On Integral Representations for Harmonic Number and Digamma Function

By Edigles Guedes

March 3, 2015

"*Then Simon Peter answered him, Lord, to whom shall we go? thou hast the words of eternal life.*" - John 6:68.

Abstract. In this paper, we demonstrate some integral representations for harmonic number and digamma function.

1. Introduction

Leonhard Euler established the following integral representation [1] for harmonic number

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx,$$

and, accordingly, we can deduce the integral representation for the digamma function [2]

$$\psi(s+1) = -\gamma + \int_0^1 \frac{1-x^s}{1-x} dx.$$

In this article, we prove that

$$H_n = 2\int_0^1 \frac{(1-x^n)(1-x^{n+1})}{1-x^2} dx,$$

$$H_n = 2m\int_0^1 \frac{x^{m-1}(1-x^{nm})\left(1-x^{(n+1)m}\right)}{1-x^{2m}} dx$$

and

$$\psi(s+1) = -\gamma - 2\int_0^1 \frac{(1-x)^s - (1-x^2)^s}{x} dx. \tag{1}$$

2. Integral Representations for Harmonic Number

Lemma 1. *If $k = 1, 2, 3, \ldots,$ then*

$$\frac{1}{k} = 2\int_0^1 x^{k-1}(1-x^k) dx. \tag{2}$$

Proof. We know the elementary identity

$$\frac{1}{k} = \frac{a}{ak-1} - \frac{1}{k(ak-1)} \tag{3}$$

and the integral representation

$$\frac{1}{k} = \int_0^1 x^{k-1} dx, \tag{4}$$

for $\mathrm{Re}(k) > 0$.

From (2) and (3), it follows that

$$\begin{aligned}\frac{1}{k} &= \int_0^1 x^{\frac{ak-1}{a}-1} dx - \int_0^1 x^{k(ak-1)-1} dx \\ &= \int_0^1 \left[x^{\frac{ak-1}{a}-1} - x^{k(ak-1)-1}\right] dx.\end{aligned} \tag{5}$$

We take $a=\frac{2}{k}$ in Eq. (4)

$$\frac{1}{k}=\int_0^1 \frac{x^{\frac{k}{2}}-x^k}{x}dx. \tag{6}$$

Let $k\to 2k$ in Eq. (5)

$$\begin{aligned}\frac{1}{k}=&2\int_0^1 \frac{x^k-x^{2k}}{x}dx\\ =&2\int_0^1 x^{k-1}(1-x^k)dx,\end{aligned}$$

which is the desired result. □

Theorem 2. *If $n=1,2,3,\ldots$, then*

$$H_n=2\int_0^1 \frac{(1-x^n)(1-x^{n+1})}{1-x^2}dx,$$

where H_n denotes the harmonic number.

Proof. We knew the definition [1]

$$H_n\equiv\sum_{k=1}^{n}\frac{1}{k}. \tag{7}$$

From (1) and (6), we conclude that

$$\begin{aligned}H_n=&\sum_{k=1}^{n}2\int_0^1 x^{k-1}(1-x^k)dx\\ =&2\int_0^1\sum_{k=1}^{n}x^{k-1}(1-x^k)dx\\ =&2\int_0^1 \frac{(1-x^n)(1-x^{n+1})}{1-x^2}dx,\end{aligned}$$

which is the desired result. □

Theorem 3. *If $n=1,2,3,\ldots$ and $m=1,2,3,\ldots$, then*

$$H_n=2m\int_0^1 \frac{x^{m-1}(1-x^{nm})\big(1-x^{(n+1)m}\big)}{1-x^{2m}}dx,$$

where H_n denotes the harmonic number.

Proof. We use the induction. Let $m=1$, thereafter,

$$\begin{aligned}H_n=&2\cdot 1\cdot\int_0^1\frac{1}{x}\sum_{k=1}^{n}(x^k-x^{2k})dx\\ =&2\int_0^1\frac{1}{x}\cdot\frac{x(1-x^n)(1-x^{n+1})}{1-x^2}dx\\ =&2\int_0^1 \frac{(1-x^n)(1-x^{n+1})}{1-x^2}dx,\end{aligned}$$

as shown in previous theorem. Assume that the formula is valid for $m=j$. Therefore, for $m=j+1$, we have

$$\begin{aligned}H_n=&2(j+1)\int_0^1\frac{1}{x}\sum_{k=1}^{n}\big(x^{k(j+1)}-x^{2k(j+1)}\big)dx\\ =&2(j+1)\int_0^1\frac{1}{x}\cdot\frac{x^{j+1}\big(1-x^{n(j+1)}\big)\big(1-x^{(n+1)(j+1)}\big)}{1-x^{2(j+1)}}dx\\ =&2(j+1)\int_0^1\frac{x^{j}\big(1-x^{n(j+1)}\big)\big(1-x^{(n+1)(j+1)}\big)}{1-x^{2(j+1)}}dx,\end{aligned}$$

consequently, setting $j+1\to m$, we obtain

$$H_n = 2m\int_0^1 \frac{x^{m-1}(1-x^{nm})\big(1-x^{(n+1)m}\big)}{1-x^{2m}}dx,$$

which is the desired result. □

3. Integral Representation for Digamma Function

Theorem 4. *If* $\mathrm{Re}(s) > -1$, *then*

$$\psi(s+1) = -\gamma - 2\int_0^1 \frac{(1-x)^s-(1-x^2)^s}{x}dx, \tag{8}$$

where $\psi(s)$ *denotes the digamma function and* γ *denotes the Euler-Mascheroni constant.*

Proof. We consider the Newton series [2] for the digamma function

$$\psi(s+1) = -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\binom{s}{k}.$$

Substitute

$$\begin{aligned}\psi(s+1) &= -\gamma - \sum_{k=1}^{\infty} (-1)^k 2\int_0^1 x^{k-1}(1-x^k)dx\binom{s}{k} \\ &= -\gamma - 2\int_0^1 \sum_{k=1}^{\infty} (-1)^k x^{k-1}(1-x^k)\binom{s}{k}dx \\ &= -\gamma - 2\int_0^1 \frac{(1-x)^s-(1-x^2)^s}{x}dx,\end{aligned}$$

which is the desired result. □

Remark 5. With the aid of Theorem 4, we evaluate some special values for digamma functions:

$$\psi\left(\frac{5}{2}\right) = -\gamma + \frac{8}{3} - 2\ln 2,$$

$$\psi\left(\frac{7}{3}\right) = -\gamma + \frac{15}{4} - \frac{\pi\sqrt{3}}{6} - \frac{3\ln 3}{2},$$

$$\psi\left(\frac{9}{4}\right) = -\gamma + \frac{24}{5} - \frac{\pi}{2} - 3\ln 2,$$

$$\psi\left(\frac{11}{5}\right) = -\gamma + \frac{35}{6} - \frac{\pi}{2}\sqrt{1+\frac{2}{\sqrt{5}}} - \frac{\sqrt{5}}{4}\ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right) - \frac{15\ln 5}{12},$$

$$\psi\left(\frac{13}{6}\right) = -\gamma + \frac{48}{7} - \frac{\pi\sqrt{3}}{2} - 2\ln 2 - \frac{3\ln 3}{2}.$$

References

[1] en.wikipedia.org/wiki/Harmonic_number, available in March 6, 2015.

[2] en.wikipedia.org/wiki/Digamma_function, available in March 6, 2015.