A ONE-DIMENSIONAL MODEL OF IRREVERSIBILITY

M. de Abreu Faro and Rodrigo de Abreu
Centro de Electrodinâmica, Instituto Superior Tècnico
Lisboa Portugal

Abstract

The main purpose of this paper is to show that it is possible to understand the origin of irreversibility with a simple one-dimensional model of a collisionless gas. We begin by studying a one-particle "gas" and then we generalise the results to an N-particle gas. The gas particles are enclosed in a cylinder and their movement is perpendicular to a frictionless piston. Only elastic collisions of the particles with the cylinder bottom and with the piston are considered.

In order to understand the origin of irreversibility we compare the solution obtained for the differential equation $f = \frac{dp}{dt}$, where the force f one the piston is due to the gravitacional field and to the particle collisions on the piston, with the solution obtained for the momentum conservation law. When the number of particle increases and are not in phase, both solution must agree most of the time in agreement with a statistical formulation. Therefore irreversibility exists in a model without friction between the piston and the cylinder wall and without heat flux between the gas and the exterior. And, although this is comonly suggested, it is not due to mathematical hypothesis like the use of a mean value for the force due to the particles collisions. In fact irreversibility has its origin in the interaction between the particles through the piston and then the statistical formulation agree with an exact and deterministic solution, most of the time and for most of the initial conditions, but not for all the time and all the initial conditions.

INTRODUCTION

One of the most interesting and enduring problems[1] in physics is the origin of irreversibility. One of the reasons for the permanence of this problem is the dificulty in conciliating Boltzmann solution, with the recurrence theorem of Poincaré. If the hypothesis of the molecular chaos or "Stosszahlansatz" were the irreversibility feature as some authors still affirmed nowadays[1], we would be faced with a real paradox, as Poincaré pointed out a long time ago. This is not the case[2]. The assumption of molecular chaos must be considered as a mathematical hypothesis[3] consistent with the tendency to the equilibrium state. With the molecular chaos hypothesis, for a finite system, we cannot find the real solution of the problem for all future because the real solution must contain an infinite number of Poincaré cycles.

Our model is a cylinder with a frictionless piston wich is sustained by a one-particle "gas". The gravitational acceleration is g, the particle mass m_2 and the piston mass m_1. The particle movement is perpendicular to the piston. The particle of mass m_2 can be divided in N particles each one with $\frac{m_2}{N}$ mass, this particles having also a movement perpendicular to the piston. This particles can collide simultaneously with the piston having an effect equal to the one-particle "gas" or can collide not simultaneously like a real gas (chaotically)[4].

By considering this one-dimensional ideal gas we can avoid the complexity due to collisions between particles and, simultaneously, have a general and exact solution method to describe the evolution of the system.

With this accurate and simple model we can explain the origin of irreversibility without needing, for that, to appeal to making mathematical hypotesis. For one-particle or N-particles defining a front surface parallel to the piston we have an equivalent effect in both situations. The equivalence is not, however, obvious, as we will see if the collisions with the piston are "chaotic", although the equilibrium pressure is the same in both situations. First we study the movement of the piston under the influence of a one-dimensional one-particle gas by using the momentum conservation law for collisions between the particle and the piston. Second, we solve the differential equation of movement of the piston by introducing the gas pressure.

Finally we compare both solutions and verify that the solution for the differential equation only can have, for most of the time, the same values as those obtained from the momentum conservation law if a large N particle number is considered. In fact it is expected that the Poincaré cycle depends on N.

In conclusion, we obtain an analytic and explicit method permitting us to compare the differential quasi-exact solution for large values of N with the exact solution for a one-particle gas. It is shown that the trajectory followed by the piston to the equilibrium point is due to the assimetric form of pressure on the moving piston. In fact the piston for a given equilibrium pressure of the gas has a pressure on it higger for a compression and lower for an expansion. This is what equation 14 reveal.

ONE-DIMENSIONAL ANALYSIS

FOR A ONE-PARTICLE GAS

Consider the System represented in Fig. 1.

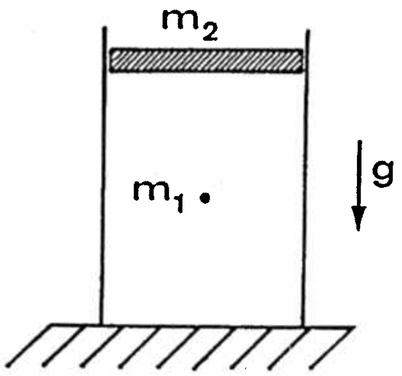

Fig.1

Fig. 1 - A Cylinder with a frictionless piston which is sustained by a one- particle "gas". The gravitational acceleration is g, the particle mass m_2 and the piston mass m_1. The particle movement is perpendicular to the piston.

The particle with mass m_2 collides with the piston which has a mass m_1. Both the piston and the particle move along the same direction.

After an elastic collision with the piston the particle rebounds to the cylinder bottom where it has another collision. The particle returns to the piston and the process is repeated. The piston moves under the particle and gravitational influences.

The velocities of the particle and the piston are considered small to avoid relativistic effects. This being so, we can use the well-known kinematics under the influence of a constant gravitational field. We obtain exact expressions for the piston and particle movements.

Collisions between the particle and the piston occur at times t_c.

At time $t=t_c - o$ we consider the following conditions.

Piston height	h	
Piston velocity Component	v'	(1a)
Particle velocity component	u	

At times $t = t_c + o$, after a collision between the particle and the piston, the conditions are:

Piston height	h	
Piston velocity component	v'	(1b)
Particle velocity component	u'	

By applying the momentum and energy conservation laws the following equations are obtained:

$$u' = [2v + (A-1)\,u] \,/\, (1+A) \tag{2a}$$

$$v' = [2Au + (1-A)\,v] \,/\, (1+A) \tag{2b}$$

with $A = m_2/m_1$.

If $v/u \ll 1$ and $A < 1$, we verify that $u' < o$. Then the time τ between the successive particle collisions with the piston and with the bottom can be obtained from:

$$h = (-u')\,\tau + \frac{1}{2} g\,\tau^2 \tag{3a}$$

because the particle leaves the piston with a velocity $(-u')$ subjected to the acceleration g.

The "landing time" τ is therefore

$$\tau = 2\,h \,/\, \sqrt{u'^2 + 2gh} + (-u')$$

The particle velocity at the bottom is

$$v_0 = (-u') + g\tau \tag{4}$$

At time <u>t</u> after the first particle-piston collision the piston height is:

$$L = h + yt - \frac{1}{2} g\; t^2. \tag{5a}$$

and the particle position is

$$m = v_0\,(t - \tau) - \frac{1}{2} g\,(t - \tau)^2. \tag{5b}$$

A new collision instant c is obtained by imposing

$$m = L \tag{6a}$$

which yields

$$c = (h + v_0 \tau + \frac{1}{2} g \tau^2) / (v_0 + g \tau - v'). \qquad (6b)$$

The piston velocity just before a new collision is

$$v = v' - g c \qquad (7a)$$

and the particle speed

$$u = vo - g (c - \tau). \qquad (7b)$$

The piston height L at $t = c$ is

$$L (c) = h + v' c - \frac{1}{2} g c^2. \qquad (7c)$$

This value L (c) is equal to m obtained from equation (5b) for $t = c$.

In conclusion (7a), (7b) and (7c) define new initial conditions v, u and h.

In the appendix A we present the program that enables us to calculate the piston and particle movements.

Because the piston has an impulsive movement (see equation 5a) we consider its mean velocity between two consecutive collisions with the particle as

$$w = [L(c) - h]/c .$$

In Fig. 2 we present the piston position L(t), the mean velocity piston w(t) and the particle velocity u(t).

In our calculations we considered that the piston was initially at rest and at a 2 m distance from the cylinder bottom. We have taken

$A = m_2/m_1 = 10^{-4}$. and $u = 100\ ms^{-1}$.

We can easily interprete what happens:

The gravitational field acts on the piston and its speed increases with a negative velocity component ($L(t) < 0$). In the meantime the collision frequency increases, as expected, because of the decreasing

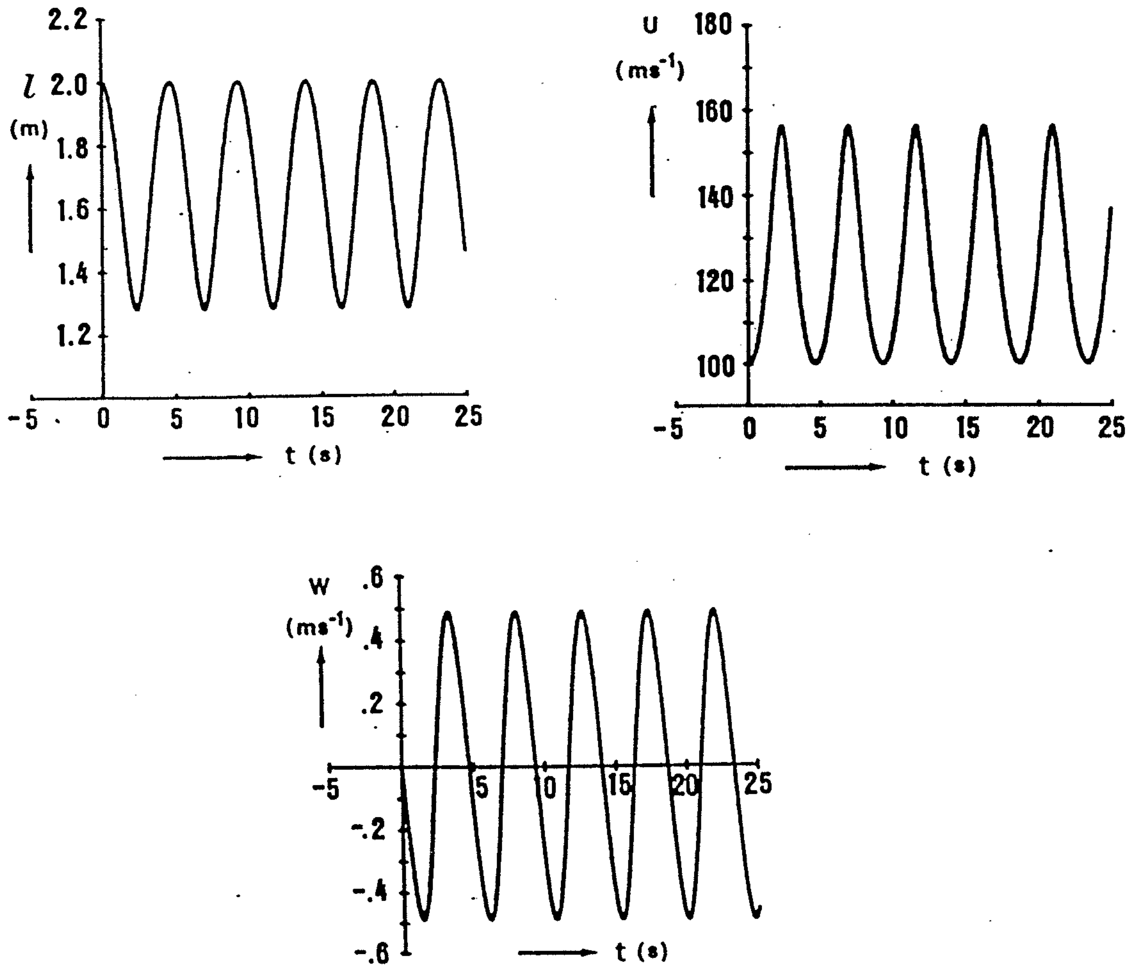

Fig.2

Fig. 2 - Piston position L(t), particle velocity component u(t) ($u(t) > 0$), and mean velocity w(t).

distance between the bottom and the piston and the increasing particle speed (as we can confirm from (2a) for $v < o$). First we have a negative piston acceleration and after the piston come to rest the movement is reversed with a positive acceleration due to the particle collisions. The particle speed decreases, as we can see at (2a) for $v > o$. The collision frequency also decreases ($L(t) > o$) and a new inversion occurs.

The piston position h and y and v return approximately to the initial values. If $A < 1$ but not too small compared to unity, we notice that all the values for L(t) lie on a curve that contains the initial value. As seen from fig.2 this L(t), u(t) and w(t) are periodic and continuous functions. This periodic functions have an important second harmonic component as it can be verified by Fourier Analysis. We verify this fact with a computer graphic plotting.

Now let us see if there are special initial conditions for which the mean velocity of the piston is zero. If.

$$w = [L(c) - h]/c = o$$

then $L(c) = h$ and (5a) enable us to obtain

$$h + yc - gc^2 / 2 = h$$

and therefore $gc = 2y$. By combining this result with (7a), we obtain

$$v = v' - gc = - v'.$$

For $L = h$ the particle velocity components is

$$u' = - u. \tag{8b}$$

If we consider (2a) and (2b) together with $v = -y$, then

$$v = -Au.$$

Obviously the "landing time" for the particle (3b) satisfies $g\tau = v' = -v = Au$ and for $u' = -u$ the initial conditions are the following:

$$v = -Au, \tag{9a}$$

$$h = \frac{Au^2}{g}\left(1 + \frac{A}{2}\right). \tag{9b}$$

In conclusion: for an initial particle velocity u and for $A = \frac{m_2}{m_1}$, (9a) and (9b) are the initial conditions for which we obtain a zero mean velocity $w = 0$. Then all the collisions particle-piston occur at the same position h and with the same velocities u and v. These conditions can then be considered as the equilibrium conditions of the system h_e, u_e and v_e. In this case the time between collisions is:

$$c = 2\tau = 2A\, u/g. \tag{9c}$$

When conditions are such that $A \ll 1$, then from (9a) and (9b) we have

$$v \simeq o, \tag{10a}$$

$$h = \frac{Au^2}{g}, \tag{10b}$$

which means that macroscopically the piston is at rest.

On the other hand we can write (10b) as:

$$m_1 g = 2m_2\, u\, \frac{1}{2h/u}. \tag{11}$$

This means that macroscopically the gravitational force balanced by the momentum variation $2m_2\, u$ during the time $2h/u$ or $c=2Au/g$ as we obtain from (9c).

The piston movement in phase space is represented by Fig.3.

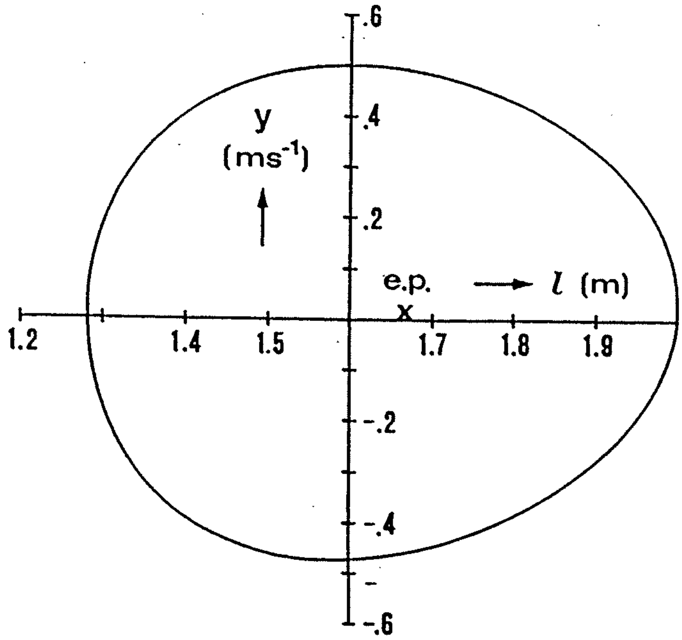

Fig.3

Fig. 3 - The piston movement in phase space.

For each cycle the energy has the following constant value

$$E_o = (m_1 + m_2)\, gh + m_2 \frac{u^2}{2} + m_1 \frac{v^2}{2}. \qquad (11a)$$

We can easily obtain for a given energy value Eo the rest state for the piston (w = o). By connecting (11a) with (9a) and (9b), there results

$$E_o = \frac{m_1}{2} \left[2(1 + A)g\, h_e + \frac{(1+A)g\ h_e}{1+ \frac{A}{2}} \right] \qquad (11b)$$

Equating (11a) to (11b) we obtain

$$g\, h_e = \frac{2(1+A)\ gh + Au^2 + v^2}{2(1+A) + (1+A)/(1+A/2)} \qquad (11c)$$

The equilibrium point $[h_e, w_e]$ belongs to the interior of the cycle and can only be attained if the initial conditions are the rest conditions . We have a periodic behaviour but we don't have an asymptotically stable behaviour. In fact, the system is maintained as a cycle but does not tend to the equilibrium point at the interior of the cycle.

II DIFFERENTIAL EQUATION SOLUTION

Designating now by x the piston heigh that we previously called L, the differential equation that describes the piston movement is:

$$\ddot{x} = a_c - g, \qquad (12)$$

where a_c is the mean acceleration due to the sucessive collisions of the one-particle gas with the piston.

The mean acceleration a_c is

$$a_c = \frac{1}{m_1}[2\, m_2\, (u - \dot{x})\ \gamma_c] \qquad (13a)$$

where $2\ m_2\ (u\ -\ \dot{x})$ is the particle momentum variation in an elastic collision with the piston and $(u - \dot{x})$ is the particle component velocity in the rest frame of the piston and γ_c is the number of collisions per unit time between the particle and the piston. Neglecting the displacement of piston between successive collisions, the time between successive collisions is $2\ x/(u-\dot{x})$ and therefore γ_c is given by $(u-\dot{x})/2x$.

We thus obtain

$$a_c = \frac{1}{m_1}\ 2\ m_2\ (u - \dot{x})\ \frac{\dfrac{1}{2x}}{\overline{u-\dot{x}}}$$

wich can still be written in the form

$$a_c = \frac{1}{m_1}\ \frac{2}{x}\ \frac{m_2\ u^2}{2}\ \left[\ 1 - \left(\frac{\dot{x}}{u}\right)\ \right]^2. \qquad (13b)$$

When conditions are such that $|\dot{x}| \ll |u|$, we can make the aproximation

$$a_c = \frac{1}{m_1} \frac{2}{x} \frac{m_2 u^2}{2} (1 - \frac{2\dot{x}}{u}), \text{ from wich} \tag{13c}$$

we may define an equivalent force on the piston

$$f_c = m_1 a_c = 2 \frac{E_k}{x} (1 - \frac{2\dot{x}}{u}). \tag{14}$$

The factor $(1 - \frac{2\dot{x}}{u})$ is a correction to the equilibrium pressure $2 \frac{E_k}{x}$. The pressure exerted on a piston at rest in the laboratory frame would simply be $2 \frac{E_k}{x}$ as seen from equation (13b) for $\dot{x} = 0$. The factor $(1 - 2 \frac{\dot{x}}{u})$ is a correction to the above pressure arising from the motion of the piston. The force expressed by equation (14) is assimetric in the sense that it depends on the sign of x. If this factor were absent the solution of equation (12) would be time reversible, while this is not the case when retaining the full expression (14).

Let us write the energy expressions for the initial state $[x_o, \dot{x}_o, u_o]$. and for a generic state $[x, \dot{x}, u]$.

In order to solve equation (12) assume that t=o is the collision instant between the particle and piston. Then, for t=o, the particle and the piston heights are the same.

By imposing energy conservation we obtain for all subsequent times

$$E_0 = (m_1 + m_2)\, g\, x_0 + m_2 \frac{u_0^2}{2} + m_1 \frac{\dot{x}_0^2}{2} = \qquad (15)$$

$$= (m_1 + m_2)\, g\, x + m_2 \frac{u^2}{2} + m_1 \frac{\dot{x}^2}{2}.$$

From this expression we are able to calculate the particle velocity. Solving eq. (15) for $E_k = m_2 \frac{u^2}{2}$ and inserting the result in eq. (14), equation (12) finally yields (16) where $A = m_2/m_1$.

$$x = \frac{1}{x}\left[\frac{2\,E_0}{m_1} - 2(1+A)\, gx - \dot{x}^2\right]\left(1 - \frac{2\dot{x}}{u}\right) - g. \qquad (16)$$

This equation has the form $\ddot{x} = f(x,\dot{x})$ which can be reduced to the autonomous system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = f(x,y). \qquad (17)$$

By using well known analysis methods we can state, using (17), that the system is an autonomous and asymptotically stable system with a singular point where $\dot{x} = o$ and $\ddot{x} = o$. This point is a focus, which means that in phase space the point $P(x, \dot{x})$ describes a spiral convergent to the equilibrium point $Q(x_e, \dot{x}_e = o)$. Physically this means that the solution to eq. (12), with f_c given by eq. (14) has a time irreversible behaviour. The equilibrium height x_e can be easily obtained from eq. (16c) setting $\dot{x} = \ddot{x} = o$

wich yelds (18a).

$$x_e = \frac{2E_0/m_1}{2(1+A)g + g} = \frac{2(1+A)\ g\ x_0 + A\ u_0^2 + \dot{x}_0^2}{2(1+A)\ g+g} \tag{18a}$$

We can also obtain the equilibrium particle speed u_e. From (16b) and (18a).

$$u_e = \sqrt{\frac{g x_e}{A}}. \tag{18b}$$

If we compare (18a) with the relation (11c), we verify that for $A \ll 1$ x_e is pratically coincident with the height h_e that we previously calculated from the collisional model discussed in section I.

Similarly u_e has a value close to that obtained from expression (10b). For example assuming $A = 10^{-4}$ we obtain for the equilibrium height the values 1.6657 using the model of section I and the value 1.6666 for the model of section II as shown in the appendices A and B.

The solution of eq. (17b) can be numerically obtained using the Runge-Kutta method and for a particular set of initial conditions is depicted in fig.4. The solution of the differential equation only agrees aproximately with the cycle defined by the collisional model in the initial cycle. The result is an oscillatory movement which is not a sinusoidal one. This solution is indeed a spiral convergent to an equilibrium point. Only the first cycle of this spiral aproximately coincides with the cycle obtained from the collisional model.

In order to estimate the oscillation period we can assume $(1 - \frac{2\dot{x}}{u}) \simeq 1$ for the initial cycles and rewrite eq. (16c) as

$$\ddot{x} = \frac{1}{x}[(2\,(1 + A)\,g + g)\,(\bar{x}_e - x) - \dot{x}^2] - g$$

where use was made of eq.(18a).

Conclusions and Discussion

With the two solutions considered, i.e. the momentum conservation solution (m.c.s.) and the differential equation solution, (d.e.s.) we can understand the origin of irreversibility.

Obviously the rigorous solution method obtained with the momentum and energy conservation principles like any other numerical solution has numerical calculation limitations like any other and therefore with the rigorous solution method we cannot have a rigorous solution. Such calculation limitations are obviously related to software and hardware limitations. In fact when the number of particles increase the software dimension also increases and the calculation time is larger. In addition to this limitation we can only process a finite number of digits. This being so, this rigorous solution method is obviously limited by the numerical calculation capacity. Therefore it is impossible to know whether we have or not a rigorous solution if the solution method is sensitive to the number of digits processed. Here we do not have this problem as we compare the rigorous solution method, although with a finite number of digits, with the d.e.s.

For N particles defining a parallel front to the piston the rigorous solution is different from the d.e.s. For a N-particles system, the d.e.s. must aproximately agree with the momentum conservation solution for most of the initial conditions and for most of the time otherwise the differential equation would be meaningless. However, this solution converges to the equilibrium point, therefore it does not contain the Poincaré cycles that a rigorous solution must exhibit. A rigorous mechanical solution (i.e the rigorous momentum conservation solution without numerical limitation) would also be expected to agree with the d.e.s. for most of the time and initial conditions. This can be easily understood considering the piston divided into N pistons each one with an associated particle (note that we are considering the particles movement perpendicular to the piston). If the pistons are independent we have for each one a solution identical with that represented in fig 3. although only in the case of N particles defining a parallel front to the piston (particles in phase) do we have equal solutions in time. If the particles are not in phase and because the pistons are connected to each other (we have in fact only one piston) the movement in phase space of each particle and each piston when the pistons are independent, is perturbed due to the connection between the pistons («Leslie» no longer knows what piton he is going to find[5-7]). Then the piston (and the particles) can explore others region positions in phase space with coordinates values similar to those corresponding to the d.e.s.

This simple model makes it possible to understand the origin of irreversibility with a "mechanical" treatment. In fact the tendency to the equilibrium point is not related to friction between the piston and the cylinder walls[5] (because we have assumed that the piston movement is frictionless) or to heat exchange between the gas and the exterior, but is rather a result of the interaction between the particles through the piston. This interaction leads the piston to the equilibrium point although the piston always returns to the neighbourhood of the initial point an infinite number of times. Then we must conclude that the differential equation solution must agree with the rigorous solution for most of the time and for most of the initial conditions but not for whole time and all initial conditions. When a statistical formulation is made, whatever it is, we have an agreement with the equilibrium tendency, but this agreement does not warrant to thinking that irreversibility is originated from mathematical hypotheses like $\vec{f} = \frac{d\vec{p}}{dt}$ with the mean value assumed in our present treatment.

The rigorous solution must also describe the tendency to equilibrium, but goes beyond that. In fact only conceptually, as a limit, can we think of a point in phase space and think of a well defined trajectory.

But as a limit this classical conceptualization of a trajectory can be conceived and therefore we can connect the differential equation with the rigorous solution. In this limit the irreversibility emerge associated with the interaction between the particles if the particles are not in phase. This interaction mechanism has obviously nothing to do, with numerical rigour and is presented independently of the number of digits processed.
Although the trajectory calculated by the rigorous solution is different from the d.e.s. for the whole future, they must agree for most of the time and this explains the physical meaning and success of the d.e.s. method. Although completely correlated, only once in a while do the particles and the piston exibit this correlation macroscopically. This being so, solutions like $\vec{f} = \frac{d\vec{p}}{dt}$ or like the Liouville solution must be considered statistical mechanical solutions. The mechanical solution is the momentum conservation solution and the physical mechanism that leads the piston trajectory to agree with the differential equation trajectory is the interaction between the particles trough the piston. This interaction is the

origin of irreversibility. Collisions are the mechanism that constructs the trajectory in phase space and the equilibrium point is one point of that trajectory for most of the initial conditions and for mos t of the time.

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10 ! APPENDIX B
20 !
30 ! Program describing piston
   movement (differential
   equation (16c)).
60 !
90 PRINTER IS 705,80
100 PLOTTER IS 705
110 PRINT "IN;SP1;IP 400,600,400
    0,4000;"
120 P=.1 @ L=2 @ V=0
130 FOR N=1 TO 4000
140 U=L
150 Y=V
160 K=U
170 H=Y
180 GOSUB 430
190 A=W
200 K=U+P*Y/2
210 H=Y+A/2
220 GOSUB 430
230 B=W
240 K=U+P*Y/2+P*A/4
250 L=Y+B/2
260 GOSUB 430
270 C=W
280 K=U+P*Y+P*B/2
290 H=Y+C
300 GOSUB 430
310 D=W
320 L=U+P*Y+P*(A+B+C)/6
330 V=Y+(A+2*B+2*C+D)/6
340 SCALE 1.2,2,-.6,.6
350 XAXIS 0,.1
360 YAXIS 1.6,.1
370 PLOT L,V
380 NEXT N
390 END
400 !
410 ! ------ SUBROTINE ------
420 !
430 W=P*((5-2*K-H*H)*(1-H/SQR((5
    -2*K-H*H)*10000))/K-1)
440 RETURN
450 !
460 ! Equilibrium point :
470 ! L = 1.66668
480 !
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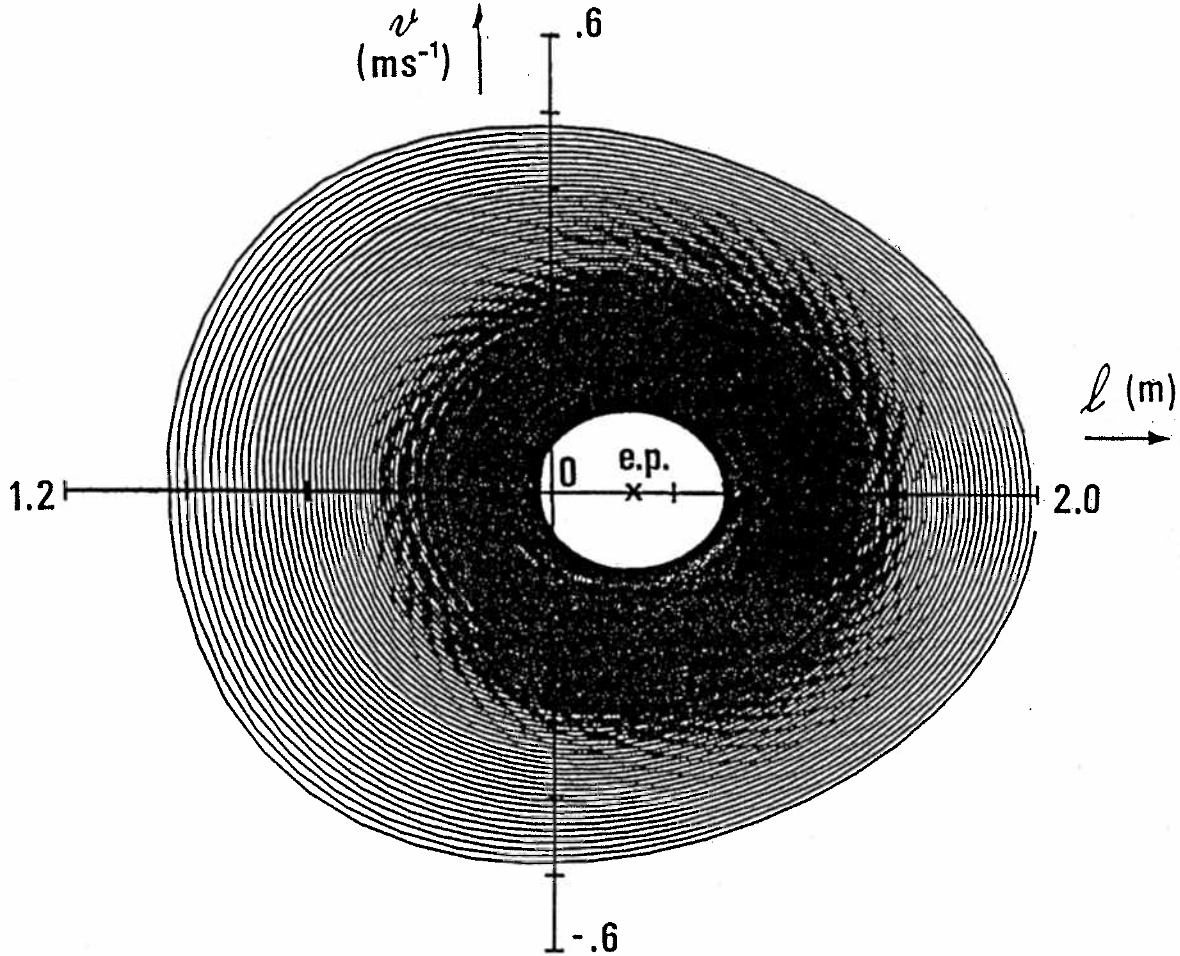

L - Piston height. In the text we have made l=x.
V - Piston velocity component.

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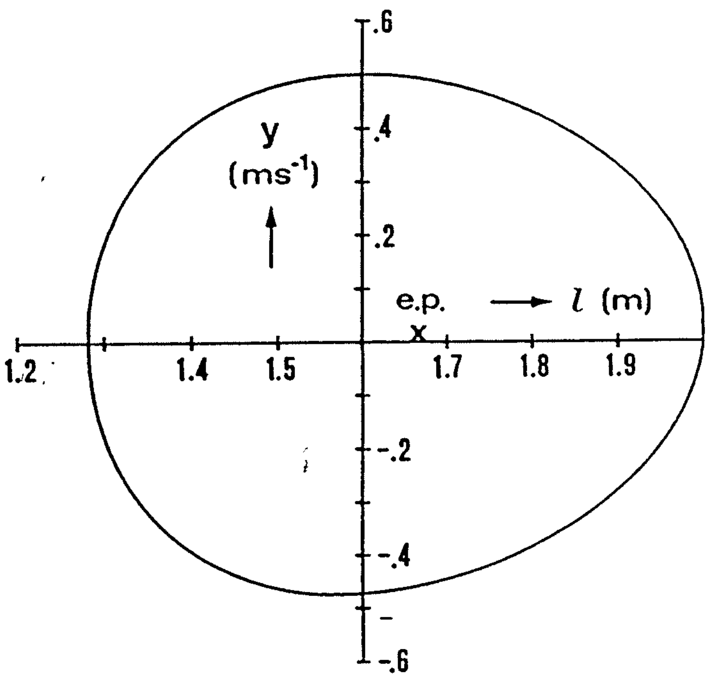
 10 ! ---- APPENDIX A ----
 20 !
 30 ! Program describing piston
 40 ! movement ( momentum -
      conservation law).
 50 !
 60 PRINTER IS 705,80
 70 PLOTTER IS 705
 80 PRINT "IN;SP1;IP 400,600,400
    0,4000;"
 90 H=2 @ U=100 @ V=0
100 A=.0001
110 G=1
120 T=0
130 FOR N=1 TO 300
140 X=(2*V+(A-1)*U)/(1+A)
150 Y=(2*A*U+(1-A)*V)/(1+A)
160 Z=2*H/(SQR(X^2+2*G*H)+(-X))
170 V0=-X+G*Z
180 C=(H+V0*Z+.5*G*Z^2)/(V0+G*Z-
    Y)
190 U=V0-G*(C-Z)
200 V=Y-G*C
210 L=H+Y*C-C^2/2
220 W=(L-H)/C
230 M=V0*(C-Z)-.5*G*(C-Z)^2
240 T=T+C
250 H=L
260 SCALE 1.2,2,-.6,.6
270 XAXIS 0,.1
280 YAXIS 1.6,.1
290 PLOT L,Y
300 NEXT N
310 END
315 !
320 ! Equilibrium point :
330 ! L = 1.66571733
340 !
```



H - Piston height at a collision time (tc).

L - Piston height at another collision instant (after the interval of time between collisions C) (t=tc+c).

W - Piston velocity component at time tc-0.

Y - Piston velocity component at time tc+0.

APPENDIX C

I. Equations (2a) and (2b) can be obtained in the following way.

The elastic collision between the mass m_1 piston and the mass m_2 particle satisfies the following equations where (v,y) and (u,x) are the velocity components for the piston (m_1) and for the particle (m_2), at the moments (t_c - 0) and (t_c + 0) (t_c is an instant of collision)

$$m_1 v + m_2 u = m_1 y + m_2 x$$

$$m_1 v^2 + m_2 u^2 = m_1 y^2 + m_2 x^2 .$$

We can write from these equations the following

$$x - y = v - u$$

$$m_2 x + m_1 y = m_1 v + m_2 u .$$

Then we obtain (2 a)

$$x = \frac{\begin{vmatrix} v-u & -1 \\ m_1 v + m_2 u & m_1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ m_2 & m_1 \end{vmatrix}}$$

or

$$x = \frac{2v + u\ (A-1)}{1+A} \qquad (2a)$$

with

$$A = \frac{m_2}{m_1} .$$

In a similar way we obtain eq. (2b)

$$y = \left[2Au + (1-A)\ v\right]/(1+A). \qquad (2b)$$

II. Equation (16c) is obtained in the following way:

The equation (13a) can be written under the form

$$a_c = \frac{1}{m_1} \frac{2}{x} \frac{m_2 u^2}{2} \left[1 - (\frac{\dot{x}}{u})\right]^2 \qquad (13a')$$

If $\left|\frac{\dot{x}}{u}\right| << 1$, we have

$$\left[1 - (\frac{\dot{x}}{u})\right]^2 \simeq (1 - \frac{2\dot{x}}{u}) .$$

Then (13a') can be written

$$a_c = \frac{1}{m_1} \frac{2}{x} \frac{m_2 u^2}{2} (1 - \frac{2\dot{x}}{u}) \qquad (13b)$$

or

$$a_c = \frac{A}{x} u^2 (1 - \frac{2\dot{x}}{u}) . \qquad (13b')$$

Now, from the energy conservation equation, we can write u^2 as a function of $\dot{x}$ (piston velocity) and x (piston position).

In fact

$$E_o = (m_1 + m_2)\ g\ x_o + m_2\ u_o^2/2 + m_1 \frac{\dot{x}_o^{\,2}}{2} =$$

$$= (m_1 + m_2)\ g\ x + m_2 \frac{u^2}{2} + m_1 \frac{\dot{x}^2}{2} .$$

Then

$$u^2 = \frac{1}{A} \left[2(1 + A)\ g\ x_o + A\ u_o^{\,2} + \dot{x}_o^{\,2} - \right.$$

$$\left. - 2\ (1 + A)\ g\ x - \dot{x}^2 \right]$$

Or

$$Au^2 = \left[2 \frac{E_o}{m_1} - 2\ (1 + A) g\ x - \dot{x}^2\right] \qquad (16b)$$

But

$$\ddot{x} = a_c - g .$$

Then we obtain (16c) substituting at 13b' equation (16b).

$$\ddot{x} = a_c - g \mp \frac{1}{x} \left[2 \frac{E_o}{m_1} - 2(1 + A)\ g\ x - \dot{x}^2\right] \times$$

$$\times \left(1 - \frac{2\dot{x}}{u} \right) - g . \qquad (16c)$$