Some Amazing Infinite Series for the Multiplication Between Sine, Hyperbolic Sine and Exponential Functions

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Therefore, as God's chosen people, holy and dearly loved, clothe yourselves with compassion, kindness, humility, gentleness and patience. - Colossians 3:12

Abstract. I prove the expansion in infinite seiries for the multiplication between sine, hyperbolic sine and exponential functions, that do not exist in the mathematical literature.

1. Introduction

In this paper, I demonstrated some infinite series, which converge rapidly, for the multiplication between sine, hyperbolic sine and exponential functions. This has led the following expansions in power series:

$$\frac{2e^{1/2}}{3\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(3k+1)!}\left(\frac{k+1}{3k+2}\right)$$

and

$$\frac{98e^{1/14}}{3\sqrt{3}}\sin\left(\frac{\sqrt{3}}{14}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{7^{3k}(3k+1)!}\left(\frac{7k+5}{3k+2}\right).$$

2. The Amazing Infinite Series

Theorem 1. *For $z\in\mathbb{R}$ and $|z|\leqslant 1,$ then*

$$\frac{2}{\sqrt{3}}\sin\left(\frac{z}{2}\right)\sinh\left(\frac{z\sqrt{3}}{2}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k z^{6k+2}}{(6k+2)!}\left[1+\frac{z^2}{6k+3}-\frac{z^2}{6k+4}\right],$$

where $\sin(z)$ denotes the sine function and $\sinh(z)$ denotes the hyperbolic sine function.

Proof. I know that

$$\sum_{k=0}^{\infty}\frac{\sin\left(\frac{\pi k}{3}\right)z^{2k}}{(2k)!}=\sin\left(\frac{z}{2}\right)\sinh\left(\frac{z\sqrt{3}}{2}\right). \tag{1}$$

On the other hand, I notice that

$$\sin\left(\frac{\pi k}{3}\right)=\frac{\sqrt{3}}{2}\times\begin{cases}1, k=1,2,7,8,13,14,19,20,\ldots\\ 0, k=0,3,6,9,12,15,18,\ldots\\ -1, k=4,5,10,11,16,17,\ldots\end{cases} \tag{2}$$

From Eq. (1) and Eq. (2), it follows that

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\sin\left(\frac{\pi k}{3}\right)z^{2k}}{(2k)!}&=\frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}(-1)^k\left[\frac{z^{6k+2}}{(6k+2)!}+\frac{z^{6k+4}}{(6k+4)!}\right] \quad (3)\\ &=\frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k z^{6k+2}}{(6k+2)!}\left[1+\frac{z^2}{(6k+4)(6k+3)}\right]\\ &=\frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k z^{6k+2}}{(6k+2)!}\left[1+\frac{z^2}{6k+3}-\frac{z^2}{6k+4}\right].\end{aligned}$$

I substitute the right hand side of the Eq. (3) in the left hand side of the Eq. (1). This completes the proof. □

Theorem 2. *For* $z \in \mathbb{R}$ *and* $|z| \leqslant 1$, *then*

$$\frac{2e^{z^2/2}}{\sqrt{3}}\sin\left(\frac{z^2\sqrt{3}}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{6k+2}}{(3k+1)!}\left[1 + \frac{z^2}{3k+2}\right],$$

where $\sin(z)$ *denotes the sine function and* e^z *denotes the exponential function.*

Proof. I know that

$$\sum_{k=0}^{\infty} \frac{\sin\left(\frac{\pi k}{3}\right) z^{2k}}{(k)!} = e^{z^2/2}\sin\left(\frac{z^2\sqrt{3}}{2}\right). \tag{4}$$

From Eq. (2) and Eq. (4), I obtain

$$\sum_{k=0}^{\infty} \frac{\sin\left(\frac{\pi k}{3}\right) z^{2k}}{(k)!} = \frac{\sqrt{3}}{2}\sum_{k=0}^{\infty} (-1)^k \left[\frac{z^{6k+2}}{(3k+1)!} + \frac{z^{6k+4}}{(3k+2)!}\right]$$

$$= \frac{\sqrt{3}}{2}\sum_{k=0}^{\infty} \frac{(-1)^k z^{6k+2}}{(3k+1)!}\left[1 + \frac{z^2}{3k+2}\right].$$

□

Remark 3. Choosing $z = 1, 1/\sqrt{2}, 1/\sqrt{3}, 1/2, 1\sqrt{5}, 1/\sqrt{6}, 1/\sqrt{7}$, in Theorem 2, I find respectively

$$\frac{2e^{1/2}}{3\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)!}\left(\frac{k+1}{3k+2}\right),$$

$$\frac{8e^{1/4}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{4}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{3k}(3k+1)!}\left(\frac{6k+5}{3k+2}\right),$$

$$6\sqrt{3}e^{1/6}\sin\left(\frac{1}{2\sqrt{3}}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{3^{3k}(3k+1)!}\left(\frac{9k+7}{3k+2}\right),$$

$$\frac{32e^{1/8}}{3\sqrt{3}}\sin\left(\frac{\sqrt{3}}{8}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^{3k}(3k+1)!}\left(\frac{4k+3}{3k+2}\right),$$

$$\frac{50e^{1/10}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{10}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{5^{3k}(3k+1)!}\left(\frac{15k+11}{3k+2}\right),$$

$$24\sqrt{3}e^{1/12}\sin\left(\frac{1}{4\sqrt{3}}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{6^{3k}(3k+1)!}\left(\frac{18k+13}{3k+2}\right),$$

$$\frac{98e^{1/14}}{3\sqrt{3}}\sin\left(\frac{\sqrt{3}}{14}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{7^{3k}(3k+1)!}\left(\frac{7k+5}{3k+2}\right).$$