Acceleration of the Infinite Series for the Hyperbolic Sine, Hyperbolic Cosine, Struve and Bessel Function of the first kind

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March 19, 2014.

I the LORD search the heart, I try the reins, even to give every man according to his ways, and according to the fruit of his doings. - Jeremiah 17:10

Abstract. I prove some accelerations of the infinite series for the hyperbolic sine, hyperbolic cosine, Struve and Bessel function of the first kind.

1. Introduction

In this paper, I demonstrated the following expansions of the infinite series:

$$\sinh(z)=\sum_{k=0}^{\infty}\frac{z^{4k+1}}{(4k+1)!}\left[1+\frac{z^2}{4k+2}-\frac{z^2}{4k+3}\right],$$

$$\cosh(z)=\sum_{k=0}^{\infty}\frac{z^{4k}}{(4k)!}\left[1+\frac{z^2}{4k+1}-\frac{z^2}{4k+2}\right],$$

$$\frac{2^v\sqrt{\pi}}{z^v}\mathbf{H}_v(z)=\sum_{k=0}^{\infty}\frac{(2k)!z^{4k+1}}{(4k+1)!}\left[\frac{1}{\Gamma\left(2k+v+\frac{3}{2}\right)}-\frac{z^2}{2\Gamma\left(2k+v+\frac{5}{2}\right)}+\frac{(2k+1)z^2}{(4k+3)\Gamma\left(2k+v+\frac{5}{2}\right)}\right]$$

and

$$\frac{2^v\sqrt{\pi}}{z^v}J_v(z)=\sum_{k=0}^{\infty}\frac{z^{4k}}{(4k)!}\left[\frac{\Gamma\left(2k+\frac{1}{2}\right)}{\Gamma(2k+v+1)}-\frac{z^2\Gamma\left(2k+\frac{3}{2}\right)}{\Gamma(2k+v+2)}+\frac{(2k+1)z^2\Gamma\left(2k+\frac{3}{2}\right)}{(4k+3)\Gamma(2k+v+2)}\right],$$

which converges rapidly.

2. Acceleration for the Hyperbolic Sine, Hyperbolic Cosine, Struve and Bessel Function of the first kind.

2.1. Hyperbolic Sine Function.

Theorem 1. *For* $z\in\mathbb{R}$, *then*

$$\sinh(z)=\sum_{k=0}^{\infty}\frac{z^{4k+1}}{(4k+1)!}\left[1+\frac{z^2}{4k+2}-\frac{z^2}{4k+3}\right],$$

where $\sinh(z)$ *denotes the hyperbolic sine function and* $k!$ *denotes the factorial function.*

Proof. In previous paper [1], I demonstrated that

$$\sin(z)=\sum_{k=0}^{\infty}\frac{z^{4k+1}}{(4k+1)!}\left[1-\frac{z^2}{4k+2}+\frac{z^2}{4k+3}\right]. \tag{1}$$

I set $z\rightarrow iz$ in Eq. (1)

$$\sin(iz)=\sum_{k=0}^{\infty}\frac{i^{4k+1}z^{4k+1}}{(4k+1)!}\left[1-\frac{i^2z^2}{4k+2}+\frac{i^2z^2}{4k+3}\right],$$

so,

$$i\sinh(z)=i\sum_{k=0}^{\infty}\frac{z^{4k+1}}{(4k+1)!}\left[1+\frac{z^2}{4k+2}-\frac{z^2}{4k+3}\right].$$

Eliminate i in the equation above; this completes the proof. □

2.2. Hyperbolic Cosine Function.

Theorem 2. *For* $z \in \mathbb{R}$, *then*

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{z^{4k}}{(4k)!}\left[1 + \frac{z^2}{4k+1} - \frac{z^2}{4k+2}\right],$$

where $\cosh(z)$ *denotes the hyperbolic cosine function and* $k!$ *denotes the factorial function.*

Proof. In previous paper [1], I demonstrated that

$$\cos(z) = \sum_{k=0}^{\infty} \frac{z^{4k}}{(4k)!}\left[1 - \frac{z^2}{4k+1} + \frac{z^2}{4k+2}\right]. \tag{2}$$

I set $z \to iz$ in Eq. (2)

$$\cos(iz) = \sum_{k=0}^{\infty} \frac{i^{4k} z^{4k}}{(4k)!}\left[1 - \frac{i^2 z^2}{4k+1} + \frac{i^2 z^2}{4k+2}\right],$$

so,

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{z^{4k}}{(4k)!}\left[1 + \frac{z^2}{4k+1} - \frac{z^2}{4k+2}\right].$$

□

2.3. Struve Function.

Theorem 3. *For* $\Re(v) > -\frac{1}{2}$ and $z \in \mathbb{R}$, *then*

$$\frac{2^v\sqrt{\pi}}{z^v}\mathbf{H}_v(z) = \sum_{k=0}^{\infty} \frac{(2k)!z^{4k+1}}{(4k+1)!}\left[\frac{1}{\Gamma\left(2k+v+\frac{3}{2}\right)} - \frac{z^2}{2\Gamma\left(2k+v+\frac{5}{2}\right)} + \frac{(2k+1)z^2}{(4k+3)\Gamma\left(2k+v+\frac{5}{2}\right)}\right],$$

where $\mathbf{H}_v(z)$ *denotes the Struve function,* $k!$ *denotes the factorial function and* $\Gamma(z)$ *denotes the gamma function.*

Proof. I put $z \to zt$ in the Eq. (1), multiply by $(1-t^2)^{v-\frac{1}{2}}$ and integrate from 0 at 1 with respect to t, thus

$$\int_0^1 (1-t^2)^{v-\frac{1}{2}}\sin(zt)\mathrm{dt} \tag{3}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(4k+1)!}\left[z^{4k+1}\int_0^1 (1-t^2)^{v-\frac{1}{2}} t^{4k+1}\mathrm{dt} - \frac{z^{4k+3}}{4k+2}\int_0^1 (1-t^2)^{v-\frac{1}{2}} t^{4k+3}\mathrm{dt} + \frac{z^{4k+3}}{4k+3}\int_0^1 (1-t^2)^{v-\frac{1}{2}} t^{4k+3}\mathrm{dt}\right]$$

$$= \Gamma\left(v+\frac{1}{2}\right)\sum_{k=0}^{\infty} \frac{(2k)!z^{4k+1}}{2(4k+1)!}\left[\frac{1}{\Gamma\left(2k+v+\frac{3}{2}\right)} - \frac{z^2}{2\Gamma\left(2k+v+\frac{5}{2}\right)} + \frac{(2k+1)z^2}{(4k+3)\Gamma\left(2k+v+\frac{5}{2}\right)}\right].$$

On the other hand, I know [2, page 328] that

$$\int_0^1 (1-t^2)^{v-\frac{1}{2}}\sin(zt)\mathrm{dt} = \frac{\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{2\left(\frac{1}{2}z\right)^v}\mathbf{H}_v(z) \tag{4}$$

I substitute the rigth hand side of the Eq. (4) into the left hand side of the Eq. (3)

$$\frac{\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{2\left(\frac{1}{2}z\right)^v}\mathbf{H}_v(z)$$

$$= \Gamma\left(v+\frac{1}{2}\right)\sum_{k=0}^{\infty} \frac{(2k)!z^{4k+1}}{2(4k+1)!}\left[\frac{1}{\Gamma\left(2k+v+\frac{3}{2}\right)} - \frac{z^2}{2\Gamma\left(2k+v+\frac{5}{2}\right)} + \frac{(2k+1)z^2}{(4k+3)\Gamma\left(2k+v+\frac{5}{2}\right)}\right].$$

Eliminate $\frac{\Gamma\left(v+\frac{1}{2}\right)}{2}$ in both members of the Equation above; this completes the proof. □

2.4. Bessel Function of the first kind.

Theorem 4. *For* $\Re(v) > -\frac{1}{2}$ and $z \in \mathbb{R}$, *then*

$$\frac{2^v\sqrt{\pi}}{z^v}J_v(z) = \sum_{k=0}^{\infty}\frac{z^{4k}}{(4k)!}\left[\frac{\Gamma\left(2k+\frac{1}{2}\right)}{\Gamma(2k+v+1)} - \frac{z^2\Gamma\left(2k+\frac{3}{2}\right)}{\Gamma(2k+v+2)} + \frac{(2k+1)z^2\Gamma\left(2k+\frac{3}{2}\right)}{(4k+3)\Gamma(2k+v+2)}\right],$$

where $J_v(z)$ *denotes the Bessel function of the first kind,* $k!$ *denotes the factorial function and* $\Gamma(z)$ *denotes the gamma function.*

Proof. I put $z \to zt$ in the Eq. (2), multiply by $(1-t^2)^{v-\frac{1}{2}}$ and integrate from 0 at 1 with respect to t, thus

$$\int_0^1 (1-t^2)^{v-\frac{1}{2}}\cos(zt)\mathrm{dt} \tag{5}$$

$$= \sum_{k=0}^{\infty}\frac{1}{(4k)!}\left[z^{4k}\int_0^1 (1-t^2)^{v-\frac{1}{2}}t^{4k}\mathrm{dt} - \frac{z^{4k+2}}{4k+1}\int_0^1 (1-t^2)^{v-\frac{1}{2}}t^{4k+2}\mathrm{dt}\right.$$

$$\left. + \frac{z^{4k+2}}{4k+2}\int_0^1 (1-t^2)^{v-\frac{1}{2}}t^{4k+2}\mathrm{dt}\right]$$

$$= \frac{\Gamma\left(v+\frac{1}{2}\right)}{2}\sum_{k=0}^{\infty}\frac{z^{4k}}{(4k)!}\left[\frac{\Gamma\left(2k+\frac{1}{2}\right)}{\Gamma(2k+v+1)} - \frac{z^2\Gamma\left(2k+\frac{3}{2}\right)}{\Gamma(2k+v+2)} + \frac{(2k+1)z^2\Gamma\left(2k+\frac{3}{2}\right)}{(4k+3)\Gamma(2k+v+2)}\right].$$

On the other hand, I know [2, page 48] that

$$\int_0^1 (1-t^2)^{v-\frac{1}{2}}\cos(zt)\mathrm{dt} = \frac{\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{2\left(\frac{1}{2}z\right)^v}J_v(z) \tag{6}$$

I substitute the rigth hand side of the Eq. (6) into the left hand side of the Eq. (5)

$$\frac{\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{2\left(\frac{1}{2}z\right)^v}J_v(z)$$

$$= \frac{\Gamma\left(v+\frac{1}{2}\right)}{2}\sum_{k=0}^{\infty}\frac{z^{4k}}{(4k)!}\left[\frac{\Gamma\left(2k+\frac{1}{2}\right)}{\Gamma(2k+v+1)} - \frac{z^2\Gamma\left(2k+\frac{3}{2}\right)}{\Gamma(2k+v+2)} + \frac{(2k+1)z^2\Gamma\left(2k+\frac{3}{2}\right)}{(4k+3)\Gamma(2k+v+2)}\right].$$

Eliminate $\frac{\Gamma\left(v+\frac{1}{2}\right)}{2}$ in both members of the Equation above; this completes the proof. □

References

[1] Guedes, Edigles, *Acceleration of the Infinite Series for the Sine and Cosine Functions*, to appear.

[2] Watson, G. N., *A Treatise on the Theory of Bessel Functions*, Merchant Books, 2008.