(R+r)\left(\frac{dR}{dt} + \frac{dr}{dt}\right)\right]}{16a^2}$$

$$\Rightarrow \qquad 2y.\frac{dy}{dt} = -\frac{4u(R+r)((R-r)^2 - 4a^2)}{16a^2} \qquad ...(9) \qquad \text{(By (3))}$$

To re-iterate, t_A and t_B are the instants at which dynamic circles emerge from sources A and B, respectively ($t_A < t_B$). Additionally, let us assume τ to be the instant at which both these expanding wavefronts come to meet at a common point V lying on the line AB. We can therefore reason that the dynamic circle arising from source A, would have grown from an initial radius $R = 0$ to $R = R(\tau)$ in the time interval spanning t_A to τ. Similarly, the dynamic circle arising from source B, would have grown from an initial radius $r = 0$ to

$r = r(\tau)$ in the time interval spanning t_B to τ. So we can integrate equation (3), keeping in mind that the speed of propagation of both dynamic circles is equal and uniform in all directions and that $t_A < t_B < \tau$:

$$\int_0^{R(\tau)} dR = \int_{t_A}^{\tau} u.dt \Rightarrow R(\tau) = u(\tau - t_A) \qquad ...(10)$$

$$\int_0^{r(\tau)} dr = \int_{t_B}^{\tau} u.dt \Rightarrow r(\tau) = u(\tau - t_B) \qquad ...(11)$$

At the instant, $t = \tau$, both the dynamic circles meet at the point V on the line $AB = 2a$. Hence,

$$R(\tau) + r(\tau) = 2a \qquad ...(12)$$

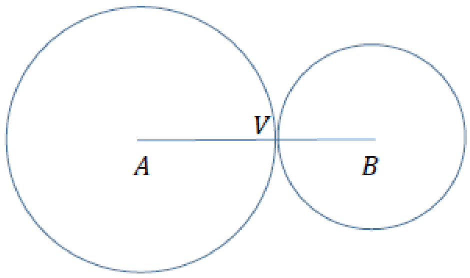

Figure 2: *Dynamic circles expand to meet at a single point V lying on the line joining A and B*

Subtracting (11) from (10),

$$R(\tau) - r(\tau) = u(t_B - t_A) = u.\Delta t_{AB} \qquad ...(13)$$

The two expanding dynamic circles will intersect each other at two points, call them P and P', after time $t > \tau$. The (x, y) co-ordinates of these point-pair intersections are given by equations (4) and (8):

$$\left(\frac{(R(t)^2 - r(t)^2)}{4a}, \pm\sqrt{-\frac{((R(t)+r(t))^2 - 4a^2)((R(t)-r(t))^2 - 4a^2)}{16a^2}}\right) \qquad ...(14)$$

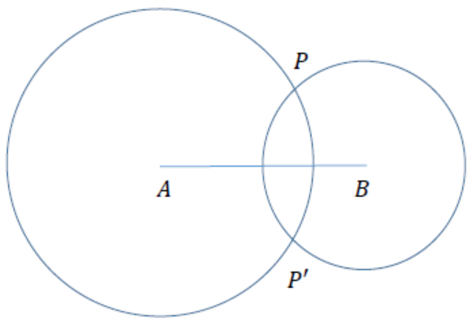

Figure 3: *Dynamic circles expand to intersect each other at two points P and P'*

The co-ordinate of the point V lying on AB can be found by substituting (12) & (13) in (14):

$$\left(\frac{u\Delta t_{AB}}{2}, 0\right) \qquad \text{...(15)}$$

Since the two dynamic circles propagate outwards at the same expansion rate u, we can expect that the instantaneous difference in their radii, $R(t) - r(t)$ to be constant with time. A formal justification of this statement can be made as follows:

$$\frac{d(R(t)-r(t))}{dt} = \frac{dR}{dt} - \frac{dr}{dt} = u - u = 0 \qquad \text{(By (3))}$$

$$\Rightarrow R(t) - r(t) = constant$$

This would imply that Equation (13) should hold true for all times, $t \geq \tau$. That is,

$$R(t) - r(t) = u(t_B - t_A) = u.\Delta t_{AB} \qquad \text{...(16)}$$

This satisfies the defining property of a hyperbola, as the locus of the point whose difference in the distances from two fixed points (foci), is a constant (Refer Defintion-4). That implies, the locus of the point of intersections of two dynamic circles emanating from sources A and B, takes the shape of a hyperbola, since the differences in their instantaneous radii have been shown to be constant. Therefore, $V\left(\frac{u\Delta t_{AB}}{2}, 0\right)$ will be the co-ordinate of the Vertex of one branch of a hyperbola, generated when source A emits a dynamic circle before source B. The Vertex of the complementary branch of the hyperbola is generated when source B emits a dynamic circle before source A and has its vertex at the co-ordinate $V'\left(-\frac{u\Delta t_{BA}}{2}, 0\right)$, since $\Delta t_{AB} = t_B - t_A = -(t_A - t_B) = -\Delta t_{BA}$. The preceding argument justifies the validity of the proposition that two dynamic circles intersect each other in a hyperbola.

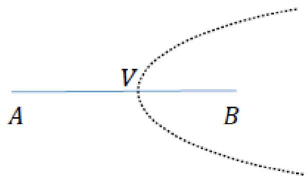

Figure 4: *Locus of the Intersection Points when Source A emits before Source B*

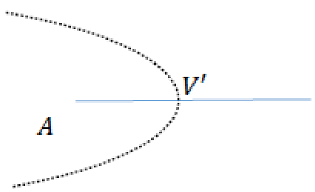

Figure 5: *Locus of the Intersection Points when Source B emits before Source A*

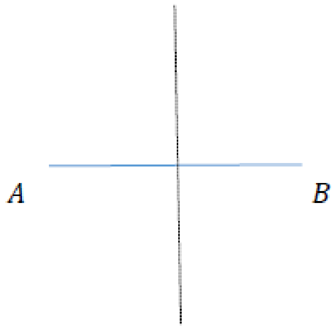

Figure 6: *Locus of the Intersection Points when Sources A and B emit simultaneously*

The general equation of a hyperbola with center at origin and transverse axis along the X-axis is:

$$\frac{x^2}{C^2} - \frac{y^2}{D^2} = 1 \qquad \text{...(17)}$$

Where C and D are the semi-lengths of the transverse and conjugate axes, respectively. The value of the constant C is already known to us from (15) since it represents the distance of the vertex of the hyperbola from the origin. That is,

$$C = \frac{u\Delta t_{AB}}{2} \qquad \text{...(18)}$$

However, the value of the constant D is yet to be determined. Once D is found and put into (17), we would have arrived at the required equation of the hyperbola. (Note that the sources $A(-a, 0)$ and $B(a, 0)$ lie at the foci of the hyperbola).

Differentiating (17) with respect to time,

$$\frac{1}{C^2} 2x\frac{dx}{dt} - \frac{1}{D^2} 2y\frac{dy}{dt} = 0$$

The above equation should hold true for all times $t \geq \tau > t_B > t_A$. This would mean that for $t = \tau$,

$$\frac{1}{C^2}.2x\frac{dx}{dt}_{t=\tau} - \frac{1}{D^2}.2y\frac{dy}{dt}_{t=\tau} = 0 \qquad \text{...(19)}$$

From Equations (6), (12) and (13),

$$2x\frac{dx}{dt}_{t=\tau} = \frac{4u(R(\tau)+r(\tau))(R(\tau)-r(\tau))^2}{16a^2} = 4u.2a.\frac{(u\Delta t_{AB})^2}{16a^2} = \frac{u^3(\Delta t_{AB})^2}{2a} \qquad \text{...(20)}$$

From Equations (9), (12) and (13),

$$2y.\frac{dy}{dt}_{t=\tau} = -\frac{4u(R(\tau)+r(\tau))\left((R(\tau)-r(\tau))^2 - 4a^2\right)}{16a^2} = -4u.\frac{2a\left((u.\Delta t_{AB})^2 - 4a^2\right)}{16a^2} = -\frac{u\left((u.\Delta t_{AB})^2 - 4a^2\right)}{2a} \quad \text{...(21)}$$

Substituting (20), (21) and (18) in Equation (19),

$$\frac{1}{\left(\frac{u\Delta t_{AB}}{2}\right)^2}\frac{u^3(\Delta t_{AB})^2}{2a} - \frac{1}{D^2}\left(-\frac{u\left((u.\Delta t_{AB})^2 - 4a^2\right)}{2a}\right) = 0$$

On algebraic simplification of the above, we get:

$$D^2 = a^2 - \frac{u^2(\Delta t_{AB})^2}{4} = a^2 - \left(\frac{u\Delta t_{AB}}{2}\right)^2 = a^2 - C^2 \qquad \text{...(22)} \quad \text{(By (18))}$$

Substituting (22) and (18) in (17), we finally arrive at,

$$\boxed{\frac{x^2}{\left(\frac{u\Delta t_{AB}}{2}\right)^2} - \frac{y^2}{a^2 - \left(\frac{u\Delta t_{AB}}{2}\right)^2} = 1}$$

This is the analytical equation of the dynamic hyperbola representing the locus of the points of intersection of two dynamic circles emanated from sources A and B at times t_A and t_B, respectively ($t_A < t_B$) . It is expressed in terms of the Inter-Source Interval Δt_{AB}, the speed of propagation u and the position of the sources $(\pm a, 0)$ with respect to the origin O, which lies midway between them.

Remarks

A. On the Dynamic Hyperbola Equation

- The vertices lie at the co-ordinate points $V\left(\frac{u\Delta t_{AB}}{2}, 0\right)$ and $V'\left(-\frac{u\Delta t_{BA}}{2}, 0\right)$
- The foci lie at the co-ordinate points $A(-a, 0)$ and $B(a, 0)$
- The center lies at the origin $O(0,0)$

B. In the converse case, where the dynamic circle from source B emerges before that from source A (i.e. $t_B < t_A$), an identical equation of a dynamic hyperbola can be derived with the sole difference that Δt_{AB} is replaced by Δt_{BA}. N.B. $\Delta t_{BA} = t_A - t_B = -(t_B - t_A) = -\Delta t_{AB}$.

$$\frac{x^2}{\left(\frac{u\Delta t_{BA}}{2}\right)^2} - \frac{y^2}{a^2 - \left(\frac{u\Delta t_{BA}}{2}\right)^2} = 1$$

C. Eccentricity of the Dynamic Hyperbola

The expressions for the squared lengths of semi-transverse axis and semi-conjugate axis were found to be $C^2 = \left(\frac{u\Delta t_{AB}}{2}\right)^2$ and $D^2 = a^2 - \left(\frac{u\Delta t_{AB}}{2}\right)^2$, respectively. The eccentricity e of a hyperbola is related to these quantities by the expression:

$$D^2 = C^2(e^2 - 1)$$

$$\Rightarrow \qquad e = \sqrt{1 + \frac{D^2}{C^2}}$$

$$\Rightarrow \qquad e = \sqrt{1 + \frac{a^2 - \left(\frac{u\Delta t_{AB}}{2}\right)^2}{\left(\frac{u\Delta t_{AB}}{2}\right)^2}}$$

$$\Rightarrow \qquad e = \frac{2a}{u\Delta t_{AB}}$$

Since the eccentricity of a hyperbola is always greater than unit ($e > 1$), it implies that,

$$u\Delta t_{AB} < 2a$$

$$\Rightarrow R(t) - r(t) < 2a \qquad \text{(By (16))}$$

The above inequality holds true when the dynamic circle from source A emerges before that from source B. However, in the converse case when the dynamic circle from source B emerges before that from source A, the following inequality should hold:

$$r(t) - R(t) < 2a$$

On combining both the above inequalities, we arrive at the Principal Condition that should be fulfilled for the generation of a dynamic hyperbola from two dynamic circles:

$$|R(t) - r(t)| < 2a$$

D. When the Inter-Source Interval is brought close to zero, that is as $\Delta t_{AB} \to 0$ or as $\Delta t_{BA} \to 0$, both vertices V and V' approach the origin $O(0,0)$ and the hyperbolic branches gradually straighten out to coincide with the Y-axis, whose equation is $x = 0$. To illustrate this, put $\Delta t_{AB} = 0$ in the dynamic hyperbola equation:

$$\frac{x^2}{0^2} - \frac{y^2}{a^2} = 1 \Rightarrow x^2 = 0^2.\left(1 + \frac{y^2}{a^2}\right) = 0 \Rightarrow x = 0$$

E. Dual Interpretations of the Quantity Δt_{AB}

(i) Δt_{AB} as the Inter-Source Interval (ISI)

Δt_{AB} represents the time interval spanning the instants between the emanation of dynamic circles from sources A and B. It is mathematically expressed as $\Delta t_{AB} = t_B - t_A$, when A generates a dynamic circle before B and as $\Delta t_{BA} = t_A - t_B$, when B generates a dynamic circle before A.

(ii) Δt_{AB} as the Time Difference of Arrival (TDOA)

Re-iterating Equation (16),

$$R(t) - r(t) = u(t_B - t_A) = u.\Delta t_{AB}$$

$$\Rightarrow \frac{R(t)-r(t)}{u} = t_B - t_A$$

$$\Rightarrow \frac{R(t)}{u} - \frac{r(t)}{u} = t_B - t_A$$

$$\Rightarrow t_{AP} - t_{BP} = t_B - t_A$$

Where,

$$t_{AP} = time\ taken\ for\ the\ dynamic\ circle\ to\ arrive\ at\ P\ from\ source\ A$$

$$t_{BP} = time\ taken\ for\ the\ dynamic\ circle\ to\ arrive\ at\ P\ from\ source\ B$$

$$t_A = Instant\ at\ which\ source\ A\ emits\ a\ dynamic\ circle$$

$$t_B = Instant\ at\ which\ source\ B\ emits\ a\ dynamic\ circle$$

Therefore, it may be concluded that the difference in the times of arrival (TDOA) of the dynamic circles from sources A and B at an arbitrary point P, is equal to the Inter-Source Interval.

$$i.e. \qquad TDOA = ISI = \Delta t$$

Applications of the Dynamic Hyperbola Theorem

1. Modeling Neurosensory Systems upon the PWC principle

The equation of the dynamic hyperbola derived here was originally used to develop two hypothetical neurocomputational models of a generic sensory system. These geometrical models, are based on the principle of Polychronous Wavefront Computation (PWC). The complete details regarding this application can be found in my prior work *("A Mathematical Treatise on Polychronous Wavefront Computation and its Application into Modeling Neurosensory Systems").* Though the PWC principle has been around in the literature since the year 2006, it was lacking a rigorous mathematical treatment. Also, one conceptual error that has been repeated in the years subsequent to the publication of the original work, is the statement that two dynamic circles intersect in a parabola. It has been very elaborately shown here that two dynamic circles infact intersect in a branch of a hyperbola.

2. Localizing the position of a Receiver/Transmitter Station in a 2D plane

The equation of the dynamic hyperbola derived here was used to develop a novel trilateration algorithm, by means of which the position of a Receiver/Transmitter Station in a 2D plane can be ascertained. The complete details regarding this application can be found in my prior work *("A Novel Trilateration Algorithm for localization of a Receiver/Transmitter Station in a 2D plane using Analytical Geometry").* The new algorithm falls under the class of Time Difference of Arrival (TDOA) algorithms. They bear the distinct advantage over Time of Arrival (TOA) algorithms, in that accurate clock synchronization between the anchor and blind nodes is unnecessary.

3. Reformulating the analysis underlying Young's Double Slit Interference Experiment

The equation of the dynamic hyperbola derived here was used to reformulate the original analysis underlying the classical double slit interference experiment of 1801. The complete details regarding this application can be found in my prior work *("A Theoretical Reformulation of the Classical Double Slit Interference Experiment").* While Young's analysis is based on a set of geometrical assumptions, collectively referred to as the Parallel Ray Approximation (PRA), the new analysis bypasses the need for PRA by exploiting the equation of the dynamic hyperbola. The predictions that ensue possess greater precision in the context of localizing fringe position on the distant screen. For instance, in the old analysis, the fringes are predicted to be equally spaced regardless of distance from the screen center. But the new analysis predicts that the fringes are infact unequally spaced. This becomes increasingly evident on moving further away from the screen center.

References

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Acknowledgements

Gloria in Excelsis Deo

Dedication

- *To my dear father (Mr. Thomas Varghese) and mother (Dr. Annie Susan Thomas) – the constancy of their love, sustains my progress.*

- *To my Mathematics teacher (Mr. Shaukat Ali) – whom I greatly honor and respect for showing me what the good life consists of.*

- *To my aunt (Ms. Susheela George) – the manifold gifts that spring from her steadfast faith in Jesus Christ, is like a rock of hope to me.*

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