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Rethinking the Numbers: Quadrature and Trisection in Actual Infinity

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Abstract: **The problems of squaring the circle or "quadrature" and trisection of an acute angle are supposed to be impossible to solve because the geometric constructibility, i.e. compass-and-straightedge construction, of irrational numbers like π is involved, and such numbers are not constructible. So, if these two problems were actually solved, it would imply that irrational numbers are geometrically constructible and this, in turn, that the infinite of the decimal digits of such numbers has an end, because it is this infinite which inhibits constructibility. A finitely infinite number of decimal digits would be the case if the infinity was the actual rather than the potential one. Euclid's theorem rules out the presence of actual infinity in favor of the infinite infinity of the potential infinity. But, space per se is finite even if it is expanding all the time, casting consequently doubt about the empirical relevance of this theorem in so far as the nexus space-actual infinity is concerned. Assuming that the quadrature and the trisection are space only problems, they should subsequently be possible to solve, prompting, in turn, a consideration of the real-world relevance of Euclid's theorem and of irrationality in connection with time and spacetime and hence, motion rather than space alone. The number-computability constraint suggests that only logically, i.e. through Euclidean geometry, this issue can be dealt with. So long as any number is expressible as a polynomial root the issue at hand boils down to the geometric constructibility of any root. This article is an attempt towards this direction after having tackled the problems of quadrature and trisection first by themselves through reduction impossible in the form of proof by contradiction, and then as two only examples of the general problem of polynomial root construction. The general conclusion is that an irrational numbers is irrational on the real plane, but in the three-dimensional world, it is as a vector the image of one at least constructible position vector, and through the angle formed between them, constructible becomes the "irrational vector" too, as a right-triangle side.So, the physical, the real-world reflection of the impossibility of quadrature and trisection should be sought in connection with spacetime, motion, and potential infinity.**

Keywords: **Actual Infinity, Geometric Constructability, Quadrature, Trisection**

"As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent mutual forces, and have marched together towards perfection." Joseph Louis Lagrange (1736-1813, [45, Preface])

1. INTRODUCTION

"Two truths cannot contradict one another." Galileo Galilei (1564-1642, [23, p.186])

At any given point in time, the universe is finite. If it expands over time, it will be an ever-changing finiteness, but still finiteness [63]. If it was infinite, the term "expansion" would be meaningless. At the other end, if it was not expanding, it would not necessarily imply that it is infinite. In any case, we take the static finiteness of space as our working hypothesis in this book.

The universe has a beginning and an end, some extreme limits, changing perhaps with the passage of time. It follows empirically that (i) any number in connection with space should have a beginning and an end and should be subsequently geometrically constructible, and (ii) any number without actualization spatially-wise and hence, in constructible number, is either a number which man is unable to compute accurately and can only approximate [60], or a number pertaining to the time component of the universe. But, space-wise, the infinite must be the finite one, the actual as opposed to potential infinite. And, the Arithmetic which is empirically relevant space-wise should be the one of this type of infinite. The greatest obstacle to the advancement of such a view of the Arithmetic, is the fundamental theorem of Arithmetic by Euclid (323-283 B.C., [17]). For us, here, the actual infinite is identified with rationality and constructibility whereas potential infinity is identified with irrationality. It is a thesis which would be heretical if judged from the viewpoint of Euclid's theorem. Let us at least challenge him as follows:

Let the natural number N be the product of all prime numbers, p_i, $i = 1,2, \dots, n < N$. Let $N' = N + l = l + \prod_1^n p_i$, where l can be any positive number but prime. N′cannot be prime either, because its division by any of the p's would leave a remainder equal to l. Actually, N′ cannot even be natural number, because by assumption there are no other p's that could form a product producing N'. This in turn, rules out l being a natural number, too; not even a fraction of integers, b/h, because then, $N' = N + l = N + (b/h) \Rightarrow hN' = hN + b$,which contradicts that N′ and by extension its multiple, cN′, cannot be natural numbers. Therefore, l must be an irrational number. One is thus inclined to conclude that if a number $N' > N$ has to exist, it will have to be an irrational number, and that the largest natural number will be the product of all primes if $l < 1$. That is, all what is left to have a workable definition of actual infinity as well as of potential infinity is to show that $l < 1$ obtains only when primes are finite in number, because if not, their product cannot be defined, which in turns implies that any other product is bound to be a partial one and hence consistent with $l > 1$.

Let $l = b$ so that $N' = b + \prod_i p_i$, where $i = 1,2, \dots, n$ and b is either the unit or a composite number. As such, N'

becomes a natural number and may be rewritten as a product of primes q_j, $j = 1,2,\dots,r < n$, which are already included in the product, $\prod_1^n p_i$: $N' = \prod_j q_j$, and $\prod_i p_i = \prod_j q_j \prod_k q_k$, $k = r+1, r+2, \dots, n$. Consequently,
$N' = b + \prod_i p_i = b + N'^{\prod_k q_k} \Rightarrow N = b/(1 - \prod_k q_k)$, and N' only if $1 \geq \prod_k q_k$, which is not true and which in turn, implies that N' is not a natural number and that $l < 1$. Would we have obtained this result if all primes were not there? The answer is negative as follows: $N' = b + \prod_\mu \nu$, with ν being μ primes and with b being any natural number. Next, let $N' = \prod_m v$and $\prod_m v = b \prod_w u$, i.e. b is the product of some of the primes v and the rest of them are designated through u. Equating N''s, yields that $b = b\prod_w u - \prod_\mu \nu \Rightarrow b = \prod_\mu \nu/(\prod_w u - 1)$. The numerator exceeds the denominator when $\prod_\mu \nu + 1 > \prod_w u \Rightarrow b + \prod_\mu \nu + 1 > b + \prod_w u$. Or, noting that, $b + \prod_\mu \nu = N' = b\prod_w u$, our inequality becomes, $b\prod_w u + 1 > b + \prod_w u \Rightarrow \prod_w u > (b-1)/(b-1) = 1$, which is true, because otherwise b would have to be negative.

Consequently, $b = l > 1$, unless all primes are taken into consideration, indeed [14, 53]. Suppose that I did not actually do so when I asserted earlier that I did, because simply the infinitely infinite of primes prevented me from doing so. But, then I should have produced l as a natural number, which I did not and hence, I did take all primes into account, telling me in turn this, that primes are finitely only infinite. This is a conclusion based on a RAI/C argument, which in this case may lack the validity of a formal proof. Nevertheless, it is a conclusion that does cast doubt as to the particular real-world context in which Euclid's theorem holds. It appears that: Actual infinite consists of the rational numbers that may be formed on the basis of natural numbers whose number is equal to the product of all primes, symbolizing it via $\boldsymbol{\varpi}$. Potential infinite consists of the potential infinite of the decimal digits that might start being added at the end of a given rational endlessly, and by the potential infinite of the order/disorder with which decimal digits would keep piling up.

That is, our potential infinity is the outcome of the interplay of these two kinds of potential infinity with regard to each of the finitely infinite rational numbers, over the whole set of rational numbers. There are as many such "two-footed" potential infinities as finite rationals. A potentially infinite number of irrationals like N' may come out of $\boldsymbol{\varpi}$. And, by "backward induction", (i.e. if $\boldsymbol{\varpi} > y \geq 0$, then $\boldsymbol{\varpi} \in \Xi$ and $y \in \Xi$ imply $\boldsymbol{\varpi} - y \in \Xi$, where Ξ is the set of rational numbers up to $\boldsymbol{\varpi}$, any other rational up to $\boldsymbol{\varpi}$ has its "own" potentially infinite irrationals. The set of potential infinities is a finitely infinite one, identical to Ξ, simply because it is subject to the superstructure of actual infinite. Before relaxing this Russell side of the set-theoretic definition [19, 34], through the introduction of time considerations, let us extend our conclusions to include complex numbers as well.

Given the fundamental theorem of Algebra, which dictates the involvement of the complex numbers in any sensible discussion of the Arithmetic, it should be remarked that our own perception of the Arithmetic here, may be extended to include imaginary numbers too, since this is only a matter of multiplication with $\iota = \sqrt{-1}$. But, as far as complex numbers are concerned, they have to be ordered by establishing ordering based not only on the magnitude of the real numbers, b and h of a complex number, $z = b + h\iota$, but also on $arg(z)$ as follows: (a) The complex numbers corresponding to the circumference of a given "complex circle", circle on the complex plane, are all smaller/larger *vis a vis* the numbers belonging to a larger/smaller complex circumference; and (b) The numbers on a given complex circumference take on their minimum and maximum values as in Fig. 1.1, which illustrates the whole "zigzag counting" of numbers that is proposed.

Let Re be the real line, Im be the imaginary axis, and the z's as follows: $(\boldsymbol{\alpha})\ Re(z_{10}) > Re(z_6) > Re(z_0)$ and $arg(z_{10}) = arg(z_6) = arg(z_0) = 0$ so that $z_{10} > z_6 > z_0$. $(\boldsymbol{\beta})\ Re(z_6) > Re(z_7) > Re(z_8) > Re(z_9) = 0andarg(z_9) = \pi/2 > arg(z_8) > arg(z_7) > arg(z_6) = 0$ so that $z_{10} > z_9 > z_8 > z_7 > z_6 > z_0$. $(\boldsymbol{\gamma})\ Re(z_5) > Re(z_4) = 0andarg(z_4) = 3\pi/2 < arg(z_5) < 2\pi$ so that $z_5 > z_4$ since in general, $b - h\iota > h\iota \Rightarrow b > 0$, which is true, and which along with the truth of the inequality, $b > b - h\iota > 0 \Rightarrow 0 > h\iota$ yields that: $z_{10} > z_9 > z_8 > z_7 > z_6 > z_5 > z_4 > z_0$. $(\boldsymbol{\delta})\ 0\ Re(z_3) = Re(z_1) > Re(z_2)$ and $\pi/2 < arg(z_5) < \pi = arg(z_2) < arg(z_1) < 3\pi/2$, which given the truth of relations, $-b + h\iota > -b \Rightarrow h\iota > 0, -b > -b - h\iota > 0 > -h\iota, -h\iota > b - h\iota \Rightarrow 0 > -b$, and $b - h\iota > -b + h\iota \Rightarrow (b/h) > \iota = \sqrt{-1}$, completes the ordering of z's as follows: $z_{10} > z_9 > z_8 > z_7 > z_6 > z_5 > z_4 > z_3 > z_2 > z_1 > z_0$.

The maximum complex number under a given circumference, is a purely imaginary number, followed by the minimum complex number of the immediately larger circumference, which is the same number but with negative sign, (minus an infinitesimally small real number right before the antipode of the maximum), and so on: zigzag counting. Complex numbers become completely ordered and the intersection of complex circles with the horizontal axis is another way of rendering the real line a complete continuum. Real numbers become thus something like the serial numbers of the complex circles; one real for each such circle, taking the absolute value of the real.

But, this is the only bijection that there can be between real and complex numbers. And, certainly, there can be none of any of these two sets of numbers with the set of the natural numbers given that infinity is the actual one. If not, an endless process of complex circle generation is

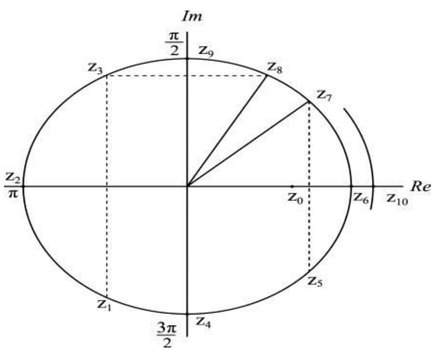

Fig. 1.1: Zigzag Counting of Numbers

triggered trying but never managing to catch up with the ever-expanding real line. The three-dimensional positioning of numbers mentioned in the Abstract, does not change this conclusion, since the sphere to which such a positioning connotes, is made up by its spherical slices, by circles. One side of the cosmos has to be seen as such a finite field from the viewpoint of constructibility. In so far as at any point in time there is some actual infinity at which the real line ends spatially-wise, complex numbers ensure the constructibility of all numbers especially when the complex or real character of the construction plane, is immaterial to the construction: Irrational numbers are irrational on the real plane, but in the three-dimensional world, they are images of position vectors, which are constructible on the complex plane.

The argument runs as follows: An irrational number is irrational and not constructible as a vector on the plane. But, in the three-dimensional space, such a vector would have one at least "shadow", which is constructible as a vector corresponding to a rational number. And, through this "shadow vector" and the angle formed between it and the "irrational vector", drawn both of these vectors as hypotenuse-side, respectively, of a right triangle, the irrational vector should be constructible by itself as well. That is, these vectors are position ones, belonging to a spherical section, a "spherical slice" of the three-dimensional complex space, and in this two-dimensional section, the hypotenuse-shadow is a circle radius whose projection on the real line corresponds to the irrational vector. The complex or real character of the construction plane is immaterial to the construction, but the fact is that the shadow is one in the complex space.

In view of the complex numbers, the size of actual infinity expands from $\boldsymbol{\varpi} \equiv \boldsymbol{\varpi}_1$ in the one-dimensional space, to the area $\pi\boldsymbol{\varpi}^2 \equiv \boldsymbol{\varpi}_2$ in the complex plane, and to the volume $4\pi\boldsymbol{\varpi}^3 \equiv \boldsymbol{\varpi}_3$ in the three-dimensional complexspace, in which one of the axes is the imaginary one; (the corresponding $\boldsymbol{\Xi}$'s are $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$ and $\boldsymbol{\Xi}_3$); all, with no breach in continuity whatsoever. An irrational number of the real line becomes constructible in the two- and three-dimensional space and hence, a rational number within this context. Consequently, irrationality, the potential infinite, has to be seen as the potentially infinite ways of vacillating between neighboring rational numbers. And, this can be the result only of motion and hence, velocity and time.

Our universe could be a Kurt Friedrich Gödel (1906-1978), for instance, type of constructible universe [15, 24], but with its recursive (and hence, time dependent) formula adjusted (becoming a differential equation) in line with chaos theory [39], with each constructible number acting as a local strange attractor and with the irrationality being described as endless oscillation about it and taking on values from the neighboring rationals. Like, for instance, oscillating about the number 1.41421 with the oscillation reaching number 1.35623, and upon return back to 1.41421, having taken on the value 1.4142135623... Or, something like this... This is how more or less the actual and the potential infinities could coexist absolved from the Russell-Epimenides of Knossos (7th-6th century B.C.) Paradox [34, 35].

Such a perception of the cosmos is important, because although a number like $(4\pi\boldsymbol{\varpi}^3/3)+1$ would not make sense within the context of actual infinity even after the introduction of the dynamic element, a potentially infinite sequence of positive integers may be defined over irrational numbers, ir, exclusively... One ir, two ir, ... $(\omega+1)ir$... , forming an ordered field. And, this sequence, abstracting from the ir, may be used to count all rational in the complex plane and composite world numbers. Consequently, the sum $(4\pi\boldsymbol{\varpi}^3/3)+1$ does make sense in so far as it may be referring as a serial number to an ir, or to a rational, which is less than $(4\pi\boldsymbol{\varpi}^3/3)$. Of course, all these number considerations cognitively, because nature knows only number one. At any given point in historic time, each and every being and phenomenon in the cosmos is unique, fabricating the new unique cosmos of the next point in historic time, with a new uniqueness of beings and phenomena, and so on. No single being and no single phenomenon is ever the same individually, because this is the rule, a prerequisite, for perpetuating the totality, which subsequently is never the same as well.

In what follows, some specific bibliographical references on the key concepts pervading the discussion and which need not be mentioned repeatedly, are: On Infinity: Maor [49], Moore [52], Rucker [57], and Zippin [69]. On Series: Bromwich [8], Knopp [41], Laugwitz [46], Manning [48], and Zygmund [70]. On Numbers and their History: Conway and Guy [12], Dickson [16], Guy [28], Ifrah [36], and Weil [66]. In Geometric Constructibility: Coxeter [13], Hobson [32], Hobson et al. [33], Kazarinoff [40], Knopp [42], Knorr [42], and Martin [50]. Of course, the whole discussion is permeated by the history on the subjects it treats, and some of the sources consulted are: On the History of Geometry: Greenberg [26], Hartshorne [29], and Wells [67]. On the History of Mathematics: Boyer and Merzbach [6], Cajori [9], Gray and Parshall [25], Gullberg [27], Jahnke [37], and Jones and Bedient [38]. And, particularly, on the History of Ancient Greek Mathematics: Christianidis [11], Heath [30] and [31], Netz [54], and Thomas [61] and [62].

Primitive Statements

Given the definitions of (a) the actual, $\boldsymbol{\varpi}$, as opposed to potential, ∞, infinity as developed earlier, (b) *apeiro*nomial as being any non-zero coefficient general polynomial of degree $\boldsymbol{\varpi}$, (c) irrational numbers as being decimal numbers with infinite non-repeating decimal digits, (d) geometric constructibility which is the ability to construct physically with a straightedge and a compass numbers as lines of a definite, exact rather than approximate length, and (e) RAI/C, which if it proves constructibility, it will be about lines and angles with a beginning and an end;

And, given the propositions that (f) all rational numbers are geometrically constructible and with a unique eventually repeating infinite decimal expansion if one's denominator involves a prime factor other than 2 or 5, and (g) all constructible numbers are algebraic numbers:

Special Theory

And, moreover, given (h) the observation that irrational numbers are mainly the sum of some infinite series, that the terms of a series may be seen as roots of an apeironomial, AP, and that the series, the sum-product or other function of roots, form subsequently an elementary symmetric apeironomial, EP, which is always equal to some AP-coefficient ratio:

It follows that any irrational number constructible via reduction impossible in the form of proof by contradiction (RAI/C): (1) is an algebraic number from the minimally irreducible polynomial having as a term the infinite series, i.e. EP, instead of the irrational number that the series represents, and (2) is a definite, exact rather than approximate number, and as such the infinite of the number of decimal digits is the actual infinite even if these digits are non-recurring.

General Theory

Finally, given that as it will be shown later, (i) all irrational numbers are constructible via RAI/C in the form of AP-coefficient ratios equaling some EP of the roots of AP: Any angle may be constructed as a line segment and vice versa:

It follows that (3) any irrational number is representable by the sum of some infinite series, it is an algebraic number from the minimally irreducible polynomial having as a term this series rather than the corresponding irrational number, and (4) the non-repetitiveness of the decimal digits of irrational numbers ceases sooner or later and the infinite of the number of these digits is the actual infinite.

In what follows, the points made by Special Theory are highlighted through the paradigms of the Squaring-Quadrature of the Circle and of the Trisection of an arbitrary acute angle. The General Theory is developed afterwards through the advancement of Theory of General Geometric Constructibility.

2. The Quadrature

"There is no place that can take away the happiness of a man, nor yet his virtue or wisdom. Anaxagoras, indeed, wrote on the squaring of the circle while in prison." Plutarch (c.46-120, [61, On Exile])

A. *A Brief Account of the Problem*

Constructing with a straightedge-ruler and a compass a square having area and perimeter equal to those of a given circle or vice versa, was deemed to be impossible by ancient Greeks: "…Bryson (of Heraclea) declared the circle to be greater than all inscribed and less than all circumscribed polygons" (Themistius, 317-c.390, [62]). That's the most that could be done with a ruler and a compass. Many attempts to refute the ancients have been made since then, but all have failed [32, 33]. In 1882, Ferdinand von Lindemann (1852-1939, [47]) proved that the squaring or quadrature of the circle is impossible, because π is a transcendental, rather than an algebraic number; that is, π is not a solution of any polynomial with rational coefficients. Hence, we cannot construct with a ruler and a compass a line segment x such that $x^2 = \pi R^2$, or setting the radius R of the circle equal to one, the number $x = \sqrt{\pi}$ is not constructible. It is quite clear that it is the inconstructibility of transcendence which is responsible for the impossibility of the Quadrature.

Is this thesis true or false? If inadequate computability is thought of corroborating some notion of potential infinity, this statement is indeed true. From the π=3.1605 of the Rhind papyrus in the 17th century B.C. [56] and the π=3.1415926535898732 of astronomer Ghiyath al-Din JamshidMas'ud al-Kashi (c.1380-1429) of Samarkand around 1430 [5, 18, 55] to the π with the 10000 decimal digits in 1958 and the π with the trillion decimal places being produced nowadays, there has always been a computation problem. Today, the problem is that real numbers are computed by finite, terminating algorithms. It is these computations that are taken to be the real numbers, not the real-real numbers *per se*. And, this presents problems like, for instance, that under the classical definition of a sequence, the set of computable numbers is not closed in so far as taking the *supremum* of a bounded sequence is concerned [7, 44]. Indeed, "He who can properly define and divide is to be considered a god" (Plato, 429-347 B.C., [68]).

Note for example that all numbers, rational and irrational, are representable through sums of infinite series. One such series is:

$$4\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right]=\pi,$$

which implies that we may write: $x^2 = (SR^2x^0)\left[1+\frac{x^2}{x}\right] + r \Rightarrow x^2 = SR^2 + r$, where S is a shorthand notation for the above series while r is a zero polynomial. Suppose that $r \neq 0$ and that $R \neq 1$ so that $r = 0$ and $x = SR'$ for some $R' \neq R$. Or, suppose that $r \neq 0$ and that $R = 1$ so that $r = 0$ and $x' = S$ for some $x' \neq x$. In either case, the fact remains that there is always some line segment $x$or x', call it uniformly y, such that $y = R\sqrt{S}$, $R > 0$. And, this is enough for us:

The transcendental number π comes up as the unique solution to the polynomial equation: $x^2 - SR^2x^0 = 0 \Rightarrow x = R\sqrt{S}$ and hence, $x = \sqrt{\pi}$ is constructible and the squaring of the circle is possible. The construction of the number $x = \sqrt{\pi}$ is possible as a line segment correspondi-ng to an angle of tangent equal to $\sqrt{\pi}$ the way it is elaborated below. From still another point of view, let the numbers of series S (inside the brackets above) be polynomial roots so that S may be seen as the elementary symmetric polynomial $S \equiv e_1(x_1, x_2, \ldots, x_\nu) = \sum_1^\nu x_i$, $i = 1,2,\ldots,\nu$, coming out as a coefficient of the following linear factorization of a monic polynomial in μ:

$$\prod_1^\nu (\mu - x_i) = \mu^\nu - e_1(x_1,\ldots,x_\nu)\mu^{\nu-1} + e_2(x_1,\ldots,x_\nu)\mu^{\nu-2} - \cdots + (-1)^\nu e_\nu(x_1,\ldots,x_\nu)$$

Consequently, $\nu-1, \nu-2, \ldots$ would be sensible if the infinite of S was the actual infinite, $\boldsymbol{\varpi}$; otherwise, $\infty - 1 = \infty - 2 = \cdots$

But, let us take the matter a little bit further. We do dismiss the transcendence of π, but do we retain its irrationality? We know from Euler that:

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$$\frac{\pi}{4}=\frac{3}{4}\frac{5}{4}\frac{7}{8}\frac{11}{12}\frac{13}{12}\frac{17}{16}\frac{19}{20}\frac{23}{24}\frac{29}{28}\frac{31}{32}\cdots$$

The numerator is always a prime number while the denominator is always a multiple of four nearest to the numerator. Let us ignore our theorem about the finiteness of the primes, which was advanced in the Introduction, and let us abide by Euclid's Theorem that prime numbers are infinite. If it were not so, π would be a rational number. But, what kind of infinity is that of the prime numbers? One way to perceive it, is to let a computer adding terms to the right of this expression of $\pi/4$ ad infinitum, independently of man's presence on this earth. Another way is to view the product of fractions as product of polynomial roots in which case the product would be the elementary symmetric polynomial $e_\nu(x_1,\ldots,x_\nu)$ with the same caveat about the infinite of ν as with regard to e_1 in connection with series S. This in turn means that infinity is the actual rather than the potential one.

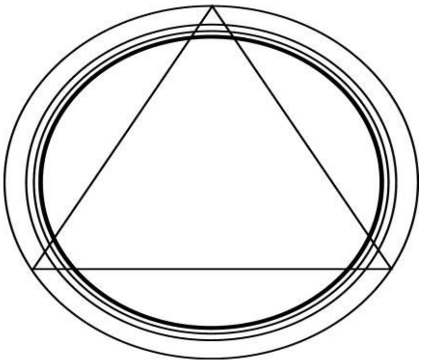

Fig. 2.1 Convergence to Finiteness

Any notion of actual infinite as signifying the presence of some extreme limit, would suffice to sustain the assertion that π is rational; rational though not computable until now. After all, what is π? It is the ratio of a circle's circumference to its diameter. That is, the ratio of the four sides of the square that squares the circle to the diameter of the circle. All of these magnitudes have endpoints; they are rational quantities and subsequently, $\pi$is the ratio of two rational quantities. $\pi$is proved to be an irrational number, because irrationality is taken to coincide with potentially infinite non-repeating decimal expansion. It is the potentiality of the example with the computer above, which is in disharmony with the physical world spatially-wise.

The key question is whether one accepts or not the truth of the statement that there is some square which has an area equal to the area of some circle. Once one does reckon this statement to be true, one puts in jeopardy any argument on the impossibility of the Quadrature. Because, a square is finite and so should a circle, or the same, 2π, be, being thereby equally constructible as a square. And, one does have to concede to the truth of this, because take, for example, the numbers $a_i=\sqrt{(3i-1)(3i+1)}/3i$ and the ratio of the perimeters of an equilateral triangle and of its circumcircle, $2\pi/3\sqrt{3}$. It may be shown that $\lim_{i\to\infty}2\pi(a_1^2a_2^2\ldots a_i^2\ldots)=3\sqrt{3}$, where $i=1,2,\ldots,\infty$ (Fig. 2.1, *Jean-Paul Delahaye*, [51]).

That is, the process of shrinking the circumcircle by multiplying its radius with the squares of the a's, *ends* by producing a circumference equal to the perimeter of the equilateral triangle. The sides of two such triangles form a hexagon from which an equal-perimeter square may be drawn, having perimeter equal to two such circumferences. In sum, there does exist some square perimeter corresponding to 2π. The end of the process of shrinking is a physical end, an end within the context of the two-dimensional space, not an end in the sphere of some abstract Platonic forms. The infinite in the lim above is the $\boldsymbol{\varpi}$ rather than the ∞.

B. Construction of Angle with Gradient Equal to $\sqrt{\pi}$

"Meton: With the straightedge I set to work, To make the circle four-cornered." *Aristophanes* (444-380 B.C., [2])

Problem:

Given line segment $\mathcal{E}$, construct with the use of a straightedge and a compass, a right triangle having $\mathcal{E}$ as one of its catheti and with the angle formed by $\mathcal{E}$ and the hypotenuse, having trigonometric tangent equal to $\sqrt{\pi}$ so that the other cathetus may be squaring the circle drawn with radius equal to $\mathcal{E}$, (or construct another line segment having length equal to the product $\mathcal{E}\sqrt{\pi}$ and being perpendicular at one of the endpoints of $\mathcal{E}$ so that the latter may be squaring the circle of radius $\mathcal{E}$).

Intuitive Observation:

Drawing a circle of circumference $L=2\pi R$, (R=radius), both L and $R=L/2\pi$ are according to traditional mathematics irrational numbers, because π is such a number, and if in general y is a rational number and z is an irrational one, the numbers $z+y$, $z-y$, $y-z$, zy, z/y, and y/z, will be irrational as well. And, from our earlier discussion follows that the irrational numbers L and R should be as constructible as rational numbers are. Methodologically, I could take any number involving π for granted such as line segment $\sqrt{\pi}$, form the hypotenuse $\sqrt{2\pi}$ from the isosceles right triangle of side $\sqrt{\pi}$, separate $\sqrt{2}$ from π on the hypotenuse with a compass, and claim that the hypotenuse is the side x of the square squaring the circle with radius equal to $\sqrt{2}$: $x^2=\pi(\sqrt{2})^2\Rightarrow x^2=2\pi\Rightarrow x=\sqrt{2\pi}\ldots$ But, contrary to common sense [4], traditional mathematics do not allow me to consider π to be constructible, and so I have to find another, indirect, implicit, way through which π will be involved in my construction. And, this way is through trigonometry, because trigonometric numbers are based on radians of a rational multiple of π in bijection with rational number of degrees.

Analysis: Consider Fig. 2.2:

(i) Let $(\Sigma\Phi/\Omega\Sigma)=\tan\Phi\Omega\Sigma=\sqrt{3}\Rightarrow\Sigma\Phi=\Omega\Sigma\sqrt{3}$ and hence, according to Pythagorean Theorem, $\Omega\Phi=\sqrt{\Sigma\Phi^2+\Omega\Sigma^2}=\sqrt{3\Omega\Sigma^2+\Omega\Sigma^2}=\Omega\Sigma\sqrt{4}=2\Omega\Sigma$. Or, if $\Omega\Sigma\equiv\mathcal{E}$, then $\Sigma\Phi=\mathcal{E}\sqrt{3}$, $\Omega\Phi=\Omega O=\Omega\Delta=\Omega Z=\Omega\Gamma=2\mathcal{E}$ and consequently, $\Omega O'=OO'=\mathcal{E}\sqrt{2}$. And, since $E\Sigma=\Omega\Sigma$, it follows that $E\Sigma=\mathcal{E}$ and $\Omega E=\mathcal{E}\sqrt{2}$, concluding thus that line segments $\Omega O'$ and ΩE are

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radiuses of a circle with center at point Ω, $(\Omega, \mathcal{E}\sqrt{2})$, Ω being also the center of the circle $(\Omega, 2\mathcal{E})$. Moreover, $\Omega C = \sqrt{2(2\mathcal{E})^2} = 2\mathcal{E}\sqrt{2}$, $OC = \Omega C - \Omega O = 2\mathcal{E}\sqrt{2} - 2\mathcal{E} = 2\mathcal{E}(\sqrt{2}-1)$ and $EO = \Omega O - \Omega E = 2\mathcal{E} - \mathcal{E}\sqrt{2} = \mathcal{E}\sqrt{2}(\sqrt{2}-1) = OC/\sqrt{2}$ so that $EO + OC = \mathcal{E}\sqrt{2} = \Omega E$. Point E lies in the middles of ΩC and triangle $\vartriangle \Phi\Omega\Delta$ is an equilateral one.

(ii) Let next $(HZ/\Omega H) = \tan Z\Omega H = \sqrt{\pi} \Rightarrow HZ = \Omega H\sqrt{\pi}$ and hence, $\Omega Z = \sqrt{HZ^2 + \Omega H^2} = \sqrt{\Omega H^2 + \pi \Omega H^2}$ $= \Omega\Sigma\sqrt{1+\pi}$. Or, if $\Omega H = H\Theta \equiv R \Rightarrow \Omega\Theta = R\sqrt{2}$and $HZ = R\sqrt{\pi}$, the above magnitudes become $\Omega Z = \Omega\Phi = \Omega O = \Omega\Delta = \Omega\Gamma = R\sqrt{1+\pi} = 2\mathcal{E}$, $\Omega O' = OO' = \Omega E = EC = R\sqrt{1+\pi}/\sqrt{2}$, $\Omega C = R\sqrt{2}\sqrt{1+\pi}$, $OC = R(\sqrt{2}-1)\sqrt{1+\pi}$, and $EO = [R(\sqrt{2}-1)\sqrt{1+\pi}]/\sqrt{2}$. We also obtain the difference $H\Sigma = \mathcal{E} - R$ and $XP \parallel \Theta E = H\Sigma\sqrt{2} = XP$ and $\Lambda\Phi = \Theta E\sqrt{2}$, where X is the midpoint of $H\Lambda$ while P is the midpoint of $\Sigma\Phi$; $H\Lambda = R\sqrt{3}$, because of the similarity of triangles $\vartriangle \Phi\Omega\Sigma$ and $\vartriangle \Lambda\Omega H$, and given that $\Omega H \equiv R$.

(iii) Furthermore, let $\mathcal{E}\sqrt{3} = HN \parallel \Omega A$ so that $(TN/HT) = \tan NHT = \sqrt{\pi} \Rightarrow TN = HT\sqrt{\pi}$ and hence, $HM = \sqrt{HT^2 + \pi HT^2} = HT\sqrt{1+\pi}$. Consequently, $\mathcal{E}\sqrt{3} = HT\sqrt{1+\pi} \Rightarrow HT = \mathcal{E}\sqrt{3}/\sqrt{1+\pi} = R\sqrt{3}/2$ and

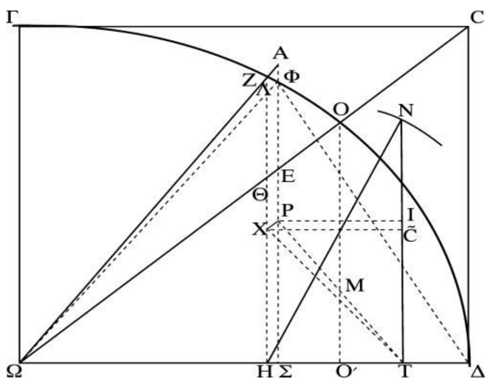

Fig. 2.2. Squaring the Circle

$(TN = (R\sqrt{3}/2)\sqrt{\pi}$. The quadrilateral $HX\tilde{C}T$ is a square having side equal to $R\sqrt{3}/2$. Moreover, in Fig. 2.2, $I\tilde{C} = H\Sigma = \mathcal{E} - R$ while equalities $HX = HT = R\sqrt{3}/2$ and $H\Theta = H\Lambda = R$ imply that $X\Theta = H\Theta - HX = R - (R\sqrt{3}/2) = H\Delta - HT = T\Delta = PE$.

(iv) Let finally, the upward extensions of ΩZ and $\Sigma\Phi$ meet at point A so that $(\Sigma A/\Omega\Sigma) = (\Sigma A/\mathcal{E}) = \sqrt{\pi} \Rightarrow \Sigma A = \mathcal{E}\sqrt{3}$ and subsequently, $\Omega A = \sqrt{\Omega\Sigma^2 + \Sigma A^2} = \sqrt{\mathcal{E}^2 + \pi\mathcal{E}^2} = \mathcal{E}\sqrt{1+\pi}$, obtaining also that $ZA = \Omega A - \Omega Z = \mathcal{E}(\sqrt{1+\pi} - 2)$ and $\Phi A = \mathcal{E}(\sqrt{\pi} - \sqrt{3}) = \Sigma A - \Sigma\Phi$.

Conclusion: The radius of circle $(\Omega, \Omega A = \mathcal{E}\sqrt{1+\pi})$ gives through circle $(\Omega, \Omega Z = R\sqrt{1+\pi} = 2\mathcal{E})$ rise to the cathetus $HZ = R\sqrt{\pi}$ that squares the circle $(\Omega, \Omega H = R)$, which has radius the other cathetus $\Omega H = R$ of the right triangle $\vartriangle \Omega ZH$; while the radius of the circle $(H, HN = \mathcal{E}\sqrt{3})$ gives rise to the cathetus $TN = HT\sqrt{\pi}$ that squares the circle $(H, HT = R\sqrt{3}/2)$, which has radius the other cathetus HT of the right triangle $\vartriangle HNT$. It follows that if one starts with the equilateral triangle $\vartriangle \Phi\Omega\Delta$ in circle $(\Omega, R\sqrt{1+\pi})$, obtain next $T\Delta = PE$ on $\Omega\Delta$, form afterwards square $HX\tilde{C}T$ from quadrilateral ΣPIT, and draw finally, from point H circle $(H, \mathcal{E}\sqrt{3})$ to meet at point N the perpendicular at point T, the result will be $\tan NHT = \sqrt{\pi}$ and similar triangles $\vartriangle NHT$, $\vartriangle Z\Omega H$, and $\vartriangle A\Omega\Sigma$, having solved through the latter triangles the stated Problem.

Construction:

(a) Given line segment $\mathcal{E} = \Omega\Sigma$, draw with center endpoint Ω, circle $(\Omega, 2\mathcal{E})$, form equilateral triangle $\vartriangle \Phi\Omega\Delta$ in the northeast quadrant $\Gamma\Omega\Delta$, draw from Φ perpendicular $\Sigma\Phi$ to side $\Omega\Delta$, and receive the bisector ΩO of the right angle $\Gamma\Omega\Delta$, where O is the intersection point of the bisector with the circumference of circle $(\Omega, 2\mathcal{E})$ while the bisector cuts also $\Sigma\Phi$ at point E. {Or, given line segment $\mathcal{E} = \Omega\Sigma$, draw with center endpoint Ω, circle $(\Omega, 2\mathcal{E})$, inscribe the northeast quadrant $\Gamma\Omega\Delta$ inside square $\Omega\Gamma C\Delta$, draw from the midpoint E of the diagonal ΩC line perpendicular to side $\Omega\Delta$ of angle $\angle\Gamma\Omega\Delta$, which perpendicular meets $\Omega\Delta$ at point Σ and cuts the circumference of circle $(\Omega, 2\mathcal{E})$ at point Φ, and form angle $\angle\Phi\Omega\Sigma$.}

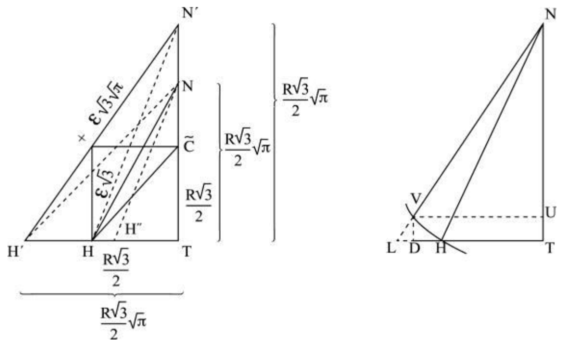

Fig.2.3.The Contradiction

(b) From the middle P of perpendicular $\Sigma\Phi$, receive distance equal to PE and transfer it on $\Omega\Delta$ as line segment $T\Delta$, drawing next at T perpendicular which cuts at point I the parallel to $\Omega\Delta$ drawn from P, forming this the parallelogram ΣPIT. With center M at the midpoint of diagonal TP of ΣPIT, draw on the left of TP, semi-circumference of radius $TP/2$, draw from P a half-line parallel to bisector ΩO (or to diagonal ΩC), which half-line meets the semi-circumference at point X and forms with ΣP angle $\angle\Sigma PX$, draw afterwards from X a parallel to ΣP, which cuts $\Omega\Delta$ at point H, and draw moreover a parallel to $\Omega\Delta$, which cuts TI at point $\tilde{C}$, receiving thus the quadrilateral $HX\tilde{C}T$.

(c) With center point H, draw circle $(H, \Sigma\Phi)$, which intersects the upward extension of $T\tilde{C}$ (or TI) at point N so that $HN = \Sigma\Phi$, draw from center Ω radius ΩZ parallel to HN, and finally, receive line segment HZ, forming the triangle $\vartriangle Z\Omega H$ and subsequently, $\vartriangle A\Omega\Sigma$, which is the sought triangle.

Proof (by Contradiction):

We have to prove three things: First, that the quadrilateral $HX\tilde{C}T$ is a square, next that $\tan NHT = \sqrt{\pi}$ and finally, that HZ is perpendicular at H:

(α) Indeed, by construction, $\angle PX\tilde{C} = 45°$, because $XP \parallel \Omega O(\parallel \Omega\Sigma)$. And, since, drawing TX, triangle $\triangle TXP$ is inscribed in circle $(M, TP/2)$, $\angle TXP = 90°$ and hence, $\angle \tilde{C}XT = \angle TXP - \angle PX\tilde{C} = 45°$, which implies that TX is a diagonal of a square.

(β1) Let next $\tan NHT = \psi \neq \sqrt{\pi}$. By construction, $\Sigma\Phi = \mathcal{E}\sqrt{3}$ and since, the upward extension of HX intersects $\Omega\Phi$ at point Λ and cuts $\Omega O(\Omega C)$ at Θ, then by the similarity of triangles $\triangle \Phi\Omega\Sigma$ and $\triangle \Lambda\Omega H$, $H\Lambda = \Omega H\sqrt{3}$ and $H\Theta = \Omega H\sqrt{2}$. Or, if $\Omega H \equiv R$, then $H\Lambda = R\sqrt{3}$, which implies that $HX = R\sqrt{3}/2$, since $XP \parallel \Omega O(\parallel \Omega\Sigma)$ and $\Sigma P = \mathcal{E}\sqrt{3}/2$ by construction. Consequently, $HT = R\sqrt{3}/2$, because $HX\tilde{C}T$ is a square. Therefore, if $\psi \neq \sqrt{\pi}$, TM should be equal to $\psi(R\sqrt{3}/2)$.

Consider now the left part of Fig. 2.3, which includes square $HX\tilde{C}T$, triangle $\triangle NHT$, and the similar triangle $\triangle N'H'T'$, which obtains through the multiplication of the sides of $\triangle NHT$ by $\sqrt{\pi}$. Let $\angle NH'T$ be the angle which is equal to $\sqrt{\pi}$, $\tan NH'T = \sqrt{\pi}$. We have $TN = \psi(R\sqrt{3}/2)$, $TH' = (R\sqrt{3}/2)\sqrt{\pi}$, and $TN = TH'\sqrt{\pi}$; inserting the first two equalities in the last one yields that $\psi(R\sqrt{3}/2) = [(R\sqrt{3}/2)\sqrt{\pi}]\sqrt{\pi} \Rightarrow \psi = \pi$, which is not true, because π is a half-circle, and which moreover implies that $TN = \pi(R\sqrt{3}/2)$, giving rise to five contradictions:

The first is that $\triangle NH'T$ is a scaled-up by $\sqrt{\pi}$ version of $\triangle \tilde{C}HT$. How do we know that the hypotenuse of the smaller triangle coincides with diagonal $\tilde{C}H$? We know it, because, given that $(TH'/TH) = (R\sqrt{3}/2)\sqrt{\pi}/(R\sqrt{3}/2) = \sqrt{\pi}$, then by the similarity of the bigger with the smaller triangle, the same proportion $\sqrt{\pi}$ should hold for the other side TN of $\triangle NH'T$. And, given the length of TN, this proportion is provided by the ratio $TN/T\tilde{C}$. If $TN = \pi(R\sqrt{3}/2)$ as it seems to obtain when $\psi = \pi$, then $T\tilde{C}$ should be equal to $(R\sqrt{3}/2)\sqrt{\pi}$ to enable subsequently the derivation of $(T\tilde{C}/TH) = \sqrt{\pi}$.

This does not contradict only that $(T\tilde{C}/TH) = 1$ by construction; it also contradicts our assumption that $\tan NH'T = \sqrt{\pi}$, because $\angle \tilde{C}HT = 45°$. The third contradiction is that if $\psi = \pi$and $TN = \pi(R\sqrt{3}/2)$, then TN should coincide with TN'; but, it does not. And, there is a fourth contradiction, because if they did coincide, then $\tan NH'T = \tan N'H'T = \sqrt{\pi}$ and since, $NH \parallel N'H'$, we would have $\psi = \sqrt{\pi}$ rather than $\psi = \pi$. And, there is a fifth contradiction, because if $\tan NHT = \psi = \pi$and $\tan\tilde{C}HT = \sqrt{\pi}$, the angle sum identity for $\tan(\tilde{C}HT + NH\tilde{C})$ would yield $\tan NH\tilde{C} = (\pi - \sqrt{\pi})/(1 + \pi\sqrt{\pi})$. Given now that $\angle\tilde{C}HT + \angle NH\tilde{C} + \angle NHH' = \pi$ and that the sum $(\tan\tilde{C}HT + \tan NH\tilde{C} + \tan NHH')$ is equal to the product $(\tan\tilde{C}HT \tan NH\tilde{C} \tan NHH')$, one obtains that
$NHH' = -\pi$, which is false.

Could it be at the other end that $\tan N'HT = \sqrt{\pi}$? We understand through similar triangle $\triangle N'HT$ and $\triangle NH''T$ that the answer is negative. We should have $(TN'/TH) = \sqrt{\pi} = (TN/TH'') = \psi(R\sqrt{3}/2)/x \Rightarrow x = TH'' = \psi(R\sqrt{3}/2)/\sqrt{\pi}$ and hence, $H''H = (R\sqrt{3}/2) - [\psi(R\sqrt{3}/2)/\sqrt{\pi}] = (R\sqrt{3}/2)[(\sqrt{\pi} - \psi)/\sqrt{\pi}]$, which given that $(TN/T\tilde{C}) = \sqrt{\pi}$, yields that $T\tilde{C} = H''H$, contradicting that $T\tilde{C} = R\sqrt{3}/2$, because $R\sqrt{3}/2 = (R\sqrt{3}/2)[(\sqrt{\pi} - \psi)/\sqrt{\pi}] \Rightarrow \sqrt{\pi} = \sqrt{\pi} - \psi \Rightarrow \psi = 0$.
Note that the same result would obtain even if we accepted that $\sqrt{\pi} \neq (TN/T\tilde{C}) = \psi$ since, we should also have that $(TH''/H''H) = \psi$ as well.

The general conclusion is that square $HX\tilde{C}T$ along with the use of proportions do establish that $\tan NHT = \sqrt{\pi}$ and consequently, that $\mathcal{E}\sqrt{3}$ is equal to the square root of the sum $\left[(R\sqrt{3}/2)^2 + [(R\sqrt{3}/2)\sqrt{\pi}]^2\right]$ from which it follows that $R\sqrt{1+\pi} = 2\mathcal{E}$.

(β2) But, do we really need $HX\tilde{C}T$ to prove that $\tan NHT = \sqrt{\pi}$? Let us disregard it for a moment, and let us experiment not only with a different hypotenuse or different horizontal triangle side, but by altering both of them the way the right-hand part of Fig. 2.3 illustrates. Suppose that the triangle with the "real $\sqrt{\pi}$" is $\triangle NVU$ rather than $\triangle NHT$, with $\tan NVU = \tan VLD = \sqrt{\pi}$, $VU = R\sqrt{3}/2$ − because this is the length we should have according to the Analysis in order to have $\sqrt{\pi}$, too − and $NV = NH = \mathcal{E}\sqrt{3}$ on $NL = \mathcal{E}\sqrt{3}\sqrt{\pi}$ so that LT is some multiple λ of $R\sqrt{3}/2 = DT = VU$.

From the differences $LD = LT - HT = LT - VU = (R\sqrt{3}/2)(\lambda - 1)$and $LV = LN - VN = \mathcal{E}\sqrt{3}(\sqrt{\pi} - 1)$, and from the similarity of triangles $\triangle VLD$ and $\triangle NLT$, we obtain the proportions:

$$\frac{\mathcal{E}\sqrt{3}\sqrt{\pi}}{\lambda(R\sqrt{3}/2)} = \frac{\mathcal{E}\sqrt{3}(\sqrt{\pi} - 1)}{(R\sqrt{3}/2)(\lambda - 1)}$$

from which it follows that:

$$\frac{\sqrt{3}\sqrt{\pi}}{\lambda} = \frac{\sqrt{\pi} - 1}{\lambda - 1} \Rightarrow \lambda^2(2\sqrt{\pi} - 1) - \lambda 2\pi + \pi = 0$$

which equation in λ yields the solutions $\lambda = \sqrt{\pi}$ and $\lambda = \sqrt{\pi}/(2\sqrt{\pi} - 1)$. The latter solution is rejected because it implies that $LT = (R\sqrt{3}/2)[\sqrt{\pi}/(2\sqrt{\pi} - 1)]$ and hence, that $LD = LT - DT = (R\sqrt{3}/2)\left\{\left[\frac{\sqrt{\pi}}{(2\sqrt{\pi}-1)}\right] - 1\right\}$, with the right-hand side becoming,
$-(R\sqrt{3}/2)[(1 + \sqrt{\pi})/(2\sqrt{\pi} - 1)] < 0$. Consequently, multiple $\lambda = \sqrt{\pi}$ reflects the angle $\angle NVU = \angle VLD$, the tangent of which has been assumed to be $\sqrt{\pi}$. It follows that $UN = (R\sqrt{3}/2)\sqrt{\pi}$ given that $VU = R\sqrt{3}/2$, and therefore, $TN = (R\sqrt{3}/2)\pi$, which if rewritten as $TN = [(R\sqrt{3}/2)\sqrt{\pi}]\sqrt{\pi}$, is consistent with $\tan NHT = \sqrt{\pi}$ and $HT = R\sqrt{3}/2$, contrary to what we have assumed.

But, more important is the observation that if $TN = [(R\sqrt{3}/2)\sqrt{\pi}]\sqrt{\pi}$, triangle $\triangle\, NLT$ should be the multiple $\sqrt{\pi}$ of the sides of another triangle, similar to $\triangle\, NLT$, with sides equal to $R\sqrt{3}/2$ and $(R\sqrt{3}/2)\sqrt{\pi}$, and a hypotenuse $\mathcal{E}\sqrt{3}$, having the angle facing the side equal to $(R\sqrt{3}/2)\sqrt{\pi}$, tangent equal to $\sqrt{\pi}$. This is a quite interesting result, because it suggests that even if the Construction was wrong, it would lead to the correction of the error by simply drawing a parallel to NL so that $\sqrt{\pi}$ may be obtained.

(γ) We must finally show that HZ is perpendicular at Hon $\Omega\Delta$. If it was at $\underline{H} \neq H$, then $tanN\underline{H}T = \sqrt{\pi}$, which contradicts that $tanNHT = \sqrt{\pi}$ unless $\underline{H}$ and H coincide. Also, if the upward extension of HX did not intersect the circumference of circle $(\Omega, 2\mathcal{E})$ at Z but at $\underline{Z}$, we should have $tan\underline{Z}\Omega H = \sqrt{\pi}$ and hence, $\underline{Z}$ and Z should coincide, given moreover that by construction, $\Omega Z \parallel HN$: The parallels ensure the verticality. If not anything else, $HZ = \sqrt{\Omega Z^2 - \Omega H^2}$, which is equal to the square root of $\left[\left(R\sqrt{1+\pi}\right)^2 - R^2\right]$, implying that $HZ = R\sqrt{\pi}$, which is true and therefore, $HZ \perp \Omega\Delta$. It follows that the sought triangle is $\triangle$ ZΩH, with its hypotenuse ΩZ being the side of the square squaring the circle with radius equal to side ΩH…Quod Erat Demonstrandum…

3. The Trisection

"Mighty is geometry; joined with art, resistless." *Euripides* (485-406 B.C., [68, p. 474, 17])

A. Preamble

The Trisection of an arbitrary acute angle by means of a straightedge and a compass was deemed by the ancient Greeks to be impossible. In Book IV of his "Mathematical Collections", Pappus of Alexandria (c. 290-c. 350) writes: "…geometers of the past who sought by planes to solve the … problem of the trisection of an angle, which is by its nature a solid problem, were unable to succeed. For they were as yet unfamiliar with the conic sections and were baffled for that reason. But later with the help of the conics they trisected the angle using the following 'vergings' for the solution..." [62]

In 1837, Pierre Laurent Wantzel (1814-1848, [65]) "proved" the impossibility formally. From the triple-angle formulas of Trigonometry, we know for angle ω that that;

$$tan^3\omega - 3tan3\omega tan^2\omega - 3tan\omega + tan3\omega = 0. \quad (1)$$

This, equation is supposed to be an irreducible polynomial equation, and cubic roots are not geometrically constructible. But, it is not an equation: Given any cubic equation, $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$, one of the conditions to have three equal roots is, $a_2^2 - 3a_3a_1 = 0$, which gives, $9(tan^2 3\omega + 1) = 0 \Rightarrow tan^2 3\omega = -1$. Neither this result, which we will have the opportunity to see it again in subsection 3.3, nor the result that $x = tan3\omega$ implied by the requirement to have three roots equal, $x = -a_2/3a_3 = 3tan3\omega/3 = tan3\omega$, is sensible. Because, simply, triple-angle formulas are identities regarding measure, keeping measures balanced like income-expenditure accounts. Expression (1) is an identity, a tautology like an accounting identity equating assets with liabilities with no *Popperian* information content whatsoever. Action may be taken once the relationship between the structure of assets and the structure of liabilities is revealed; once equations are formed. And, constructibility means action as will be explained in subsection 3.3.

But, for now, one should note that we are dealing with trigonometric numbers, which are irrational numbers based on rational multiples of a circle or of π. Expressing angle ω in terms of π, $\omega = \pi/z$, one obtains from the infinite product formulas that:

$$sin\omega = (\pi/z)\prod_{i=1}^{\infty}\{1 - [(\pi/z)^2/\pi^2 i^2]\}$$
$$= (\pi/z)\prod_{i=1}^{\infty}[1 - (1/z^2 i^2)].$$

Consequently, the Trisection might be viewed as a variant of the Quadrature the way we developed it earlier. Yet, based on the proposed utilization method of RAI/C, a much simpler Trisection is given below right away.

B. Construction of Trisector of Acute Angle

*Problem:*Trisect a given acute angle ω, with the aid of a straightedge and a compass.

*Analysis:*Suppose that we have trisected angle $\angle\Theta\Omega\Sigma = \omega$ in Fig. 3.1, which also contains the following elements: Bisector ΩT forms with half-line ΩB, angle $\angle T\Omega B = \epsilon + (\omega/2) = 45°$. Also, $\omega_1 = \omega_4 = (\omega/3)$and $\omega_2 = \omega_3 = (\omega/6)$. The right triangle $\triangle\, I\Omega\Gamma$, which is formed having hypotenuse the line segment ΩI of trisector ΩO, is an isosceles triangle

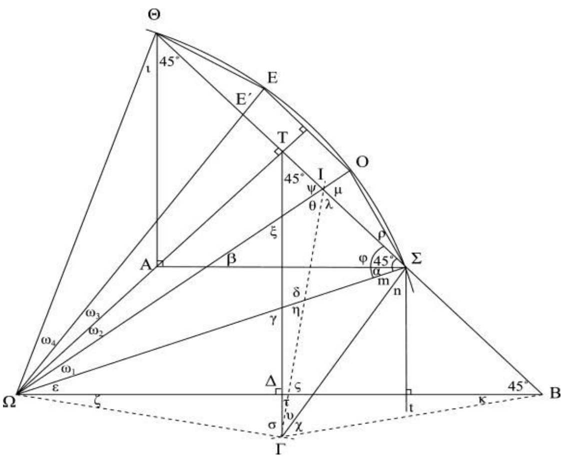

Fig.3.1. Construction of Trisector

as is triangle $\triangle\, T\Omega B$ so that $\Omega\Gamma = \Gamma I = \Gamma B$, $\sigma + \tau = 90°$, $\sigma = \psi = \tau + \upsilon + \chi$, $\zeta = \kappa = \tau = (\omega/6)$. Given now that $\alpha = \iota = \epsilon \equiv b$, $2b + \omega = 90°$, $\varphi + (\omega/2) = 90°$, $\psi + (\omega/6) = 90°$, and $\psi = \varphi + (\omega/3)$, routine calculations of triangle angles yield also the following list of angles: $\beta = 45° - (\omega/6)$, $\gamma = 90° - b = 45° + (\omega/2) = b + \omega$, $\delta = 90° + b + (\omega/6)$, $\eta = 45° + (\omega/3)$, $\theta = b + (\omega/2) = 45°$, $\lambda = 45° + (\omega/6)$, $\mu = \nu = \varphi + (\omega/3)$, $\rho = \omega/3$, $t = 90° - (\omega/6) = \varphi + (\omega/3) = s$, $m + n = b + \omega$, $\upsilon + \chi = 90° - (\omega/3)$, and $\xi = 45° + (\omega/6) = 90° + b + (\omega/3)$.

That is, according to this analysis, trisection imposes that $\Sigma O = \Sigma I$. Nevertheless, the Analysis does not determine angles m, n, υ, and χ. It appears through the sum $m + n = \alpha + \omega$ that $m = \omega$and $n = \alpha \equiv b$, and this is what will be assumed below.

*Construction of TrisectorΩO (ΩI):*Given acute angle $\angle\Theta\Omega\Sigma = \omega$ to trisect, draw bisector ΩT and form next based on it, angle $\epsilon + (\omega/2) = 45°$ and the isosceles right triangles $\triangle\, \Theta A\Sigma$, $\triangle\, T\Omega B$, and $\triangle\, T\Omega\Delta$. From point Σ, draw a line parallel to $\Omega\Theta$ and meeting the downward extension of $T\Delta$ at point Γ. The hypotenuse ΩI of the isosceles triangle $\triangle\, \Omega\Gamma I$ formed having side equal to $\Omega\Gamma$, constitutes a segment of the sought trisector ΩO of $\angle\Theta\Omega\Sigma = \omega$.

*Proof (by Contradiction):*In Fig. 3.1, we have by construction, $\Omega\Gamma = \Gamma I = \Gamma B$, $\sigma + \tau = 90°$, $\sigma = \psi = \tau + \upsilon + \chi$, $\zeta = \kappa = \tau = \omega_2$, $\alpha = \iota = \epsilon \equiv b$, $2b + \omega = 90°$, $\varphi + (\omega/2) = \varphi + \omega_1 + \omega_2 = 90°$, $\varphi + \omega_1 = \psi$, and $\psi + \omega_2 = 90°$, where the distinction between $\omega_1$and ω_2 has been based on the construction of ΩI. Given these relationships, simple calculations of triangle angles yield all of the angles mentioned in the Analysis, with $\omega_1$and ω_2 being now in the place of $\omega/3$ and $\omega/6$, respectively. I have to show that $\mu = \nu = \varphi + \omega_1 = \varphi + \omega_2 + (\omega_1/2) \Rightarrow \omega_1 = 2\omega_2$. Suppose that this equality does not hold and that $\Sigma O \neq \Sigma I$. Suppose that some other chord, not ΣO, is equal to ΣI. But, then, $\rho \neq \omega_2 + (\omega_1/2)$, which would be absurd if that other chord was the one connected with the trisectorQuod Erat Demonstrandum…

C. The Trigonometry of Constructibility It does not take only a cubic trigonometric equation to have a trisection equation. From FranciscusVieta's (1540-1603) recurrence formulas, we have $\tan(\nu+1)\omega = (\tan\nu\omega + \tan\omega)/(1 - \tan\nu\omega\tan\omega)$, or letting $\tan(\nu + 1)\omega \equiv \alpha$ and $\tan\omega = x$, and using the recurrence formula for $\tan\nu\omega$,

$$\alpha - x\alpha\frac{tan(\nu-1)\omega + x}{1 - xtan(\nu-1)\omega} - \frac{tan(\nu-1)\omega + x}{1 - xtan(\nu-1)\omega} - x = 0,$$

and using again the recurrence formula for $tan(\nu - 1)\omega$,

$$\alpha - x\alpha\frac{\frac{tan(\nu-2)\omega+x}{1-xtan(\nu-2)\omega} + x}{1 - \frac{tan(\nu-2)\omega+x}{1-xtan(\nu-2)\omega}x} - \frac{\frac{tan(\nu-2)\omega+x}{1-xtan(\nu-2)\omega} + x}{1 - \frac{tan(\nu-2)\omega+x}{1-xtan(\nu-2)\omega}x} - x = 0,$$

or letting $tan(\nu - 2)\omega \equiv y$, and after some operations,

$$x^3 - 3\frac{\alpha - y}{1 + \alpha y}x^2 - 3x + \frac{\alpha - y}{1 + \alpha y} = 0. \quad (2)$$

This cubic equation is neither an equation for ν-section, because one should have x^ν rather than x^3, nor an equation for the trisection of an angle equal to 3ω, because then $(\alpha - y)/(1 + \alpha y) = \alpha \Rightarrow y(\alpha^2 + 1) = 0$ and hence, that either $y = 0$ or $\alpha^2 = -1$, which are both absurd results. And, even more so absurd would be to let $(\alpha - y)/(1 + \alpha y) \equiv A$ so that to make (2) look like a "genuine" trisection equation: $x^3 - 3Ax^2 - 3x + A = 0$.

In general, any trigonometric polynomial equation can be anything but an equation unless we "lock" it both sides: x as well as ν. Just use the recurrence formula for $tan(\nu - 2)\omega$ above to get a fourth-degree equation. A trigonometric polynomial equation is in essence the result of a system of equations determined by the recurrence, and which equations in so far as the trisection is concerned, are two: Just insert $\nu = 2$ in $(\alpha - y)/(1 + \alpha y) = [tan(\nu + 1)\omega - tan(\nu - 1)\omega]/[1 - tan(\nu + 1)\omega tan(\nu - 1)\omega]$ to get:

$$\frac{tan3\omega - tan\omega}{1 - tan3\omega tan\omega} = \frac{\alpha - x}{1 - \alpha x},$$

which when inserted in (2), gives the quartic equation:

$$\alpha x^4 + 4x^3 - 6\alpha x^2 - 4x + \alpha = 0. \quad (3)$$

This precisely solvable equation is the equation of trisection. We are not looking for ωgiven 3ω. We are trying to find a way to construct ω given the measures of both $\omega$and 3ω. And, (3) says that the only way to do it is to bisect both angles in an *embedded bisection* fashion, i.e. through the bisection of 3ω and of the middle ω. And, indeed, this is how we managed trisection in the previous subsection. A trisection equation should be addressing the issue of the constructibility of the trisector whereas the cubic equations coming out of the triple-angle formulas are equations of the measure of ω. The latter would suffice if one fixed the 3ω which is geometrically trisectable by one's trisection method.

To confirm this, note that the recurrent formula for $\nu = 2$ and $z \equiv tan2\omega$, gives $x = (\alpha - z)/(1 + \alpha z)$, which when inserted in the place of x in the equation $x^3 - 3\alpha x^2 - 3x + \alpha = 0$ coming out of the triple-angle formula, yields the cubic equation in $tan2\omega$:

$$(2 + \alpha^4 - 3\alpha^2)z^3 + 3\alpha(1 + 3\alpha^2)z^2 + 3(1 + \alpha^4)z - 2\alpha(1 + \alpha^2) = 0,$$

or by noting that

$z \equiv tan2\omega = 2tan\omega/(1 - tan^2\omega) \equiv 2x/(1 - x^2)$, the following sixth-degree equation in $tan\omega$ obtains:

$$2\alpha(1 + \alpha^2)x^6 + 3(1 + \alpha^4)x^5 - 12\alpha^3x^4 + 2(1 - 6\alpha^2 - \alpha^4)x^3 + 12\alpha^3x^2 + 3(1 + \alpha^4)x - 2\alpha(1 + \alpha^2) = 0,$$

regarding the bisection of the three ω's. It is an equation about constructability and not about measure.

4. POLYNOMIALS

"There is more danger of numerical sequences continued indefinitely than of trees growing up to heaven. Each will some time reach its greatest height." Friedrich Ludwig GottlobFrege (1848-1925, [22, p. 204])

A. The Abel-Ruffini Theorem

In 1798, Paolo Ruffini (1765-1822) published a book [58], starting its introductory section as follows: "The algebraic solution of general equations of degree greater than four is always impossible. Behold a very important theorem which I believe I am able to assert (if I do not err): to present the proof of it is the main reason for publishing this volume. The immortal Lagrange, with his sublime reflections, has provided the basis of my proof." In 1824, and after Ruffini [59] too, Niels Hendrik Abel (1802-1829) was opening his Memoir on algebraic equations, proving the impossibility of a solution of the general equation of the fifth degree as follows: "The mathematicians have been very much absorbed with finding the general solution of algebraic equations, and several of them have tried to prove the impossibility of it. However, if I am not mistaken, they have not as yet succeeded. I therefore dare hope that the mathematicians will receive this memoir with good will, for its purpose is to fill this gap in the theory of algebraic equations." And, in 1826, his work on the impossibility of a "quintic formula" appeared in "Crelles's Journal" officially [1]. In 1846, in another journal, in the eleventh volume of the Journal de Mathématiques Pures et Appliquées, the "OEuvres Mathématiques d'Évariste Galois" appeared (pp. 381-444), confirming the Abel-Ruffini impossibility theorem and marking the development of the so-called Galois (1811-1832) theory on the relations among the roots of polynomials. In my opinion, the following considerations should also be taken into account as I consider the subject of polynomials and series to be the bridge between Geometry and Arithmetic, the key behind the physical link of Arithmetic and hence, of paramount importance.

B. A General Theory of Geometric Constructibility

Any general polynomial,

$$a_\nu x^\nu + a_{\nu-1}x^{\nu-1} + a_{\nu-2}x^{\nu-2} + \cdots + a_1 x + a_0 = 0 \Rightarrow$$
$$x^\nu + \frac{a_{\nu-1}}{a_\nu}x^{\nu-1} + \frac{a_{\nu-2}}{a_\nu}x^{\nu-2} + \cdots + \frac{a_1}{a_\nu}x + \frac{a_0}{a_\nu} = 0,$$

may be rewritten by virtue of Vieta's formulas regarding the relations between polynomial roots [66], x_i, $i = 1,2,\dots,\nu$, and polynomial coefficients, a_i plus a_0, as follows:

$$x^\nu - e_1(x_1,\dots,x_\nu)x^{\nu-1} + e_2(x_1,\dots,x_\nu)x^{\nu-2} - \cdots + (-1)^\nu e_\nu(x_1,\dots,x_\nu) = 0,$$

where$e_i(x_1,\dots,x_\nu)$ are elementary symmetric polynomials as follows:

$$e_1(x_1,\dots,x_\nu) = \sum_{i=1}^{\nu} x_i = -\frac{a_{\nu-1}}{a_\nu},$$

$$e_2(x_1,\dots,x_\nu) = \sum_{1<i<j<\nu} x_i x_j = \sum_{i=1}^{\nu-1} x_i \left(\sum_{j=i+1}^{\nu} x_j\right) = \frac{a_{\nu-2}}{a_\nu},$$

$$e_3(x_1,\dots,x_\nu) = \sum_{1<i<j<k<\nu} x_i x_j\, x_k = \sum_{i=1}^{\nu-2} x_i \left[\sum_{j=i+1}^{\nu-1} x_j \left(\sum_{k=j+1}^{\nu} x_k\right)\right] = -\frac{a_{\nu-3}}{a_\nu},$$

… … … … … … …

$$e_\nu(x_1,\dots,x_\nu) = \prod_{i=1}^{\nu} x_i = (-1)^\nu \frac{a_0}{a_\nu},$$

given that $e_0(x_1,\dots,x_\nu) = 1$, anyway. It follows that any e_i and thereby, sum-product relation among polynomial roots may be represented geometrically by the trigonometric tangent of the acute angles formed by the cathetia_i, $i = 1,2,\dots,\nu-1$, and a_ν with the hypotenuse in a right triangle: $a_i/a_\nu = tan\theta_i$, $i = 1,2,\dots,\nu-1$.

The ratio of any two line segments viewed as ratio of catheti and thereby, any acute angle θ, reflect actual infinity, some elementary symmetric polynomial, some relationship among the roots of some polynomial, either on the real or on the complex plane depending on whether the a's can be complex numbers, too. Moreover, identifying $x_i$with $tan\omega_i$, yields:

$$e_0 = 1 = tan\theta_0 = tan45°,$$

$$e_1 = \sum_i x_i = \sum_i tan\omega_i = \frac{a_{\nu-1}}{a_\nu} = tan\theta_1,$$

$$e_2 = \sum_{i<j} x_i x_j = \sum_{i<j} tan\omega_i\, tan\omega_j = \frac{a_{\nu-2}}{a_\nu} = tan\theta_2,$$

$$e_3 = \sum_{i<j<k} x_i x_j x_k = \sum_{i<j<k} tan\omega_i\, tan\omega_j tan\omega_k = \frac{a_{\nu-3}}{a_\nu} = tan\theta_3,$$

… … … … … … …

$$e_\nu = \prod_i x_i = \prod_i tan\omega_i = \frac{a_0}{a_\nu} = tan\theta_\nu.$$

Consequently, letting ν be some infinitely large number so that i might as well be tending to $\nu \approx \infty$,

$$e_1 = \lim_{i\to\nu\approx\infty} \sum_i x_i = \lim_{i\to\nu\approx\infty} \sum_i tan\omega_i = \frac{a_{\nu-1}}{a_\nu} = tan\theta_1,$$

$$e_2 = \lim_{i,j\to\nu\approx\infty} \sum_{i<j} x_i x_j = \lim_{i,j\to\nu\approx\infty} \sum_{i<j} tan\omega_i tan\omega_j = \frac{a_{\nu-2}}{a_\nu} = tan\theta_2,$$

… … … … … … …

$$e_\nu = \lim_{i\to\nu\approx\infty} \prod_i x_i = \lim_{i\to\nu\approx\infty} \prod_i tan\omega_i = \frac{a_0}{a_\nu} = tan\theta_\nu.$$

The ratios of catheti$a$ and acute angles θ are the limits of some infinite sums and products of roots of *apeiro*nomials, of polynomials in which $\nu \approx \infty$. There are no non-convergent series in finite space however infinite it may be. Is there such an actual infinity? There is in as much as it is true that Vieta's polynomial (as opposed to recursive) formulas are readily susceptible to geometrical interpretation: The size of ν affects the number and values of ratios and angles but never their geometrical hypostasis, their spatial underpinnings.

Any literally line segment and angle in the universe is geometrically constructible, but the discussion of the subject in hand would be incomplete if we did not address the matter of the constructability of ω's from a's or θ's, which are readily available. Towards this end, let, for simplicity, $\omega_i = i\omega$, $i = 1,2,\dots,\nu$. One might use then Vieta's recursive formula for $tan(i\omega)$ to obtain a "pseudo-trigonometric polynomial" in terms of $tan\omega$ through preferably $e_1 = \sum_i tan(i\omega) = tan\theta$ to minimize the complexity of the operations. Consider, for example, the quadratic equation $a_2x^2 + a_1x + a_0 = 0$, with

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$e_1 = tan\omega_1 + tan\omega_2 = tan\omega + [2tan\omega/(1 - tan^2\omega)] = -(a_1/a_2)$ from which it follows that $a_2 tan^3\omega + a_1 tan^2\omega - 3a_2 tan\omega - a_1 = 0$. If instead of $-(a_1/a_2)$ we hd used $tan\theta$, our cubic equation in $tan\omega$ would have been: $tan^3\omega + tan\theta tan^2\omega - 3tan\omega + tan\theta = 0$. In any case, we have a polynomial in $tan\omega$, which in general is,

$$\beta_3 x^3 + \beta_2 x^2 - \beta_1 x - \beta_0 = 0 \Rightarrow x^2 = \frac{\beta_1 x + \beta_0}{\beta_3 x + \beta_2},$$

or even more generally,

$$x^{\nu-1} = \frac{x^{\nu-3}(\beta_{\nu-2}x + \beta_{\nu-3})}{\beta_\nu x + \beta_{\nu-1}} + \frac{x^{\nu-5}(\beta_{\nu-4}x + \beta_{\nu-5})}{\beta_\nu x + \beta_{\nu-1}} + \cdots + \frac{\beta_1 x + \beta_0}{\beta_\nu x + \beta_{\nu-1}}, \quad (1)$$

where $x \equiv tan\omega$, and with the negative signs assumed for convenience. But, note that the last term on the right of the latter equation, equation (1), is the same as the right-hand term from the cubic equation, on which therefore we shall focus given that all we need to determine $tan\omega$ is only one term.

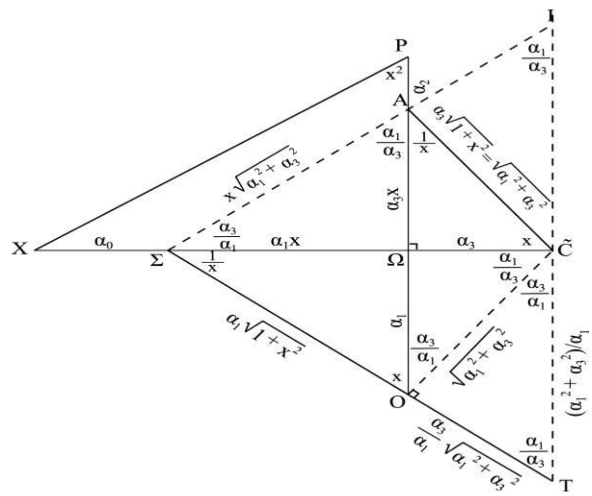

Fig.4.1. Construction of Cubic Equation Roots

Fig. 4.1 depicts x, starting from the right triangle $\triangle X P \Omega$ and drawing the other triangles based on the values of the a's. In the right triangle $\triangle \tilde{C}\Sigma I$, we have, $\tilde{C}I = (\beta_3/\beta_1)(\beta_1 x + \beta_3)$ and consequently, $\Sigma I^2 = \tilde{C}I^2 + \tilde{C}\Sigma^2$, or

$$\Sigma I^2 = \left(\frac{\beta_3}{\beta_1}\right)^2 (\beta_1 x + \beta_3)^2 + (\beta_1 x + \beta_3)^2 \Rightarrow$$

$$\Sigma I = x\sqrt{\beta_1^2 + \beta_3^2} + \frac{\beta_3}{\beta_1}\sqrt{\beta_1^2 + \beta_3^2} = \Sigma A + AI.$$

That is, $AI = OT$. If line segments OT and $O\Sigma$ are collinear, x will be equal to β_1/β_3, given that $\angle \tilde{C}OT = 90°$. They are collinear by construction, but suppose that they are not. Suppose that we started the construction of Figure 4.1 from triangle $\triangle \Omega O \tilde{C}$ and drew the rest of it on the assumption that $x = \beta_1/\beta_3$. Suppose that the extension of OT needed to form the isosceles triangle whose base is TI, meets side ΣI at point Σ' rather than at Σ. But, $\Sigma' I \neq \Sigma I$and $tan I\Sigma'\tilde{C} \neq \beta_3/\beta_1$, contrary to what we have calculated in order to have the isosceles triangle. Hence, $x = \beta_1/\beta_3$, indeed; and, in general, $x_1 = \beta_1/\beta_\nu$, which is obtained through the last term on the right of (1). Just replace in Fig. 4.1, $\beta_3$by β_ν; both are given constants, anyway.

Do we have another version of the Rational Roots Theorem? No, by no means, since we are talking about trigonometric numbers, having accepted methodologically their irrationality, and by discovering later that they are not irrational at all. And, if this is not enough to persuade the skeptics, our results emerge, methodologically again, only after the roots having been found, because otherwise no elementary polynomial may be defined and no analogous result may be obtained the way we do obtain it, here. And, if neither this was enough, note that x may be specified in an alternative way, implying $\beta_\nu^2 = \beta_0\beta_1$, as follows.

Consider, for instance, an equation of 4th degree:

$$\beta_4 x^4 + \beta_3 x^3 - \beta_2 x^2 - \beta_1 x - \beta_0 = 0 \Rightarrow x^3 = \frac{x(\beta_2 x + \beta_1)}{\beta_4 x + \beta_3} + \frac{\beta_0}{\beta_4 x + \beta_3}.$$

Since, $SO = (\beta_0/\beta_4)(\beta_0 + \beta_4)$ in Fig. 4.2, the hypotenuse

$$\Theta O^2 = SO^2 + S\Theta^2 = \left(\frac{\beta_0}{\beta_4}\right)^2 (\beta_0 + \beta_4)^2 + (\beta_0 + \beta_4)^2 \Rightarrow$$

$$\Theta O = \sqrt{\beta_0^2 + \beta_4^2} + \frac{\beta_0}{\beta_4}\sqrt{\beta_0^2 + \beta_4^2} = \Theta E + EO.$$

That is, $EO = \Theta H$. If line segments ΘH and $\Theta\Gamma$ (and $\Theta\Upsilon$) are collinear, forming the right triangle $\triangle \Gamma E H$, we should have $(\beta_4/\beta_0)(1/x) = 1 \Rightarrow x = \beta_4/\beta_0$, which follows also from having EH∥SΘ both cut by HΓ. Suppose that these line segments are not collinear and that the leftward extension of ΘY meets HO at H' rather than at H so that EH stops being parallel to ΘS. Note thatalthough ΔΓEH stops being a right triangle too, the angle with vertex Θ, continues being a right angle.

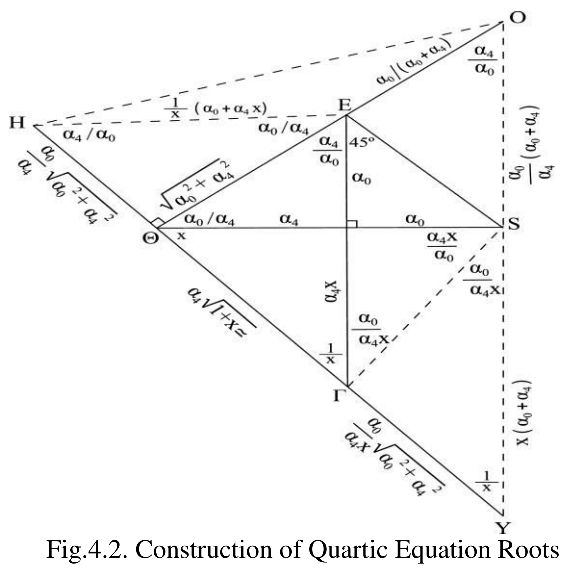

Fig.4.2. Construction of Quartic Equation Roots

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Consequently, an assumption about $H' \neq H$ would be inconsistent with $EO = \Theta H$. After all, collinearity is the case by construction, since H is the point at which the leftward extension of $\Theta\Upsilon$ meets the parallel from $E$to ΘS; and H has been connected then with O. It follows that $x = \beta_4/\beta_0$, and in general, $x = \beta_\nu/\beta_0$. And, as soon as $x = \beta_1/\beta_\nu$ is also true, it follows that $\beta_\nu^2 = \beta_0\beta_1$.

Consequently, by the right triangle altitude theorem, β_ν is the length of the altitude, ΛL of a right triangle, $\triangle K\Lambda M$, separating the hypotenuse, KM, in the two parts $KL = \beta_1$ and $ML = \beta_0$, and with the solution for $x = tan\omega$ given either by the angle $\angle K\Lambda L = \omega$ or by the angle $\angle LM\Lambda = \omega$ as depicted by Fig. 4.3. This value of x may be one only out of ν roots, but x is connected with an acute angle, ω, and only the angle which corresponds to the positive quadrant, i.e. $\angle K\Lambda L$ or $\angle LM\Lambda$, is sensible as a solution. In a few words, if P is a νth degree univariate polynomial in x and the roots of $P = 0$ are sought, and if x is replaced by the trigonometric tangent of some angle ω, the problem of finding the roots is reduced to finding the solution of $P = 0$ via the two-dimensional problem of constructing the geometric mean of the constant term and β_1, $\beta_\nu = \sqrt{\beta_0\beta_1}$, or of constructing a square area, β_ν^2, equal to the area of a given rectangle, parallelogram or triangle, having length sides β_0 and β_1: We do not even need β_ν or for that matter, any other coefficient beyond these two to solve a univariate polynomial equation.

Moreover, recall that β_0/β_ν has been identified in general with $tan\theta$ so that in our example, $\beta_1/\beta_3 = tan\theta$ or that angle θ is given by the other two acute angles, $\angle\Lambda KM = \angle M\Lambda L$, of the right triangle $\triangle K\Lambda M$: $\omega + \theta = 90°$. Also, as soon as $e_1 = tan\theta = a_{\nu-1}/a_\nu$, it follows that $\beta_0/\beta_\nu = a_{\nu-1}/a_\nu$, which in the context of $\triangle K\Lambda M$, implies in turn that $\beta_0 = a_{\nu-1}$ and $\beta_\nu = a_\nu$. The remaining of the roots of the original apeironomial

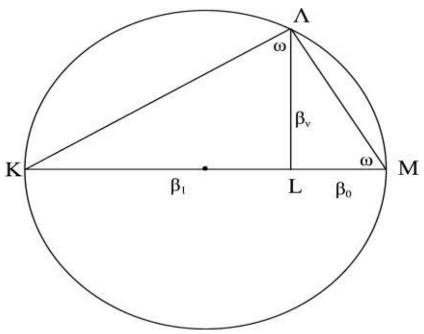

Fig.4.3. Roots as Geometric Means

equation may be identified through the use of Vieta's recursive formula. And, of course, the same approach may be followed in connection with any other formula giving rise to a series.

The grand point is that considerations such as these connote a theory of general geometric constructibility, which rejects the notion of irrationality spatially-wise. The involvement of imaginary numbers following the fundamental theorem of algebra does not change this conclusion since the instruments of constructibility are the compass and the unmarked ruler, equally applicable to either the real or the complex plane. As a matter of fact neither the Quadrature nor the Trisection advanced earlier is confined to the real plane. The problem of the Quadrature, for example, could be restated as follows: Given a real number (line segment) R, find a complex number (position vector) $z = R + x\iota$, $\iota = \sqrt{-1}$, such that $tan\phi = \sqrt{\pi}$, where $\phi = arg(z) = arctan(x/R)$. Or, in polar coordinates: Given a real number R, find a complex number z such that $mod z = r = \sqrt{R^2 + x^2}$ and

$$r\iota\left(tan\phi/\sqrt{1+tan^2\phi}\right) + r\left(1/\sqrt{1+tan^2\phi}\right) =$$
$$r(cos\phi + \iota sin\phi) = r[\iota(\sqrt{\pi}/\sqrt{1+\pi}) + (1/\sqrt{1+\pi})].$$

Indeed, the construction of solutions in terms of right triangle angles directs one to the geometric depiction of derivatives in discrete form. That is, we construct the answer to the question: Which is the value of the derivative, solving... To understand what precisely is solved, note that the unknown derivatives may be viewed as those of the characteristic equation of a linear homogeneous differential equation as follows:

$$f^\nu + \frac{a_{\nu-1}}{a_\nu}f^{\nu-1} + \frac{a_{\nu-2}}{a_\nu}f^{\nu-2} + \cdots + \frac{a_0}{a_\nu}f = 0 =$$
$$e^{xt}\left(x^\nu + \frac{a_{\nu-1}}{a_\nu}x^{\nu-1} + \frac{a_{\nu-2}}{a_\nu}x^{\nu-2} + \cdots + \frac{a_1}{a_\nu}x + \frac{a_0}{a_\nu}\right).$$

Consequently, our constructions give the values of the derivatives that solve the differential equation whose characteristic equation is the polynomial equation which is given to be sold; and so do elementary symmetric polynomials.

C. More on Constructibility

Is there any irrational that cannot be constructed? In so far as space is concerned, the answer is negative, because tangent runs from zero to infinity while secant runs from one to infinity: All irrational numbers are there; even infinity by itself is there. Infinity, the cosmos, is constructible, and this is why it has to be the actual, the proper infinity. We have one more proof that spatial infinity has to be the actual one. Spatially-wise, there is no such thing as irrationality, because simply a never ending non-repeating decimal part of a decimal number could not be constructible: When and where our line segment would end? Irrationality should be attributed to computation inadequacies and/or non-spatial considerations like time as a physical phenomenon. The difficulty of constructing irrationals lies in the difficulty of determining which exactly rationals (α's) give rise to them. This is the reason in the first place the Quadrature above has been so cumbersome.

One might object to the constructibility of numbers like $\sqrt{2}$ or π [20, 21]. Consider, for instance, $\sqrt{2}$, which had prompted much skepticism on the part of Pythagoreans. If its construction was not possible as a hypotenuse of an isosceles right triangle of unit legs, triangle inscribable into semi-circumference, the proposition that an angle inscribed in a semicircle is a right angle, would not hold. This proposition and hence, the axiom of parallel lines would be violated. It would be impossible to construct the unit per se as the hypotenuse of another isosceles right triangle of legs equal to $1/\sqrt{2}$. And, the construction of this leg-side in turn, as the hypotenuse of still another isosceles right triangle of legs equal to $1/\sqrt{4}$, and so on, since none of these sides-hypotenuses could constitute circle diameter.

What would ensure that such triangles are right triangles once the axiom of parallel lines is rejected? One might replace this axiom by setting some magnitude equal to the unit and prompting subsequently the emergence of number $\sqrt{2}$, too. But, how, construction-wise, if one did not also postulate some axiom analogous to that of parallel lines? The fact, yes fact, that $\sqrt{2}$ is constructible, that it has a beginning and an end, stems if not anything else from the fact also that constructible are numbers greater that $\sqrt{2} = 1.41421$ ..., numbers like 1.5. As soon as $\sqrt{2} < 1.5$, if their construction started from a single point, the construction representing $\sqrt{2}$ should have an end before the end of the construction representing 1.5. And, hence, the number of the decimal digits capturing $\sqrt{2}$ should have an end as well, even if the axiom of parallel lines was disregarded, and we defined instead some magnitude to be our unit. After all, the notion of Dedekind cut per se relies on general number constructibility: Cut of the real line in two distinct half-lines. If it were not so, where would the cut capturing an irrational number be placed? Unless irrationality captures the cut per se, the abrupt disruption of continuity when time is introduced in the discussion.

But, in so far as space alone is concerned, we have to distinguish between infinite but countable decimal digits and infinite uncountable digits accompanying the integer of a decimal number. Toward this end, consider the sequence of sides-hypotenuses, $1/\sqrt{2}$, $1/\sqrt{2}\sqrt{2}$,…, $1/\sqrt{2}^{\kappa}$, where κ is an integer. This sequence tends to zero. How could one start constructing the unit out of zero? The key to the answer is the word "tends"; zero should be out of reach, never reached, because only then, out of something, not out of nothing, one might start constructing the unit. Decimal digits keep coming one after the other, impossible practically to calculate their number, but they have to stop at the gate of zero. Otherwise, the unit would not be constructible.

Or, consider the example of the number π. The infinite division of polygon sides must have an end if the points comprising a circle circumference and not thin air, a complete vacuum, is to be produced. Consequently, the decimal digits of π must have an end. In general, given constructability per se, and the constructibility of a number greater than another number with infinite decimal digits, it follows logically that the latter number should be constructible as well. Decimal digits must have an end; they are infinite but countably so. The ad infinitum counting must come to a halt to allow the construction of a number which is smaller than a greater known to be constructible number. All numbers with infinite decimals are countably infinite, because there is always a greater number known to be constructible. There are no uncountably infinite decimals.

What we have, in other words, is infinity in the Aristotelian sense of actual as opposed to potential infinity. Any in general irrational number is one with an actualization in nature and hence, with a number of decimal digits in line with the Aristotelian notion of actual infinity; with decimals that sooner or later become repeating. It all comes down to the fifth axiom of Euclid: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." The first four are: "Let the following be postulated"': 1. "To draw a straight line from any point to any point." 2."To produce [extend] a finite straight line continuously in a straight line." 3."To describe a circle with any centre and distance [radius]." 4."That all right angles are equal to one another." [13, 30, 31]

Consider Fig. 4.4. If the two lines ϵ and η could not meet at point A, no circle $(O, OA = OB)$ could be drawn, because no line AΓ could be drawn too, in violation of axiom 3, which refers to any circle of any radius. As a matter of fact, no circle at all could be drawn, because one must always be able to draw from a point like A a line like ϵ, intersecting a radius like OB. But, also, axiom 4 would be violated, because angles a and b would have to be right angles, and $a \neq b$. Axiom 5 follows from and completes axioms 3 and 4 in fully describing the plane, the two-dimensional space, following axioms 1 and 2, which fully describe the one-dimensional space. More precisely, axiom 5 ensures continuity in the two-dimensional space the same way axiom 2 ensures continuity in the one-dimensional space. If lines like ϵ could not meet the horizontal axis, no circle at all could be drawn; there would be no two-dimensional space, contrary to what axioms 3 and 4 postulate.

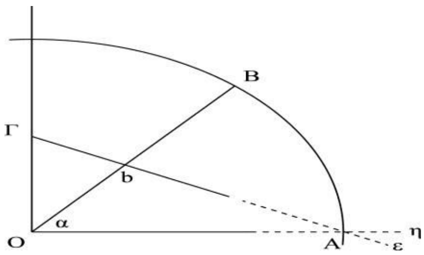

Fig.4.4.The Euclidean Axiom

This is the reason Euclid formulated axiom 5 the way he did and not as a Playfair or other similar axiom; the wording was chosen carefully in serving the purpose of this axiom. But, axiom 5 does much more than completely defining along with axioms 3 and 4 the plane. It puts an end to infinity: this "*produced indefinitely*" has an end, be it one next to the origin of axes, O, or to zillions miles away from it, the end being point A, the intersection point, because intersection takes place in the infinity, after indefinite extension of ϵ and η. If space ended before the intersection, neither axiom 4 nor axiom 5 would hold; and if space ended after A, the extension of ϵ and η would have not been indefinite, because intersection at A occurs after such an extension.

The finiteness of the infinity is in the core of geometry, and this is the reason it underlines non-Euclidean geometries as well. These geometries replace the

JohnPlayfair (1748-1819)-axiom side of axiom 5, but retain the actual infinity side, and this is the reason they continue being geometries, i.e. studies of space, each viewing it analytically from its own standpoint given that space in reality is only one. For example, from the viewpoint of geometric constructibility, space has to be the composite one advanced herein and in accordance with axiom 5. Any other axiomatic theoretical construction dismissing axiom 5 altogether, simply is not geometry. When Aristotle (384-322 B.C. [3, ch. 6]) said: "For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different", he said it literally: He did not say that the cosmos is limitless but that the limits are in continuous change, and trying to catch up with them is futile. This position is very important analytically, because it implies that statically viewed, the cosmos is susceptible to scientific inquiry including the dynamics inside its borders...

5. Conclusion

"The essence of mathematics lies precisely in its freedom." Georg Cantor (1845-1918, [10, p.12])

At a given point in historic time, the Cosmos might be viewed statically, and hence, as a finite spatial entity separate from motion-cum-time considerations whatsoever. Because, if statically the Cosmos was not finite, in what the dynamics introduced by historic time would lie if not in qualifying at least finiteness? Consequently, if numbers are to be empirically relevant space-wise, their multitude and decimal expansion should be finite, too. Especially so with regard to space as such, because simply the only empirical content numbers may be deriving in such a space would be that from serving the construction of space. All numbers are rational numbers given also that all what the passage of historical time might signify, would be replicated stationarity, expanded borders of the universe, but still *borders, limits*, (if expansion does take place). Within this space-only context, the irrationality of the irrational numbers is relevant rationality, the relevance depending on which moment is chosen to define finiteness, to define the borders of the universe, the end of the accumulation of decimal digits.

From the viewpoint of the Arithmetic, if any number, n, is a number to which some infinite series, $S = \sum_{i=0}^{\infty} \rho_i$, converges, and, if a partial sum of the terms of S consists of roots of some general polynomial equation in x, $\sum_{k=0}^{K} a_k x^k$: Then, by Vieta's formulas, this sum, $\sum_{k=1}^{K} \rho_k$, may be identified with the elementary symmetric polynomial, e_1, equaling the coefficient ratio, a_{K-1}/a_K. It also follows that $n = a_{K-1}/a_K$ is the sum of S, and that any in general number, real or complex, may be characterized likewise.

But, then again, the remaining of the terms of S should have to be seen as polynomial equation roots as well. By doing so, however, all roots, all terms of S should have to be seen in connection with a single polynomial of length equal to the "length" of S, which length cannot be broken down to form "partial K-length polynomials"". If such a breakdown were allowed, say from 0 to K, from $K + 1$ to $2K$, and so on, it would follow that $n = a_{K-1}/a_K = a_{2K-1}/a_{2K}$ ..., imposing a certain structure on α's, and there is no *a priori* reason to assume so.

Now, given (i) this, (ii) that behind any number n there is some infinite series S converging to n, (iii) that behind any S there is some polynomial equation, having as roots the terms of S, (iv) that as such, S is also an elementary symmetric polynomial, e_1, (v) that e_1 equals the ratio of the last but one to the last polynomial coefficient, and (vi) that this ratio may be identified geometrically with the trigonometric tangent of an acute angle of a right triangle: It follows that the length of the polynomial behind an S and hence, the infinite of the horizon of S, should be the actual infinity, i.e. a polynomial of finitely infinite degree, or *apeiro*nomial, giving rise to a finitely infinite series. There can be no infinitely infinite e's and an infinitely infinite captured by an acute angle! Consequently, geometrically, in space, all decimal expansions are terminating, all series are convergent, all numbers are rational, and all numbers are constructible the way the problem of the Quadrature was solved.

Indeed, an infinite series might be viewed alternatively as a Brook Taylor (1685-1731) series approximation and as a solution to a differential equation of finite order near some ordinary or singular point. An infinite series in space is always an approximation of the finite, converging to it. But, what kind of space is that? To answer this question, note, for example, that empirically, $\sqrt{2}$ is a two-dimensional figure, the hypotenuse of a right triangle of unit-length catheti, with vertical projection onto the horizontal axis, on the one-dimensional real line, equal to one, and with its "circular" only projection (as circle radius rotating clockwise until it cuts the real axis) being a *Richard Dedekind* (1831-1916) cut. All "irrational"' numbers have exactly this property. They are Dedekind cuts only as circular projections from the two- to the one-dimensional space. A Dedekind cut is a one-dimensional notion. A cut on an axis does not correspond to a hole on a plane. Otherwise, space would be full of such holes given that the bulk of real numbers are irrational.

Yet, this is one only of the two scales in explaining geometrical, spatial, rationality. It is not enough, because note that on the real plane, $\sqrt{2}$ would arise from $y = \sqrt{1 + x}$, $x = 1,2, ...$ The number $\sqrt{2}$ on the axes continues being $\sqrt{2}$ on the plane. In the complex plane, however, our hypotenuse would be the position vector $z = 1 + 1\iota$, $\iota = \sqrt{-1}$. Now, this is exactly what the two-dimensional constructions of real-line irrationals are, namely position vectors on the complex plane, with their circular only projections on the real axis being Dedekind cuts. Roughly put, irrational are the numbers, which are constructible as complex numbers but non-constructible as reals given that mostly rationals from the reals are constructible.

And, all complex numbers are constructible, including those with irrational (in real line terms) real and/or imaginary parts, because, once for example $z = 1 + 1\iota$ is constructed, its circular projection onto the imaginary axis is $\iota\sqrt{2}$ and hence, $w = 1 + \iota\sqrt{2}$ may be constructed too, with Dedekind cut the irrational $\sqrt{3} = \sqrt{1+\xi^2}$. $\xi \equiv \sqrt{2}$. Moreover, equally constructible are all those complex numbers which are consistent as points with a given circumference like those for the projection to $\sqrt{2}$ or $\sqrt{3}$ on the real line; all these complex numbers, a great many of them with irrational both real and imaginary parts, have the same circular projection. To each real number, ν, rational or not, correspond so many complex numbers as points on the circumference, $2\pi\nu$ complex numbers, cutting the real line at that particular real number in the complex plane. So, when one manages to construct an irrational number, one does so, on the complex plane.

They say that there is at least one number between any two consecutive real numbers; there are $2\pi\nu$, actually, but complex numbers and no more than $2\pi\nu$, because a circle circumference locks at where it starts. And, there is not any other number between any two consecutive complex numbers, because simply discontinuities are inconsistent with constructibility. There can always be on the complex plane at least one position vector, $v = R + \iota x$, with R being real line rational number and with circular projection corresponding to a real line irrational number, and such that $x = Rtan\omega$ may be constructed given R and given the trigonometric tangent of the angle ω formed by the modulus of v with the given R: Construct $x = Rtan\omega$, with general solution method as in Section 2 regarding the example of $tan\omega = \sqrt{\pi}$ (of which the $\omega = x\pi/2a$ of the quadratrix of *Hippias ofElia* (c.460-c.390 BC), with a being the side of Hippias' square, or the ω's of the various spirals, or ..., are special cases).

Construction-wise, the real or complex character of the plane is immaterial, because the instruments of construction are the unmarked ruler and the compass. But, arithmetically, our cosmos, its spatial component, consists of one real plane where only rational numbers are constructible, and of a third imaginary axis-dimension, making possible the construction of what on the real plane would be irrational numbers, in which case, irrationals are also rationals in this complex, composite, or even better, constructible cosmos: The general conclusion is that an irrational numbers is irrational on the real plane, but in the three-dimensional world, it is as a vector the image of one at least constructible position vector, and through the angle formed between them, constructible becomes the "irrational vector" too, as a right-triangle side. At any given point in historic time, it is that collection of such numbers to which the process of forming "bigger" and "bigger" proper subsets of them would end, in a Dedekind-*Bernard Bolzano* (1781-1848) fashion.

This, in sum, appears to me to be the Arithmetic behind the way the problem of the Quadrature was solved, namely via trigonometric tangents. It cannot be explained otherwise why this problem has been solved, and why the angle has been trisected, just when both of these tasks have been proved to be impossible. Responsible for the impossibility must have been the persistence on potential infinity on all domains and the inappropriate development of the notion of actual infinity. Actual is the static, and potential is the dynamic, both in a world in which complex numbers are as fundamental in knowing this world as real numbers are. Put differently, once a two at least dimensional world is recognized, it will inevitably lead to the constitution of a third imaginary axis, because such is the world of Geometry, of nature spatially-wise, where there can be no irrationality of numbers. And, when calendar, historic time is introduced into this picture, it will only replicate it, it will not prompt any dynamics, unless time is taken to mean something more than clock-ticking, however elaborate concepts like reference frames may be making this ticking.

Indeed, "genuinely" irrational numbers, endless non-repeating decimal expansions, non-convergent infinite series, do have to exist in cosmos, and hence, they can only be attributed to the presence of time. How, exactly, given that the influence of the passage of historic time is identified with the ever changing finiteness of space? To answer this question, time has to be introduced into a static space and hence, geometrically, by considering motion in it through the straightedge and the compass. By doing so, we really fill in our empty so far space with "stuff" to which the assumed motion may be referring, and another concept of time different from the concept of historic time emerges analytically, simply, because it is a quantity found to be described by irrational numbers, while historic time and space are described by rational numbers.

From the viewpoint of methodology: What complicates the countability of numbers is the recurrence or not and the finite or infinite of the decimal digits of decimal numbers. The only criterion to decide about both of them is the physical relevance of numbers in connection with motion in space and time given that space is what it is. And, the only way to examine this relevance is through Geometry as the most accurate representation of the properties of space. When a mass is set in motion, the accompanying change of physical time in space takes place linearly, and Euclidean Geometry applies to it. But, motion of matter in space is non-linear, and non-Euclidean Geometry bears on it. Genuine irrationality derives empirical content neither from space alone nor from time alone, but from this precisely interaction between the rectilinear and the curvilinear, which physically is brought about by motion.

Hence, the notion of potential infinity which is advanced herein, is the potential infinite emanating out of each of the finitely infinite rational numbers by the potential infinite of the decimal digits that might start being added at the end of a given rational endlessly, and by the potential infinite of the order/disorder with which decimal digits would keep piling up. That is our potential infinity is the outcome of the interplay of these two kinds of potential infinity with regard to each of the finitely infinite rational numbers, over the whole set of rational numbers. There are as many potential infinities as finite rationals. The set of potential infinities is a finitely infinite one, simply because it is subject to the superstructure of actual infinite. The actual may be compromised with the potential, relaxing this *Bertrand Russell* (1872-1970) side of their set-theoretic relationship, by noting that this relationship holds at a given point in historic time and hence, should be attributed to age time.

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