New Integral Representation for Inverse Sine Function, the Rate of Catalan's Constant by Archimedes Constant and Other Functions

By Edigles Guedes

February 22, 2015

"*Devise not evil against thy neighbour, seeing he dwelleth securely by thee.*" – Proverbs 3:29.

Abstract. In present article, we developed infinite series representations for inverse sine function and other functions. Our main goal is to get the hypergeometric representation for Catalan constant and hyperbolic sine function; and new integral representation for inverse sine function.

1. Introduction

In this paper, our main goal, is to prove the following hypergeometric representations:

$$ {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},1;1\right)=\frac{4G}{\pi}, $$

and

$$ {}_2F_3\left(1,\frac{9}{4};\frac{5}{4},\frac{3}{2},2;x^2\right)=\frac{\sinh^2 x+2x\sinh 2x}{5x^2}; $$

and the integral representation for inverse sine function:

$$ \sin^{-1}\alpha=\frac{2\alpha}{\pi}\int_0^1\int_0^1\int_0^1\frac{x^{\alpha z}y^{-\alpha z}}{-\ln xy\sqrt{1-z^2}}\mathrm{dx}\,\mathrm{dy}\mathrm{dz}. $$

2. Lemmas and Theorems

Lemma 1. *If* $a\in\mathbb{R}-\{0\}$ *and* $b\in\mathbb{R}-\{0\}$, *then*

$$ \frac{\sqrt{ab}}{a+b}=\frac{1}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2} \tag{1} $$

and

$$ \frac{a+b}{\sqrt{ab}}=\frac{2}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}}. \tag{2} $$

Proof. It is well knew the Babylonian identity [1, page 119]

$$ ab=\frac{1}{4}[(a+b)^2-(a-b)^2] \tag{3} $$

Make the following manipulation algebraic

$$ ab=\left(\frac{a+b}{2}\right)^2\left[1-\left(\frac{a-b}{a+b}\right)^2\right], $$

thence,

$$ \sqrt{ab}=\frac{a+b}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}\Rightarrow\frac{\sqrt{ab}}{a+b}=\frac{1}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}, $$

and inverting both members, we encounter

$$\frac{a+b}{\sqrt{ab}}=\frac{2}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}},$$

which are the desired result. □

Lemma 2. *If* $a\in\mathbb{R}-\{0\}$ *and* $b\in\mathbb{R}-\{0\}$, *then*

$$\frac{\sqrt{ab}}{a+b}=-\frac{1}{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n}(2n-1)}\left(\frac{a-b}{a+b}\right)^{2n} \tag{4}$$

and

$$\frac{a+b}{\sqrt{ab}}=2\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n}}\left(\frac{a-b}{a+b}\right)^{2n}. \tag{5}$$

Proof. We calculate that

$$\sqrt{1-z^2}=-\sum_{n=0}^{\infty}\binom{2n}{n}\frac{z^{2n}}{2^{2n}(2n-1)} \tag{6}$$

and

$$\frac{1}{\sqrt{1-z^2}}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{z^{2n}}{2^{2n}}. \tag{7}$$

Take $z=\frac{a-b}{a+b}$ in (6) and (7); then, replace in (1) and (2), respectively, completing the proof. □

Lemma 3. *If* $a\in\mathbb{R}-\{0\}$ *and* $b\in\mathbb{R}-\{0\}$, *then*

$$\begin{aligned}\frac{\sqrt{ab}}{(a+b)ab}&=\frac{2}{\pi}\int_0^{\pi}\frac{1}{(a+b)^2-(a-b)^2\cos^2\theta}d\theta=\frac{2}{\pi}\int_{-1}^{1}\frac{1}{[(a+b)^2-(a-b)^2u^2]\sqrt{1-u^2}}\mathrm{du}\\&=\frac{4}{\pi}\int_0^{1}\frac{1}{[(a+b)^2-(a-b)^2u^2]\sqrt{1-u^2}}\mathrm{du},\end{aligned} \tag{8}$$

where $\cos\theta$ *denotes the cosine function.*

Proof. In [2, page 423], we encounter

$$\int_0^{2\pi}\cos^{2n}\theta d\theta=\binom{2n}{n}\frac{2\pi}{2^{2n}}\Rightarrow\binom{2n}{n}\frac{1}{2^{2n}}=\frac{1}{2\pi}\int_0^{2\pi}\cos^{2n}\theta d\theta \tag{9}$$

I substitute (8) in (5), and obtain

$$\begin{aligned}\frac{a+b}{\sqrt{ab}}&=\frac{1}{\pi}\int_0^{2\pi}\frac{(a+b)^2}{(a+b)^2-(a-b)^2\cos^2\theta}d\theta\\ \Rightarrow\frac{1}{(a+b)\sqrt{ab}}&=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{(a+b)^2-(a-b)^2\cos^2\theta}d\theta\\ \Rightarrow\frac{\sqrt{ab}}{(a+b)ab}&=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{(a+b)^2-(a-b)^2\cos^2\theta}d\theta\\ \Rightarrow\frac{\sqrt{ab}}{(a+b)ab}&=\frac{2}{\pi}\int_0^{\pi}\frac{1}{(a+b)^2-(a-b)^2\cos^2\theta}d\theta.\end{aligned} \tag{10}$$

Let $\theta=\cos^{-1}u$ in (10); thereafter,

$$\begin{aligned}\frac{\sqrt{ab}}{(a+b)ab}&=\frac{2}{\pi}\int_{-1}^{1}\frac{1}{[(a+b)^2-(a-b)^2u^2]\sqrt{1-u^2}}\mathrm{du}\\&=\frac{4}{\pi}\int_0^{1}\frac{1}{[(a+b)^2-(a-b)^2u^2]\sqrt{1-u^2}}\mathrm{du},\end{aligned}$$

which are the results desired. □

Lemma 4. *If $x \in \mathbb{R} - \{0\}$ and $y \in \mathbb{R}$, then*

$$\begin{aligned}\frac{1}{x\sqrt{x^2-y^2}} &= \frac{1}{\pi}\int_0^{\pi} \frac{1}{x^2 - y^2\cos^2\theta}d\theta = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{(x^2-y^2u^2)\sqrt{1-u^2}}\text{du} \\ &= \frac{2}{\pi}\int_0^1 \frac{1}{(x^2-y^2u^2)\sqrt{1-u^2}}\text{du}\end{aligned} \tag{11}$$

Proof. Let $a+b=x$ and $a-b=y$ in Lemma 3. □

Theorem 5. *If $x \in \mathbb{R}$, then*

$$\begin{aligned}\sin^{-1}x &= \frac{1}{\pi}\int_0^{\pi} \tanh^{-1}(x\cos\theta)\sec\theta\, d\theta = \frac{1}{\pi}\int_{-1}^{1} \frac{\tanh^{-1}(ux)}{u\sqrt{1-u^2}}\text{du} \\ &= \frac{2}{\pi}\int_0^1 \frac{\tanh^{-1}(ux)}{u\sqrt{1-u^2}}\text{du},\end{aligned}$$

where $\sin^{-1}x$ denotes the inverse sine function, $\tanh^{-1}x$ denotes the inverse tangent hyperbolic function, $\cos x$ denotes the cosine function and $\sec x$ denotes the secant function.

Proof. In Lemma 4, we set $x=1$, $y=t$ in the second integral, and obtain

$$\frac{1}{\sqrt{1-t^2}} = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{(1-t^2u^2)\sqrt{1-u^2}}\text{du}. \tag{12}$$

Integrate (12) from 0 at x with respect to t, thus

$$\begin{aligned}\int_0^x \frac{\text{dt}}{\sqrt{1-t^2}} &= \frac{1}{\pi}\int_{-1}^{1}\int_0^x \frac{\text{dt}}{1-t^2u^2}\cdot\frac{\text{du}}{\sqrt{1-u^2}} \\ &= \frac{1}{\pi}\int_{-1}^{1} \frac{\tanh^{-1}(ux)}{u\sqrt{1-u^2}}\text{du}.\end{aligned} \tag{13}$$

On the other hand, it is well know that

$$\int_0^x \frac{\text{dt}}{\sqrt{1-t^2}} = \sin^{-1}x. \tag{14}$$

From (13) and (14), it follows that

$$\sin^{-1}x = \frac{1}{\pi}\int_{-1}^{1} \frac{\tanh^{-1}(ux)}{u\sqrt{1-u^2}}\text{du},$$

thereout, we deduced that

$$\sin^{-1}x = \frac{2}{\pi}\int_0^1 \frac{\tanh^{-1}(ux)}{u\sqrt{1-u^2}}\text{du}.$$

In Lemma 4, we set $x=1$, $y=t$, in the first integral, and find

$$\frac{1}{\sqrt{1-t^2}} = \frac{1}{\pi}\int_0^{\pi} \frac{1}{1-t^2\cos^2\theta}d\theta. \tag{15}$$

Integrate (15) from 0 at x with respect to t, thus

$$\begin{aligned}\int_0^x \frac{\text{dt}}{\sqrt{1-t^2}} &= \frac{1}{\pi}\int_0^{\pi}\int_0^x \frac{1}{1-t^2\cos^2\theta}\text{dt}\, d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \tanh^{-1}(x\cos\theta)\sec\theta d\theta.\end{aligned} \tag{16}$$

From (14) and (16), we have the results desired. □

Corollary 6. *We have*

$${}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, 1; 1\right) = \frac{4G}{\pi},$$

where G denotes the Catalan constant and π denotes the Archimedes constant.

Proof. Divide both sides of the Theorem 5 by $x\sqrt{1-x^2}$ and integrate from 0 at 1 with respect to x, thus

$$\int_0^1 \frac{\sin^{-1}x}{x\sqrt{1-x^2}}\mathrm{dx} = \frac{2}{\pi}\int_0^1 \frac{1}{u\sqrt{1-u^2}}\left(\int_0^1 \frac{\tanh^{-1}(ux)}{x\sqrt{1-x^2}}\mathrm{dx}\right)\mathrm{du}. \tag{17}$$

On the other hand, we know that

$$\tanh^{-1}(ux) = \sum_{k=0}^{\infty} \frac{(ux)^{2k+1}}{2k+1}. \tag{18}$$

From (17) and (18), it follows that

$$\begin{aligned}
\int_0^1 \frac{\sin^{-1}x}{x\sqrt{1-x^2}}\mathrm{dx} &= \frac{2}{\pi}\int_0^1 \frac{1}{u\sqrt{1-u^2}}\left(\int_0^1 \frac{1}{x\sqrt{1-x^2}}\sum_{k=0}^{\infty}\frac{(ux)^{2k+1}}{2k+1}\mathrm{dx}\right)\mathrm{du}\\
&= \frac{2}{\pi}\int_0^1 \frac{1}{u\sqrt{1-u^2}}\left(\sum_{k=0}^{\infty}\frac{u^{2k+1}}{2k+1}\int_0^1 \frac{x^{2k}}{\sqrt{1-x^2}}\mathrm{dx}\right)\mathrm{du}\\
&= \frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{u\sqrt{1-u^2}}\left(\sum_{k=0}^{\infty}\frac{u^{2k+1}\Gamma\left(k+\frac{1}{2}\right)}{(2k+1)\Gamma(k+1)}\right)\mathrm{du}\\
&= \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)}{(2k+1)\Gamma(k+1)}\int_0^1 \frac{u^{2k}}{\sqrt{1-u^2}}\mathrm{du}\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma^2\left(k+\frac{1}{2}\right)}{(2k+1)\Gamma^2(k+1)}\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma(2k+1)\Gamma^2\left(k+\frac{1}{2}\right)}{\Gamma(2k+2)\Gamma^2(k+1)}\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(1)_{2k}\Gamma(1)\left(\frac{1}{2}\right)_k^2\Gamma^2\left(\frac{1}{2}\right)}{(2)_{2k}\Gamma(2)(1)_k^2\Gamma^2(1)}\\
&= \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(1)_{2k}\left(\frac{1}{2}\right)_k^2}{(2)_{2k}(1)_k^2}\\
&= \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k(1)_k 2^{2k}\left(\frac{1}{2}\right)_k^2}{(1)_k\left(\frac{3}{2}\right)_k 2^{2k}(1)_k^2}\\
&= \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k(1)_k\left(\frac{1}{2}\right)_k^2}{(1)_k\left(\frac{3}{2}\right)_k(1)_k^2}\\
&= \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^3}{\left(\frac{3}{2}\right)_k(1)_k k!}\\
&= \frac{\pi}{2}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},1;1\right).
\end{aligned} \tag{19}$$

On the other hand, we know that

$$\int_0^1 \frac{\sin^{-1}x}{x\sqrt{1-x^2}}\mathrm{dx} = 2G. \tag{20}$$

From (19) and (20), it follows that

$$\begin{aligned}
&\frac{\pi}{2}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},1;1\right) = 2G\\
&\Rightarrow {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},1;1\right) = \frac{4G}{\pi},
\end{aligned}$$

which is the desired result. □

Corollary 7. *If* $\alpha \in \mathbb{R}$, *then*

$$\sin^{-1}\alpha = \frac{2\alpha}{\pi}\int_0^1\int_0^1\int_0^1 \frac{x^{\alpha z}y^{-\alpha z}}{-\ln xy\sqrt{1-z^2}}\mathrm{dx\,dydz},$$

where $\sin^{-1}\alpha$ *denotes the inverse sine function and* $\ln x$ *denotes the natural logarithm function.*

Proof. Since [3, page 54, formula 1.622.7], we encounter

$$\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right). \tag{21}$$

From Theorem 5 and (17), we have

$$\begin{aligned}\sin^{-1}z &= \frac{1}{2\pi}\int_0^\pi \ln\left(\frac{1+z\cos\theta}{1-z\cos\theta}\right)\sec\theta\, d\theta = \frac{1}{2\pi}\int_{-1}^1 \frac{\ln\left(\frac{1+zt}{1-zt}\right)}{u\sqrt{1-u^2}}\mathrm{dt}\\ &= \frac{1}{\pi}\int_0^1 \frac{\ln\left(\frac{1+zt}{1-zt}\right)}{t\sqrt{1-t^2}}\mathrm{dt}.\end{aligned} \tag{22}$$

On the other hand, in [4, page 10], we get

$$\int_0^1\int_0^1 \frac{x^{u-1}y^{v-1}}{-\ln xy}\mathrm{dx\,dy} = \frac{1}{u-v}\ln\left(\frac{u}{v}\right) \Rightarrow \ln\left(\frac{u}{v}\right) = (u-v)\int_0^1\int_0^1 \frac{x^{u-1}y^{v-1}}{-\ln xy}\mathrm{dx\,dy}. \tag{23}$$

Let $u = 1+zt$ and $v = 1-zt$ in (23)

$$\ln\left(\frac{1+zt}{1-zt}\right) = 2zt\int_0^1\int_0^1 \frac{x^{zt}y^{-zt}}{-\ln xy}\mathrm{dx\,dy}. \tag{24}$$

From (22) and (24), we obtain

$$\begin{aligned}\sin^{-1}z &= \frac{2z}{\pi}\int_0^1 \frac{1}{\sqrt{1-t^2}}\int_0^1\int_0^1 \frac{x^{zt}y^{-zt}}{-\ln xy}\mathrm{dx\,dydt}\\ &= \frac{2z}{\pi}\int_0^1\int_0^1\int_0^1 \frac{x^{zt}y^{-zt}}{-\ln xy\sqrt{1-t^2}}\mathrm{dx\,dydt}.\end{aligned} \tag{25}$$

Changing z into α and t into z in (25), we conclude this proof. □

Corollary 8. *If* $x \in \mathbb{R}$, *then*

$$\cosh x = \frac{2}{\pi}\int_0^1 \frac{\mathrm{du}}{(1-u^2\tanh^2 x)\sqrt{1-u^2}},$$

where $\cosh x$ *denotes the hyperbolic cosine function and* $\tanh x$ *denotes the hyperbolic tangent function.*

Proof. Changing x into $\tanh x$ in Theorem 5, we find

$$\sin^{-1}(\tanh x) = \frac{2}{\pi}\int_0^1 \frac{\tanh^{-1}(u\tanh x)}{u\sqrt{1-u^2}}\mathrm{du}. \tag{26}$$

The derivative with respect to x in both members of (26), give us

$$\begin{aligned}\frac{d[\sin^{-1}(\tanh x)]}{dx} &= \frac{2}{\pi}\int_0^1 \frac{d[\tanh^{-1}(u\tanh x)]}{dx}\cdot\frac{\mathrm{du}}{u\sqrt{1-u^2}}\\ \Rightarrow \operatorname{sech} x &= \frac{2}{\pi}\int_0^1 \frac{u\cdot\operatorname{sech}^2 x}{1-u^2\tanh^2 x}\cdot\frac{\mathrm{du}}{u\sqrt{1-u^2}}\\ \Rightarrow \cosh x &= \frac{2}{\pi}\int_0^1 \frac{\mathrm{du}}{(1-u^2\tanh^2 x)\sqrt{1-u^2}},\end{aligned} \tag{27}$$

which is the desired result. □

Corollary 9. *If* $x \in \mathbb{R}$*, then*

$$\cosh x = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \tanh^{2n} x,$$

where $\cosh x$ *denotes the hyperbolic cosine function and* $\tanh x$ *denotes the hyperbolic tangent function.*

Proof. In the previous Corollary, we consider the infinite series expansion

$$\frac{1}{1-u^2\tanh^2 x} = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} u^n \tanh^n x. \tag{28}$$

Multiply (19) by $\frac{2}{\pi\sqrt{1-u^2}}$ and integrate from 0 at 1 with respect to u, thus

$$\begin{aligned}\cosh x &= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} \left(\int_0^1 \frac{u^n}{\sqrt{1-u^2}} \mathrm{du} \right) \tanh^n x \\ &= \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{[1+(-1)^n]\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \tanh^n x \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \tanh^{2n} x.\end{aligned}$$

□

Theorem 10. *If* $\alpha \in \mathbb{R}_{\geq 0}$*, then*

$$\begin{aligned}\frac{(1+\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)} &= \frac{1}{\pi} \int_0^{\pi} {}_2F_1\left(1, \frac{\alpha+1}{2}; \frac{\alpha+3}{2}; \cos^2\theta\right) d\theta = \frac{2}{\pi\sqrt{\pi}} \int_{-1}^{1} \frac{{}_2F_1\left(1, \frac{\alpha+1}{2}; \frac{\alpha+3}{2}; u^2\right)}{\sqrt{1-u^2}} \mathrm{du} \\ &= \frac{4}{\pi\sqrt{\pi}} \int_0^1 \frac{{}_2F_1\left(1, \frac{\alpha+1}{2}; \frac{\alpha+3}{2}; u^2\right)}{\sqrt{1-u^2}} \mathrm{du},\end{aligned}$$

where $\Gamma(\alpha)$ *denotes the Gamma function,* ${}_2F_1(a,\ b;\ c;\ z)$ *dentoes the Gaussian hypergeometric function and* $\cos\theta$ *denotes the cosine function.*

Proof. Multiply (12) by t^α and integrate from 0 at 1 with respect to t, thus

$$\begin{aligned}\int_0^1 \frac{t^\alpha \,\mathrm{dt}}{\sqrt{1-t^2}} &= \frac{1}{\pi} \int_{-1}^{1} \int_0^1 \frac{t^\alpha \,\mathrm{dt}}{1-t^2u^2} \cdot \frac{\mathrm{du}}{\sqrt{1-u^2}} \\ &= \frac{1}{\pi} \int_{-1}^{1} \frac{{}_2F_1\left(1, \frac{\alpha+1}{2}; \frac{\alpha+3}{2}; u^2\right)}{(1+\alpha)\sqrt{1-u^2}} \mathrm{du}.\end{aligned} \tag{29}$$

On the other hand, it is well know that

$$\int_0^1 \frac{t^\alpha \,\mathrm{dt}}{\sqrt{1-t^2}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{2}\right)}{2\Gamma\left(\frac{\alpha}{2}+1\right)}. \tag{30}$$

From (29) and (30), it follows that

$$\frac{(1+\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)} = \frac{2}{\pi\sqrt{\pi}} \int_{-1}^{1} \frac{{}_2F_1\left(1, \frac{\alpha+1}{2}; \frac{\alpha+3}{2}; u^2\right)}{\sqrt{1-u^2}} \mathrm{du},$$

thereof, we deduced that

$$\frac{(1+\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)} = \frac{4}{\pi\sqrt{\pi}} \int_0^1 \frac{{}_2F_1\left(1, \frac{\alpha+1}{2}; \frac{\alpha+3}{2}; u^2\right)}{\sqrt{1-u^2}} \mathrm{du}.$$

Multiply (15) by t^α and integrate from 0 at 1 with respect to t, thus

$$\begin{aligned}\int_0^1 \frac{t^\alpha\,\mathrm{dt}}{\sqrt{1-t^2}} =& \frac{1}{\pi}\int_0^\pi\int_0^1 \frac{t^\alpha}{1-t^2\cos^2\theta}\mathrm{dt}\,d\theta\\ =&\frac{1}{\pi}\int_0^\pi \frac{{}_2F_1\left(1,\frac{\alpha+1}{2};\frac{\alpha+3}{2};\cos^2\theta\right)}{1+\alpha}d\theta.\end{aligned} \tag{31}$$

From (30) and (31), we have the results desired. □

Corollary 11. *If $\alpha\in\mathbb{R}_{\geq 0}$, then*

$$\begin{aligned}\frac{(1+\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)} =& \frac{2}{\sqrt{\pi}}\sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{\alpha+3}{2}\right)_n}\cdot\frac{1}{n!}\\ =&\frac{2}{\sqrt{\pi}}{}_2F_1\left(\frac{1}{2},\frac{\alpha+1}{2};\frac{\alpha+3}{2};1\right),\end{aligned}$$

where $\Gamma(\alpha)$ denotes the Gamma function and ${}_2F_1(a,b;c;z)$ dentoes the Gaussian hypergeometric function.

Proof. By Theorem 6, we have

$$\begin{aligned}\frac{(1+\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)} =& \frac{4}{\pi\sqrt{\pi}}\int_0^1 \frac{{}_2F_1\left(1,\frac{\alpha+1}{2};\frac{\alpha+3}{2};u^2\right)}{\sqrt{1-u^2}}\mathrm{du}\\ =&\frac{4}{\pi\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1-u^2}}\sum_{n=0}^\infty \frac{(1)_n\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{\alpha+3}{2}\right)_n}\cdot\frac{u^{2n}}{n!}\mathrm{du}\\ =&\frac{4}{\pi\sqrt{\pi}}\sum_{n=0}^\infty \frac{(1)_n\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{\alpha+3}{2}\right)_n}\cdot\frac{1}{n!}\int_0^1\frac{u^{2n}}{\sqrt{1-u^2}}\mathrm{du}\\ =&\frac{4}{\pi\sqrt{\pi}}\sum_{n=0}^\infty \frac{(1)_n\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{\alpha+3}{2}\right)_n}\cdot\frac{1}{n!}\cdot\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{2\Gamma(n+1)}\\ =&\frac{2}{\pi}\sum_{n=0}^\infty \frac{(1)_n\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{\alpha+3}{2}\right)_n}\cdot\frac{1}{n!}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\\ =&\frac{2}{\pi}\sum_{n=0}^\infty \frac{(1)_n\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{\alpha+3}{2}\right)_n}\cdot\frac{1}{n!}\cdot\frac{\left(\frac{1}{2}\right)_n\Gamma\left(\frac{1}{2}\right)}{(1)_n\Gamma(1)}\\ =&\frac{2}{\pi}\sum_{n=0}^\infty \frac{(1)_n\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{\alpha+3}{2}\right)_n}\cdot\frac{1}{n!}\cdot\frac{\left(\frac{1}{2}\right)_n\sqrt{\pi}}{(1)_n}\\ =&\frac{2}{\sqrt{\pi}}\sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{\alpha+3}{2}\right)_n}\cdot\frac{1}{n!}\\ =&\frac{2}{\sqrt{\pi}}{}_2F_1\left(\frac{1}{2},\frac{\alpha+1}{2};\frac{\alpha+3}{2};1\right),\end{aligned}$$

which are the desired results. □

Corollary 12. *If $x\in\mathbb{R}-\{0\}$, then*

$${}_2F_3\left(1,\frac{9}{4};\frac{5}{4},\frac{3}{2},2;x^2\right)=\frac{\sinh^2 x+2x\sinh 2x}{5x^2}.$$

Proof. Let $\alpha=2k$ in Corollary 11, hence,

$$\frac{\Gamma\left(\frac{2k+1}{2}\right)}{\Gamma(k+1)}+2k\frac{\Gamma\left(\frac{2k+1}{2}\right)}{\Gamma(k+1)}=\frac{2}{\sqrt{\pi}}\sum_{n=0}^\infty\frac{\left(\frac{1}{2}\right)_n\left(\frac{2k+1}{2}\right)_n}{\left(\frac{2k+3}{2}\right)_n}\cdot\frac{1}{n!} \tag{32}$$

Divide both members of (32) by $\Gamma\left(\frac{2k+1}{2}\right)$, and encounter

$$\frac{1}{k!}+\frac{2k}{k!}=\frac{2}{\Gamma\left(\frac{2k+1}{2}\right)\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{2k+1}{2}\right)_n}{\left(\frac{2k+3}{2}\right)_n}\cdot\frac{1}{n!}. \tag{33}$$

Changing k into $2k$ in (33), multiply by $\frac{1}{2}\cdot(2x)^{2k}$ and sum from 1 at infinity with respect to k, and we have

$$\begin{aligned}
\frac{1}{2}\sum_{k=1}^{\infty}\frac{(2x)^{2k}}{(2k)!}+2\sum_{k=1}^{\infty}\frac{k(2x)^{2k}}{(2k)!}&=\sum_{k=1}^{\infty}\frac{(2x)^{2k}}{\Gamma\left(\frac{4k+1}{2}\right)\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{4k+1}{2}\right)_n}{\left(\frac{4k+3}{2}\right)_n}\cdot\frac{1}{n!}\\
\Rightarrow\sinh^2 x+2x\sinh 2x&=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}\sum_{k=1}^{\infty}\frac{\left(\frac{4k+1}{2}\right)_n(2x)^{2k}}{\left(\frac{4k+3}{2}\right)_n\Gamma\left(\frac{4k+1}{2}\right)}\\
&=\frac{16x^2}{3\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n n!}\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{9}{4}\right)_k\left(\frac{n}{2}+\frac{5}{4}\right)_k x^{2k}}{\left(\frac{5}{4}\right)_k\left(\frac{5}{4}\right)_k\left(\frac{7}{4}\right)_k\left(\frac{n}{2}+\frac{9}{4}\right)_k k!}\\
&=\frac{16x^2}{3\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{9}{4}\right)_k\left(\frac{5}{4}\right)_k\Gamma\left(2k+\frac{7}{2}\right)x^{2k}}{\left(\frac{5}{4}\right)_k\left(\frac{5}{4}\right)_k\left(\frac{7}{4}\right)_k\left(\frac{9}{4}\right)_k\Gamma(2k+3)k!}\\
&=\frac{16x^2}{3\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(1)_k\Gamma\left(2k+\frac{7}{2}\right)x^{2k}}{\left(\frac{5}{4}\right)_k\left(\frac{7}{4}\right)_k\Gamma(2k+3)k!}\\
&=\frac{16x^2}{3\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{7}{2}\right)_{2k}\Gamma\left(\frac{7}{2}\right)x^{2k}}{\left(\frac{5}{4}\right)_k\left(\frac{7}{4}\right)_k(3)_{2k}\Gamma(3)k!}\\
&=\frac{16\cdot 15\cdot x^2}{3\cdot 8\cdot 2}\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{7}{2}\right)_{2k}x^{2k}}{\left(\frac{5}{4}\right)_k\left(\frac{7}{4}\right)_k(3)_{2k}k!}\\
&=5x^2\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{7}{2}\right)_{2k}x^{2k}}{\left(\frac{5}{4}\right)_k\left(\frac{7}{4}\right)_k(3)_{2k}k!}\\
&=5x^2\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{7}{4}\right)_k\left(\frac{9}{4}\right)_k 2^{2k}x^{2k}}{\left(\frac{5}{4}\right)_k\left(\frac{7}{4}\right)_k\left(\frac{3}{2}\right)_k(2)_k 2^{2k}k!}\\
&=5x^2\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{9}{4}\right)_k x^{2k}}{\left(\frac{5}{4}\right)_k\left(\frac{3}{2}\right)_k(2)_k k!}\\
&=5x^2{}_2F_3\left(1,\frac{9}{4};\frac{5}{4},\frac{3}{2},2;x^2\right),
\end{aligned} \tag{34}$$

which is the desired result. □

References

[1] Havil, Julina, *Gamma: Exploring the Euler's Constant*, Princenton University Press, 2003.

[2] Needham, Tristan, *Visual Complex Analysis*, Oxford University Press, 2000.

[3] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, 2000.

[4] Guillera, Jesús and Sondow, Jonathan, *Doube Integrals and Infinite Products for Some Classical Constants via Analytic Continuations of Lerch's Transcendent*, arXiv:math/0506319v3, 5 Aug 2006.