From Three Body to Many Body Motion, Classical

Anamitra Palit

palit.anamitra@gmail.com

Abstract

Three body motion or many body motion in relation to gravitation and other types of interaction has always occupied an enviable position in the history of physics in view of its complexity especially in comparison with the simpler two body motion. The challenge offered by these problems have been a continuous source of inspiration for mathematicians and physicists to work out solutions. This article is a modest endeavor to analyze and solve the problem as far possible .A method for solving the four body and in general the many body problem has been discussed. Certain peculiarities of the two body motion have been pointed out highlighting them in connection with three body motion and with many body motion.

Introduction

The two body problem(1) was formulated by Johannes Kepler in 1609 and solved by Sir Isaac Newton in the year 1687.Ever since the three body problem(2) stood as a formidable challenge motivating physicists and mathematicians to take on it by various methods(3): to derive solutions to the problem under suitable restrictions. In this article the method of differential equations have been applied for analysis and solving the three body and the four body problem in the most general situation. A simple numerical technique for the many body problem has been suggested. Certain subtle aspects of the two body motion have been pointed out together with nature's ingenious method of handling them in the context of the many body scenario that exists in the universe in an obvious manner.

Basic Calculations

We have the gravitating masses m_A, m_B and m_C at the corners of the triangle ABC. Their locations are denoted by the position vectors $\vec{r}_A, \vec{r}_B$ and $\vec{r}_C$.

Equations of motion as reckoned from an inertial frame (4) of reference:

$$m_A \frac{d^2\vec{r}_A}{dt^2} = G\frac{m_A m_B}{{r_{AB}}^3}\vec{r}_{AB} + G\frac{m_A m_C}{{r_{AC}}^3}\vec{r}_{CB} \quad (1.1)$$

$$m_B \frac{d^2\vec{r}_B}{dt^2} = G\frac{m_B m_C}{{r_{BC}}^3}\vec{r}_{BC} + G\frac{m_B m_A}{{r_{BA}}^3}\vec{r}_{BA} \quad (1.2)$$

$$m_C \frac{d^2\vec{r}_C}{dt^2} = G\frac{m_C m_A}{{r_{CA}}^3}\vec{r}_{CA} + G\frac{m_C m_B}{{r_{CB}}^3}\vec{r}_{CB} \quad (1.3)$$

In the above or elsewhere in the article

$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i;\ r_{ij} = |\vec{r}_{ij}| = |\vec{r}_j - \vec{r}_i|$

Or,

$\frac{d^2\vec{r}_A}{dt^2} = G\frac{m_B}{{r_{AB}}^3}\vec{r}_{AB} + G\frac{m_C}{{r_{AC}}^3}\vec{r}_{CB}$ (2.1)

$\frac{d^2\vec{r}_B}{dt^2} = G\frac{m_C}{{r_{BC}}^3}\vec{r}_{BC} + G\frac{m_A}{{r_{BA}}^3}\vec{r}_{BA}$ (2.2)

$\frac{d^2\vec{r}_C}{dt^2} = G\frac{m_A}{{r_{CA}}^3}\vec{r}_{CA} + G\frac{m_B}{{r_{CB}}^3}\vec{r}_{CB}$ (2.3)

Or ,

$$\frac{d^2(\vec{r}_A - \vec{r}_B)}{dt^2} = \frac{G(m_A+m_B)}{{r_{AB}}^3}\vec{r}_{AB} - \frac{Gm_C}{{r_{BC}}^3}\vec{r}_{BC} - \frac{Gm_C}{{r_{AC}}^3}\vec{r}_{CA}$$

$$\frac{d^2(\vec{r}_B - \vec{r}_C)}{dt^2} = \frac{G(m_B+m_C)}{{r_{BC}}^3}\vec{r}_{BC} - \frac{Gm_A}{{r_{CA}}^3}\vec{r}_{CA} - \frac{Gm_A}{{r_{AB}}^3}\vec{r}_{AB}$$

$$\frac{d^2(\vec{r}_C - \vec{r}_A)}{dt^2} = \frac{G(m_C+m_A)}{{r_{CA}}^3}\vec{r}_{CA} - \frac{Gm_B}{{r_{AB}}^3}\vec{r}_{AB} - \frac{Gm_B}{{r_{BC}}^3}\vec{r}_{BC}$$

Or

$-\frac{d^2\vec{r}_{AB}}{dt^2} = \frac{G(m_A+m_B)}{{r_{AB}}^3}\vec{r}_{AB} - \frac{Gm_C}{{r_{BC}}^3}\vec{r}_{BC} - \frac{Gm_C}{{r_{AC}}^3}\vec{r}_{CA}$ (3.1)

$-\frac{d^2\vec{r}_{BC}}{dt^2} = \frac{G(m_B+m_C)}{{r_{BC}}^3}\vec{r}_{BC} - \frac{Gm_A}{{r_{CA}}^3}\vec{r}_{CA} - \frac{Gm_A}{{r_{AB}}^3}\vec{r}_{AB}$ (3.2)

$-\frac{d^2\vec{r}_{CA}}{dt^2} = \frac{G(m_C+m_A)}{{r_{CA}}^3}\vec{r}_{CA} - \frac{Gm_B}{{r_{AB}}^3}\vec{r}_{AB} - \frac{Gm_B}{{r_{BC}}^3}\vec{r}_{BC}$ (3.3)

In the above we have two independent variables and two independent equations [since $\vec{r}_{AB} + \vec{r}_{BC} + \vec{r}_{CA} = 0$; It is also important to note that in the above equations from 3,1 to 3,3 there is no explicit involvement of $\vec{r}_A, \vec{r}_B$or $\vec{r}_C$. Rather we are concerned with the vectors

$\vec{r}_{AB}$, $\vec{r}_{BC}$ and $\vec{r}_{CA}$.Again adding (3.1) and (3.3) we obtain (3.3). Thus we have two vector unknowns and two independent vector equations. That boils down to six scalar unknowns and six scalar equations.

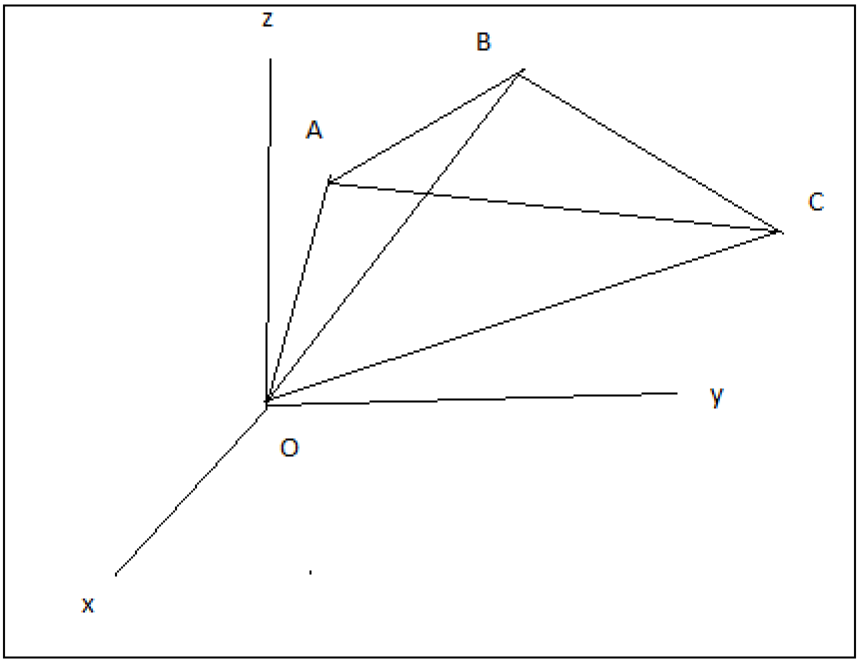

Figure I

[In the above figure OA=$\vec{r}_A$;OB=$\vec{r}_B$; OC=$\vec{r}_C$

The triangle ABC goes on rotating tumbling, changing shape and size as the motion proceeds]

Or,

$$\frac{d^2\vec{r}_{AB}}{dt^2} = -G\frac{(m_A + m_B + m_C)}{{r_{AB}}^3}\vec{r}_{AB} +$$

$$Gm_C\left(\frac{\vec{r}_{AB}}{{r_{AC}}^3} + \frac{\vec{r}_{BC}}{{r_{BC}}^3} + \frac{\vec{r}_{CA}}{{r_{CA}}^3}\right) \quad (4.1)$$

$$\frac{d^2\vec{r}_{BC}}{dt^2} = -G\frac{(m_A + m_B + m_C)}{{r_{BC}}^3}\vec{r}_{BC} +$$

$$Gm_A\left(\frac{\vec{r}_{AB}}{{r_{AB}}^3} + \frac{\vec{r}_{BC}}{{r_{BC}}^3} + \frac{\vec{r}_{CA}}{{r_{CA}}^3}\right) \quad (4.2)$$

$$\frac{d^2\vec{r}_{CA}}{dt^2} = -G\frac{(m_A + m_B + m_C)}{{r_{AB}}^3}\vec{r}_{AB} +$$

$$Gm_B\left(\frac{\vec{r}_{AB}}{{r_{AC}}^3} + \frac{\vec{r}_{BC}}{{r_{BC}}^3} + \frac{\vec{r}_{CA}}{{r_{CA}}^3}\right) \quad (4.3)$$

We write :

$$M = m_A + m_B + m_C$$

And

$\vec{X} = \frac{\vec{r}_{AB}}{{r_{AB}}^3} + \frac{\vec{r}_{BC}}{{r_{BC}}^3} + \frac{\vec{r}_{CA}}{{r_{CA}}^3}$

Our equations are

$\frac{d^2\vec{r}_{AB}}{dt^2} = -G\frac{M}{{r_{AB}}^3}\vec{r}_{AB} + Gm_C\vec{X}$ (5.1)

$\frac{d^2\vec{r}_{BC}}{dt^2} = -G\frac{M}{{r_{BC}}^3}\vec{r}_{BC} + Gm_A\vec{X}$ (5.2)

$\frac{d^2\vec{r}_{CA}}{dt^2} = -G\frac{M}{{r_{CA}}^3}\vec{r}_{CA} + Gm_B\vec{X}$ (5.3)

We solve for $\vec{K}$ in the following equation:

$$\frac{\vec{r}_{AB} - \vec{K}}{|\vec{r}_{AB} - \vec{K}^3} + \frac{\vec{r}_{BC} - \vec{K}}{|\vec{r}_{BC} - \vec{K}|^3} + \frac{\vec{r}_{CA} - \vec{K}}{|\vec{r}_{CA} - \overrightarrow{K}|^3} = 0$$

We transfer the origin to the tip of the vector $\vec{K}$

Now our equations are:

$\frac{d^2\vec{R}_{AB}}{dt^2} = -G\frac{M}{{R_{AB}}^3}\vec{R}_{AB}$ (6.1)

$\frac{d^2\vec{R}_{BC}}{dt^2} = -G\frac{M}{{R_{BC}}^3}\vec{R}_{BC}$ (6.2)

$\frac{d^2\vec{R}_{CA}}{dt^2} = -G\frac{M}{{R_{CA}}^3}\vec{R}_{CA}$ (6.3)

Where,

$$\vec{R}_{ij} = \vec{r}_{ij} - \vec{K}$$

Equations (10),(11) and (12) represent three two body motions: $\vec{f}(t,x,y,z)$ will be different in (10),(11) and (12) due to initial conditions

Solutions have to be expressed as:

$\vec{R}_{ij} = \vec{r}_{ij} - \vec{K} = \vec{f}(t,x,y,z)$(7)

Or,

$\vec{r}_{AB} - \vec{K} = \overrightarrow{f_1}(t,x,y,z)$ (8.1)

$\vec{r}_{BC} - \vec{K} = \overrightarrow{f_2}(t,x,y,z)$ (8.2)

$\vec{r}_{CA} - \vec{K} = \overrightarrow{f_3}(t,x,y,z)$ (8.3)

With given choice of $\vec{K}$ we have $\vec{X} = 0$. So as an alternative procedure we may transfer the origin to the tip of $\vec{X}$ taking care of the aspect of dimensions.

Let us take equation (5.1) by way of example

$$\frac{d^2\vec{r}_{AB}}{dt^2} = -G\frac{M}{{r_{AB}}^3}\vec{r}_{AB} + Gm_C\vec{X}$$

We multiply both sides of the above by a constant D which has the dimension of length/acceleration

$$D\frac{d^2\vec{r}_{AB}}{dt^2} = -GD\frac{M}{{r_{AB}}^3}\vec{r}_{AB} + Gm_C D\vec{X}$$

$$D\frac{d^2\vec{r}_{AB}}{dt^2} = -GD\frac{M}{{r_{AB}}^3}\vec{r}_{AB} + \vec{J}$$

Where,

$$\vec{J} = Gm_C D\vec{X}$$

$\vec{J}$ has the dimension of length. We may now transfer the origin to the tip of $\vec{J}$ to obtain equations of the type (6.1) (6.2) and (6.3).

A Special Formula for Three Body Motion

Referring to Figure I, the relative acceleration of mass m_A along the directed line segment $\overrightarrow{AB}$, that is, in the direction of $\vec{r}_{AB} = \vec{r}_B - \vec{r}_A$

$\vec{A}_{r:AB} = G\frac{m_B}{{r_{AB}}^3}\vec{r}_{AB} + G\frac{m_A}{{r_{AB}}^3}\vec{r}_{AB} - G\frac{m_C}{{r_{CA}}^3}\vec{r}_{CA}Cos(BAC) - G\frac{m_C}{{r_{BC}}^3}\vec{r}_{BC}Cos(ABC)$ (9)

The first two terms on the right side describe the relative acceleration between m_A and m_B due to mutual attraction between them.

To understand the first two terms let consider two masses m_A and m_B in absence of the third.

Acceleration of m_A with respect to an inertial frame:$G\frac{m_B}{{r_{AB}}^3}\vec{r}_{AB}$

Acceleration of m_B with respect to the same frame:$G\frac{m_A}{{r_{AB}}^3}\vec{r}_{BA}$

Relative acceleration of m_A with respect to m_B with respect to the same frame:$G\frac{m_A}{{r_{AB}}^3}\vec{r}_{BA}$

Relative acceleration of m_A with respect tom_B with respect to the same frame:$G\frac{m_A}{{r_{AB}}^3}\vec{r}_{BA}$

Relative acceleration of m_A with respect to m_B due to mutual interaction between them:

$G\frac{m_B}{{r_{AB}}^3}\vec{r}_{AB} - G\frac{m_A}{{r_{AB}}^3}\vec{r}_{BA} = G\frac{m_B}{{r_{AB}}^3}\vec{r}_{AB} + G\frac{m_A}{{r_{AB}}^3}\vec{r}_{AB}$

In the presence of a third body the mutual attraction /interaction between m_A and m_B will not be affected.

The third mass will cast its own the accelerations on m_A and m_B with respect to the reference frame[inertial] being considered. That will affect the relative acceleration between m_A and m_B without affecting physically their mutual interaction

The third term in relation (9) modifies the acceleration of m_A along AB due to the force exerted by m_C on m_A. The fourth term modifies the acceleration of m_B along BA due to the force exerted by m_C on m_B.

Similarly, acceleration of mass m_B along the direction $\vec{r}_{BC} = \vec{r}_C - \vec{r}_B$ is given by:

$\vec{A}_{r:BC} = G\frac{m_C}{{r_{BC}}^3}\vec{r}_{BC} + G\frac{m_B}{{r_{BC}}^3}\vec{r}_{BC} - G\frac{m_A}{{r_{AB}}^3}\vec{r}_{AB}Cos(ABC) - G\frac{m_A}{{r_{CA}}^3}\vec{r}_{CA}Cos(BCA)$ (10)

Acceleration of mass m_C along the direction $\vec{r}_{CA} = \vec{r}_A - \vec{r}_C$ is given by:

$\vec{A}_{r:CA} = G\frac{m_A}{{r_{CA}}^3}\vec{r}_{CA} + G\frac{m_C}{{r_{CA}}^3}\vec{r}_{CA} - G\frac{m_B}{{r_{BC}}^3}\vec{r}_{BC}Cos(BCA) - G\frac{m_B}{{r_{AB}}^3}\vec{r}_{AB}Cos(BCA)$ (11)

But

$$\frac{d^2\vec{r}_{AB}}{dt^2} + \frac{d^2\vec{r}_{BC}}{dt^2} + \frac{d^2\vec{r}_{CA}}{dt^2} = 0$$

[since $\vec{r}_{AB} + \vec{r}_{BC} + \vec{r}_{CA}$=0]

The radial and cross radial components of $\frac{d^2\vec{r}_{AB}}{dt^2} + \frac{d^2\vec{r}_{BC}}{dt^2} + \frac{d^2\vec{r}_{CA}}{dt^2}$ should be zero individually: That Implies

$$\vec{A}_{r:AB} + \vec{A}_{r:BC} + \vec{A}_{r:CA} = 0$$

[NB: $\frac{d^2\vec{r}}{dt^2}$ has both radial and cross radial components: $\vec{r}_{AB}, \vec{r}_{BC}$ and $\vec{r}_{CA}$ lie on the same plane , the mentioned vectors being the sides of the triangle ABC. Their directions are not skewed against each other. That provides us with a distinct advantage.The accelerations $\frac{d^2\vec{r}_{AB}}{dt^2}, \frac{d^2\vec{r}_{BC}}{dt^2}$ and $\frac{d^2\vec{r}_{CA}}{dt^2}$ do not have compnents normal to the plane of the triangle due to the strong form of Newton's Third Law]

Therefore,

$$G\frac{m_A + m_B}{{r_{AB}}^3}\vec{r}_{AB} + G\frac{m_B + m_C}{{r_{BC}}^3}\vec{r}_{BC} + G\frac{m_C + m_A}{{r_{CA}}^3}\vec{r}_{CA} =$$

$$G\frac{m_C}{{r_{CA}}^3}\vec{r}_{CA}Cos(BAC) + G\frac{m_C}{{r_{BC}}^3}\vec{r}_{BC}Cos(ABC) + +G\frac{m_A}{{r_{AB}}^3}\vec{r}_{AB}Cos(ABC) + G\frac{m_A}{{r_{CA}}^3}\vec{r}_{CA}Cos(BCA) +$$

$$+G\frac{m_B}{{r_{BC}}^3}\vec{r}_{BC}Cos(BCA) + G\frac{m_B}{{r_{AB}}^3}\vec{r}_{AB}Cos(BAC)$$

Or,

$\frac{m_A}{r_{AB}{}^3}\vec{r}_{AB}(1 - CosB) + \frac{m_B}{r_{AB}{}^3}\vec{r}_{AB}(1 - CosA) + \frac{m_B}{r_{BC}{}^3}\vec{r}_{BC}(1 - CosC) + \frac{m_C}{r_{BC}{}^3}\vec{r}_{BC}(1 - CosB) + \frac{m_C}{r_{CA}{}^3}\vec{r}_{CA}(1 - CosA) + \frac{m_A}{r_{CA}{}^3}\vec{r}_{CA}(1 - CosC) = 0$ (13)

Relative Angular Momentum

Multiplying (3.1) by $-\vec{r}_{AB}$ we have

$$-\vec{r}_{AB} \times \left(-\frac{d^2\vec{r}_{AB}}{dt^2}\right) = -\frac{Gm_C}{r_{BC}{}^3}(-\vec{r}_{AB}) \times \vec{r}_{BC} - \frac{Gm_C}{r_{AC}{}^3}(-\vec{r}_{AB}) \times \vec{r}_{CA} = 0$$

[Since $\vec{r}_{AB} + \vec{r}_{BC} + \vec{r}_{CA} = 0 \;\Rightarrow \frac{d^2\vec{r}_{AB}}{dt^2} + \frac{d^2\vec{r}_{BC}}{dt^2} + \frac{d^2\vec{r}_{CA}}{dt^2} = 0$]

Or,

$$\vec{r}_{AB} \times \frac{d^2\vec{r}_{AB}}{dt^2} = -\frac{Gm_C}{r_{BC}{}^3}\vec{r}_{BA} \times \vec{r}_{BC} - \frac{Gm_C}{r_{AC}{}^3}(-\vec{r}_{AB}) \times (-\vec{r}_{AC}) = 0$$

Or,

$$\vec{r}_{AB} \times \frac{d^2\vec{r}_{AB}}{dt^2} = -\frac{Gm_C}{r_{BC}{}^3}\vec{r}_{BA} \times \vec{r}_{BC} - \frac{Gm_C}{r_{AC}{}^3}\vec{r}_{AB} \times \vec{r}_{AC}$$

Referring to the triangle ABC in Figure 1

$$\vec{r}_{AB} \times \frac{d^2\vec{r}_{AB}}{dt^2} = \frac{Gm_C}{r_{BC}{}^3}\Delta(ABC)\hat{n} - \frac{Gm_C}{r_{AC}{}^3}\Delta(ABC)\hat{n}$$

Where $\hat{n}$ is the downward unit normal[into the paper] referring to the figure below

$\frac{1}{Gm_C}\vec{r}_{AB} \times \frac{d^2\vec{r}_{AB}}{dt^2} = \frac{1}{r_{BC}{}^3}\Delta(ABC)\hat{n} - \frac{1}{r_{AC}{}^3}\Delta(ABC)\hat{n}$ (14)

There are two similar relations.

Adding them you have:

$\frac{1}{m_C}\vec{r}_{AB} \times \frac{d^2\vec{r}_{AB}}{dt^2} + \frac{1}{m_A}\vec{r}_{BC} \times \frac{d^2\vec{r}_{BC}}{dt^2} + \frac{1}{m_B}\vec{r}_{CA} \times \frac{d^2\vec{r}_{CA}}{dt^2} = 0$ (15)

But

$$\frac{d}{dt}\left(\vec{r}_{AB} \times \frac{d\vec{r}_{AB}}{dt}\right) = \frac{d\vec{r}_{AB}}{dt} \times \frac{d\vec{r}_{AB}}{dt} + \vec{r}_{AB} \times \frac{d^2\vec{r}_{AB}}{dt^2} = \vec{r}_{AB} \times \frac{d^2\vec{r}_{AB}}{dt^2}$$

Similarly,

$$\frac{d}{dt}\left(\vec{r}_{BC} \times \frac{d\vec{r}_{BC}}{dt}\right) = \vec{r}_{BC} \times \frac{d^2\vec{r}_{BC}}{dt^2}$$

And

$$\frac{d}{dt}\left(\vec{r}_{CA} \times \frac{d\vec{r}_{CA}}{dt}\right) = \vec{r}_{CA} \times \frac{d^2\vec{r}_{CA}}{dt^2}$$

Therefore,

$$\frac{1}{m_C}\frac{d}{dt}\left(\vec{r}_{AB} \times \frac{d\vec{r}_{AB}}{dt}\right) + \frac{1}{m_A}\frac{d}{dt}\left(\vec{r}_{BC} \times \frac{d\vec{r}_{BC}}{dt}\right) + \frac{1}{m_B}\frac{d}{dt}\left(\vec{r}_{CA} \times \frac{d\vec{r}_{CA}}{dt}\right) = 0$$

Or,

$$\frac{d}{dt}\left[\frac{1}{m_C}\vec{r}_{AB} \times \frac{d\vec{r}_{AB}}{dt} + \frac{1}{m_A}\vec{r}_{BC} \times \frac{d\vec{r}_{BC}}{dt} + \frac{1}{m_B}\vec{r}_{CA} \times \frac{d\vec{r}_{CA}}{dt}\right] = 0$$

Or,

$$\frac{1}{m_C}\vec{r}_{AB} \times \frac{d\vec{r}_{AB}}{dt} + \frac{1}{m_A}\vec{r}_{BC} \times \frac{d\vec{r}_{BC}}{dt} + \frac{1}{m_B}\vec{r}_{CA} \times \frac{d\vec{r}_{CA}}{dt} = constant\ vector \quad (16)$$

But the constant vector on the right hand side has to be the null vector.

Both $\vec{r}_{AB}$ and $\frac{d\vec{r}_{AB}}{dt}$ will lie in the plane of the triangle ABC (shown in the figure)with the comprehension that the size and the orientation of the triangle could change continuously in the general case. Therefore $\vec{r}_{AB} \times \frac{d\vec{r}_{AB}}{dt}$ is directed at right angles top the plane of ABC. The same holds for $\vec{r}_{BC} \times \frac{d\vec{r}_{BC}}{dt}$ and $\vec{r}_{CA} \times \frac{d\vec{r}_{CA}}{dt}$.So on the left side of the above we have a vector instantaneously normal to the plane of ABC. On the RHS we have a vector which is constant with respect to time. The only option would be to have the constant vector to be the null vector so that the orientation of the triangle may change as the three body motion proceeds.

Finally we have,

$$\frac{1}{m_C}\vec{r}_{AB} \times \frac{d\vec{r}_{AB}}{dt} + \frac{1}{m_A}\vec{r}_{BC} \times \frac{d\vec{r}_{BC}}{dt} + \frac{1}{m_B}\vec{r}_{CA} \times \frac{d\vec{r}_{CA}}{dt} = 0 \quad (17)$$

In so far as the initial configuration is considered the above formula may not be satisfied due to the presence of other types of forces[other than gravity] due to collisions etc

Do we get back the relation (G) as soon as the extraneous forces stop acting or within retarded time effects even if the distortion is very large? Is there a possibility of past history or memory being retained……

Energy considerations:

Infinitesimal work is given by

$$dW = \vec{F}_{AB}.d\vec{r}_{AB} + \vec{F}_{AC}.d\vec{r}_{AC} + \vec{F}_{BA}.d\vec{r}_{BA} + \vec{F}_{BC}.d\vec{r}_{BC} + \vec{F}_{CA}.d\vec{r}_{CA} + \vec{F}_{CB}.d\vec{r}_{CB}$$

$\vec{F}_{ij}$:Is the force on the ith particle due to the jth one.It acts from body at i to the one at j along the straight line joining them(due to the strong form of the third law). His is different from the net force from i to j. The net force from I to j includes a component of interaction from the third particle.

Velocity of the particle at A may not be in the direction of either AB or AC. But while considering work, components of $\vec{v}_A$ along AB and AC are important. Components of $\vec{v}_A$perpendicular to the forces along AB and AC are not essential for evaluation of infinitesimal work on the body at A in time dt.

[Example:$dW_{AB} = \vec{F}_A.\vec{v}_A dt = (\vec{F}_{AB} + \vec{F}_{AC}).\vec{v}_A dt = \vec{F}_{AB}.\vec{v}_A dt + \vec{F}_{AC}.\vec{v}_A dt =$ $|\vec{F}_{AB}||\vec{v}_A|Cos\alpha dt + |\vec{F}_{AC}||\vec{v}_A|Cos\beta dt = (appropriate\ sign)|\vec{F}_{AB}||\vec{r}_{AB}| +$ $(appropriate\ sign)|\vec{F}_{AC}||\vec{r}_{AC}| = \vec{F}_{AB}.d\vec{r}_{AB} + \vec{F}_{AC}.d\vec{r}_{AC}; \alpha: angle\ between\ \vec{v}_A\ and\ \vec{F}_{AB};$

$\beta: angle\ between\ \vec{v}_A\ and\ \vec{F}_{AC}]$

]

$$dW = \frac{Gm_Am_B}{{r_{AB}}^2}dr_{AB} + \frac{Gm_Am_C}{{r_{AC}}^2}dr_{AC} + \frac{Gm_Bm_A}{{r_{BA}}^2}dr_{BA} + \frac{Gm_Bm_C}{{r_{BC}}^2}dr_{BC} + \frac{Gm_Cm_A}{{r_{CA}}^2}dr_{CA} + \frac{Gm_Cm_A}{{r_{CA}}^2}dr_{CA}$$

$$W = \int_i^f \frac{Gm_Am_B}{{r_{AB}}^2}dr_{AB} + \int_i^f \frac{Gm_Am_C}{{r_{AC}}^2}dr_{AC} + \int_i^f \frac{Gm_Bm_A}{{r_{BA}}^2}dr_{BA} + \int_i^f \frac{Gm_Bm_C}{{r_{BC}}^2}dr_{BC} + \int_i^f \frac{Gm_Cm_A}{{r_{CA}}^2}dr_{CA} + \int_i^f \frac{Gm_Cm_A}{{r_{CA}}^2}dr_{CA}$$

Each integration extends from initial point of the concerned particle to the final point along the path followed by it. As we can see from above that work done is independent of path.

$$W = Gm_Am_B\left(\frac{1}{r_{AB:i}} - \frac{1}{r_{AB:f}}\right) + Gm_Am_C\left(\frac{1}{r_{AC:i}} - \frac{1}{r_{AC:f}}\right) + Gm_Bm_C\left(\frac{1}{r_{BC:i}} - \frac{1}{r_{BC:f}}\right) + Gm_Bm_A\left(\frac{1}{r_{BA:i}} - \frac{1}{r_{BA:f}}\right) + Gm_Cm_A\left(\frac{1}{r_{CA:i}} - \frac{1}{r_{CA:f}}\right) + Gm_Cm_B\left(\frac{1}{r_{CB:i}} - \frac{1}{r_{CB:f}}\right)$$

$$W = 2Gm_Am_B\left(\frac{1}{r_{AB:i}} - \frac{1}{r_{AB:f}}\right) + 2Gm_Bm_C\left(\frac{1}{r_{BC:i}} - \frac{1}{r_{BC:f}}\right) + 2Gm_Cm_A\left(\frac{1}{r_{CA:i}} - \frac{1}{r_{CA:f}}\right)$$

But work done by each particle =change of its kinetic energy.

Therefore

$$Gm_Am_B\left(\frac{1}{r_{AB:i}}-\frac{1}{r_{AB:f}}\right)+2Gm_Bm_C\left(\frac{1}{r_{BC:i}}-\frac{1}{r_{BC:f}}\right)+2Gm_Cm_A\left(\frac{1}{r_{CA:i}}-\frac{1}{r_{CA:f}}\right)$$
$$=\frac{1}{2}m_A\left[\left(\frac{d\vec{r}_A}{dt}\right)_f^2-\left(\frac{d\vec{r}_A}{dt}\right)_i^2\right]+\frac{1}{2}m_B\left[\left(\frac{d\vec{r}_B}{dt}\right)_f^2-\left(\frac{d\vec{r}_B}{dt}\right)_i^2\right]$$
$$+\frac{1}{2}m_C\left[\left(\frac{d\vec{r}_C}{dt}\right)_f^2-\left(\frac{d\vec{r}_C}{dt}\right)_i^2\right]$$

Or,

$$2Gm_Am_B\frac{1}{r_{AB:i}}+2Gm_Bm_C\frac{1}{r_{BC:i}}+2Gm_Cm_A\frac{1}{r_{CA:i}}+\frac{1}{2}m_A\left(\frac{d\vec{r}_A}{dt}\right)_i^2+\frac{1}{2}m_B\left(\frac{d\vec{r}_B}{dt}\right)_i^2+\frac{1}{2}\left(\frac{d\vec{r}_C}{dt}\right)_i^2=$$
$$2Gm_Am_B\frac{1}{r_{AB:f}}+2Gm_Bm_C\frac{1}{r_{BC:f}}+2Gm_Cm_A\frac{1}{r_{CA:f}}+\frac{1}{2}m_A\left(\frac{d\vec{r}_A}{dt}\right)_f^2+\frac{1}{2}m_B\left(\frac{d\vec{r}_B}{dt}\right)_f^2+$$
$$\frac{1}{2}m_C\left(\frac{d\vec{r}_C}{dt}\right)_f^2 \quad (18)$$

Four Body Motion

Gravitating masses m_A, m_B, m_C and m_D are located at the points A, B, C and D denoted by the position vectors $\vec{r}_A$, $\vec{r}_B, \vec{r}_C$ and $\vec{r}_D$

Equations representing their interaction:

$$m_A\frac{d^2\vec{r}_A}{dt^2}=Gm_Am_B\frac{\vec{r}_{AB}}{{r_{AB}}^3}+Gm_Am_C\frac{\vec{r}_{AC}}{{r_{AC}}^3}+Gm_Am_D\frac{\vec{r}_{AD}}{{r_{AD}}^3} \quad (19.1)$$

$$m_B\frac{d^2\vec{r}_B}{dt^2}=Gm_Bm_C\frac{\vec{r}_{BC}}{{r_{BC}}^3}+Gm_Bm_A\frac{\vec{r}_{BA}}{{r_{BA}}^3}+Gm_Bm_D\frac{\vec{r}_{BD}}{{r_{BD}}^3} \quad (19.2)$$

$$m_C\frac{d^2\vec{r}_C}{dt^2}=Gm_Cm_A\frac{\vec{r}_{CA}}{{r_{CA}}^3}+Gm_Cm_B\frac{\vec{r}_{CB}}{{r_{CB}}^3}+Gm_Cm_D\frac{\vec{r}_{CD}}{{r_{CD}}^3} \quad (19.3)$$

$$m_D\frac{d^2\vec{r}_D}{dt^2}=Gm_Dm_A\frac{\vec{r}_{DA}}{{r_{DA}}^3}+Gm_Dm_B\frac{\vec{r}_{DB}}{{r_{DB}}^3}+Gm_Dm_C\frac{\vec{r}_{DC}}{{r_{DC}}^3} \quad (19.4)$$

$[\vec{r}_{ij}=\vec{r}_j-\vec{r}_i\,]$

In the above adding the right hand sides we obtain the null vector. This is in conformity with the fact that the net force on the system is zero; $m_A\frac{d^2\vec{r}_A}{dt^2}+m_B\frac{d^2\vec{r}_B}{dt^2}+m_C\frac{d^2\vec{r}_C}{dt^2}+m_D\frac{d^2\vec{r}_D}{dt^2}=0$

Or,

$$\frac{d^2\vec{r}_A}{dt^2}=Gm_B\frac{\vec{r}_{AB}}{{r_{AB}}^3}+Gm_C\frac{\vec{r}_{AC}}{{r_{AC}}^3}+Gm_D\frac{\vec{r}_{AD}}{{r_{AD}}^3} \quad (20.1)$$

$$\frac{d^2\vec{r}_B}{dt^2}=Gm_C\frac{\vec{r}_{BC}}{{r_{BC}}^3}+Gm_A\frac{\vec{r}_{BA}}{{r_B}^3}+Gm_D\frac{\vec{r}_{BD}}{{r_{BD}}^3} \quad (20.2)$$

$$\frac{d^2\vec{r}_C}{dt^2} = Gm_A\frac{\vec{r}_{CA}}{{r_{CA}}^3} + m_B\frac{\vec{r}_{CB}}{{r_{CB}}^3} + Gm_D\frac{\vec{r}_{CD}}{{r_{CD}}^3} \qquad (20.3)$$

$$\frac{d^2\vec{r}_D}{dt^2} = Gm_A\frac{\vec{r}_{DA}}{{r_{DA}}^3} + Gm_B\frac{\vec{r}_{DB}}{{r_{DB}}^3} + Gm_C\frac{\vec{r}_{DC}}{{r_{DC}}^3} \qquad (20.4)$$

Or,

$$\frac{d^2\vec{r}_{AB}}{dt^2} = -G(m_A+m_B)\frac{\vec{r}_{AB}}{{r_{AB}}^3} + Gm_C\left[\frac{\vec{r}_{BC}}{{r_{BC}}^3}+\frac{\vec{r}_{CA}}{{r_{CA}}^3}\right] + Gm_D\left[\frac{\vec{r}_{BD}}{{r_{BD}}^3}+\frac{\vec{r}_{DA}}{{r_{DA}}^3}\right] \qquad (21.1)$$

$$\frac{d^2\vec{r}_{BC}}{dt^2} = -G(m_B+m_C)\frac{\vec{r}_{BC}}{{r_{BC}}^3} + Gm_A\left[\frac{\vec{r}_{CA}}{{r_{CA}}^3}+\frac{\vec{r}_{AB}}{{r_{AB}}^3}\right] + Gm_D\left[\frac{\vec{r}_{CD}}{{r_{DB}}^3}+\frac{\vec{r}_{DB}}{{r_{DB}}^3}\right] \qquad (21.2)$$

$$\frac{d^2\vec{r}_{CD}}{dt^2} = -G(m_C+m_D)\frac{\vec{r}_{CD}}{{r_{CD}}^3} + Gm_A\left[\frac{\vec{r}_{DA}}{{r_{DA}}^3}+\frac{\vec{r}_{AB}}{{r_{AB}}^3}\right] + Gm_B\left[\frac{\vec{r}_{DB}}{{r_{DB}}^3}+\frac{\vec{r}_{CB}}{{r_{CB}}^3}\right] \qquad (21.3)$$

$$\frac{d^2\vec{r}_{DA}}{dt^2} = -G(m_D+m_A)\frac{\vec{r}_{DA}}{{r_{DA}}^3} + Gm_C\left[\frac{\vec{r}_{AC}}{{r_{AC}}^3}+\frac{\vec{r}_{CD}}{{r_{CD}}^3}\right] + Gm_B\left[\frac{\vec{r}_{AB}}{{r_{AB}}^3}+\frac{\vec{r}_{BD}}{{r_{BD}}^3}\right] \qquad (21.4)$$

$$\frac{d^2\vec{r}_{BD}}{dt^2} = -G(m_B+m_D)\frac{\vec{r}_{BD}}{{r_{BD}}^3} + Gm_A\left[\frac{\vec{r}_{DA}}{{r_{DA}}^3}+\frac{\vec{r}_{AB}}{{r_{AB}}^3}\right] + Gm_C\left[\frac{\vec{r}_{DC}}{{r_{BC}}^3}+\frac{\vec{r}_{CB}}{{r_{CD}}^3}\right] \qquad (21.5)$$

$$\frac{d^2\vec{r}_{CA}}{dt^2} = -G(m_C+m_A)\frac{\vec{r}_{CA}}{{r_{CA}}^3} + Gm_B\left[\frac{\vec{r}_{AB}}{{r_{AB}}^3}+\frac{\vec{r}_{BC}}{{r_{BC}}^3}\right] + Gm_D\left[\frac{\vec{r}_{DC}}{{r_{DC}}^3}+\frac{\vec{r}_{CB}}{{r_{CB}}^3}\right] \qquad (21.6)$$

In the above we have three independent variables and three independent equations: we may consider der the independent variables to be $\vec{r}_{AB}, \vec{r}_{BC}$ and $\vec{r}_{CD}$: $\vec{r}_{AB}+\vec{r}_{BC}+\vec{r}_{CD}+\vec{r}_{DA}=0; \vec{r}_{AB}+\vec{r}_{BC}+\vec{r}_{CA}=0; \vec{r}_{BC}+\vec{r}_{CD}+\vec{r}_{DB}=0$. Adding the left side of (3.1) through (3.4) gives us zero etc

:Or,

$$\frac{d^2\vec{r}_{AB}}{dt^2} = -G(m_A+m_B+m_C+m_D)\frac{\vec{r}_{AB}}{{r_{AB}}^3} + Gm_C\left[\frac{\vec{r}_{AB}}{{r_{AB}}^3}+\frac{\vec{r}_{BC}}{{r_{BC}}^3}+\frac{\vec{r}_{CA}}{{r_{CA}}^3}\right] + Gm_D\left[\frac{\vec{r}_{AB}}{{r_{AB}}^2}+\frac{\vec{r}_{BD}}{{r_{BD}}^3}+\frac{\vec{r}_{DA}}{{r_{DA}}^3}\right] \qquad (22.1)$$

$$\frac{d^2\vec{r}_{BC}}{dt^2} = -G(m_A+m_B+m_C+m_D)\frac{\vec{r}_{BC}}{{r_{BC}}^3} + Gm_A\left[\frac{\vec{r}_{BC}}{{r_{BC}}^3}+\frac{\vec{r}_{CA}}{{r_{CA}}^3}+\frac{\vec{r}_{AB}}{{r_{AB}}^3}\right] + Gm_D\left[\frac{\vec{r}_{BC}}{{r_{BC}}^3}+\frac{\vec{r}_{CD}}{{r_{DB}}^3}+\frac{\vec{r}_{DB}}{{r_{DB}}^3}\right] \qquad (22.2)$$

$$\frac{d^2\vec{r}_{CD}}{dt^2} = -G(m_A+m_B+m_C+m_D)\frac{\vec{r}_{CD}}{{r_{CD}}^3} + Gm_A\left[\frac{\vec{r}_{CD}}{{r_{CD}}^3}+\frac{\vec{r}_{DA}}{{r_{DA}}^3}+\frac{\vec{r}_{AB}}{{r_{AB}}^3}\right] + Gm_B\left[\frac{\vec{r}_{CD}}{{r_{CD}}^3}+\frac{\vec{r}_{DB}}{{r_{DB}}^3}+\frac{\vec{r}_{CB}}{{r_{CB}}^3}\right] \qquad (22.3)$$

$$\frac{d^2\vec{r}_{DA}}{dt^2} = -G(m_A+m_B+m_C+m_D)\frac{\vec{r}_{DA}}{{r_{DA}}^3} + Gm_C\left[\frac{\vec{r}_{DA}}{{r_{DA}}^3}+\frac{\vec{r}_{AC}}{{r_{AC}}^3}+\frac{\vec{r}_{CD}}{{r_{CD}}^3}\right] + Gm_B\left[\frac{\vec{r}_{DA}}{{r_{DA}}^3}+\frac{\vec{r}_{AB}}{{r_{AB}}^3}+\frac{\vec{r}_{BD}}{{r_{BD}}^3}\right] \qquad (22.4)$$

$\frac{d^2\vec{r}_{BD}}{dt^2} = -G(m_A + m_B + m_C + m_D)\frac{\vec{r}_{BD}}{r_{BD}{}^2} + Gm_A\left[\frac{\vec{r}_{BD}}{r_{BD}{}^3} + \frac{\vec{r}_{DA}}{r_{DA}{}^3} + \frac{\vec{r}A_{AB}}{r_{AB}{}^3}\right] + Gm_C\left[\frac{\vec{r}_{BD}}{r_{BD}{}^3} + \frac{\vec{r}_{\boldsymbol{DC}}}{r_{DC}{}^3} + \frac{\vec{r}_{CB}}{r_{CB}{}^3}\right]$ (22.5)

$$\frac{d^2\vec{r}_{CA}}{dt^2} = -G(m_A + m_B + m_C + m_D)\frac{\vec{r}_{CA}}{r_{CA}{}^3} + Gm_B\left[\frac{\vec{r}_{CA}}{r_{CA}{}^3} + \frac{\vec{r}_{AB}}{r_{AB}{}^3} + \frac{\vec{r}_{BC}}{r_{BC}{}^3}\right] + Gm_D\left[\frac{\vec{r}_{CA}}{r_{CA}{}^3} + \frac{\vec{r}_{AD}}{r_{AD}{}^3} + \frac{\vec{r}_{DC}}{r_{DC}{}^2}\right]$$

(22.6)

Or,

$\frac{d^2\vec{r}_{AB}}{dt^2} = -GM\frac{\vec{r}_{AB}}{r_{AB}{}^2} + Gm_C\vec{X} + Gm_D\vec{Y}$ (23.1)

$\frac{d^2\vec{r}_{BC}}{dt^2} = -GM\frac{\vec{r}_{BC}}{r_{BC}{}^2} + Gm_A\vec{X} + Gm_D\vec{Q}$ (23.2)

$\frac{d^2\vec{r}_{CD}}{dt^2} = -GM\frac{\vec{r}_{CD}}{r_{CD}{}^2} + Gm_A\vec{P} + Gm_B\vec{Q}$ (23.3)

$\frac{d^2\vec{r}_{DA}}{dt^2} = -GM\frac{\vec{r}_{DA}}{r_{DA}{}^2} + Gm_B\vec{Y} + Gm_C\vec{P}$ (23.4)

$\frac{d^2\vec{r}_{BD}}{dt^2} = -GM\frac{\vec{r}_{BD}}{r_{BD}{}^2} + Gm_A\vec{Y} - Gm_C\vec{Q}$ (23.5)

$\frac{d^2\vec{r}_{CA}}{dt^2} = -GM\frac{\vec{r}_{CA}}{r_{CA}{}^2} + Gm_B\vec{X} - Gm_D\vec{P}$ (23.6)

Where

$$M = m_A + m_B + m_C + m_D$$

$\vec{X} = \frac{\vec{r}_{AB}}{r_{AB}{}^3} + \frac{\vec{r}_{BC}}{r_{BC}{}^3} + \frac{\vec{r}_{CA}}{r_{CA}{}^3}$;

$\vec{Y} = \frac{\vec{r}_{AB}}{r_{AB}{}^3} + \frac{\vec{r}_{BD}}{r_{BD}{}^3} + \frac{\vec{r}_{DA}}{r_{DA}{}^3}$

$\vec{P} = \frac{\vec{r}_{DA}}{r_{DA}{}^3} + \frac{\vec{r}_{AC}}{r_{AC}{}^3} + \frac{\vec{r}_{CD}}{r_{CD}{}^3}$

$\vec{Q} = \frac{\vec{r}_{DB}}{r_{DB}{}^3} + \frac{\vec{r}_{BC}}{r_{BC}{}^3} + \frac{\vec{r}_{CD}}{r_{CD}{}^3}$

Here we have to transform equation by equation from(23.1) to (23.6) instead of a single transformation on all of them at once:

We start with equation (23.1)

$$\frac{d^2\vec{r}_{AB}}{dt^2} = -GM\frac{\vec{r}_{AB}}{{r_{AB}}^3} + Gm_C\vec{X} + Gm_D\vec{Y}$$

Or,

$$\frac{d^2\vec{r}_{AB}}{dt^2} = -GM\frac{\vec{r}_{AB}}{{r_{AB}}^2} + Gm_C\left[\frac{\vec{r}_{AB}}{{r_{AB}}^3} + \frac{\vec{r}_{BC}}{{r_{BC}}^3} + \frac{\vec{r}_{CA}}{{r_{CA}}^3}\right] + Gm_D\left[\frac{\vec{r}_{AB}}{{r_{AB}}^3} + \frac{\vec{r}_{BD}}{{r_{BD}}^3} + \frac{\vec{r}_{DA}}{{r_{DA}}^3}\right] \quad (24)$$

We solve for $\vec{K}$ from the following equation

$$Gm_C\left[\frac{\vec{r}_{AB}-\vec{K}}{|r_{AB}-\vec{K}|^3} + \frac{\vec{r}_{BC}-\vec{K}}{|r_{BC}-\vec{K}|^3} + \frac{\vec{r}_{CA}-\vec{K}}{|r_{CA}-\vec{K}^3}\right] + Gm_D\left[\frac{\vec{r}_{AB}-\vec{K}}{|r_{AB}-\vec{K}|^3} + \frac{\vec{r}_{BD}-\vec{K}}{|r_{BD}-\vec{K}|^3} + \frac{\vec{r}_{DA}-\vec{K}}{|r_{DA}-\vec{K}|^3}\right] = 0 \quad (25)$$

Then we transfer the origin to the tip of $\vec{K}$. That converts (23.1) into a two body equation:

$$\frac{d^2(\vec{r}_{AB}-\vec{K})}{dt^2} = -GM\frac{(\vec{r}_{AB}-\vec{K})}{|\vec{r}_{AB}-\vec{K}|^3} \quad (26)$$

Or,

$$\frac{d^2\vec{R}}{dt^2} = -GM\frac{\vec{R}}{|\vec{R}|^3} \quad (27)$$

$\vec{R}$=$\vec{r}_{AB} - \vec{K}$

This transformation will make equations (23.2) to (23.6) more complicated. We do not solve them now We solve only the two body equation and re transform to the original variables

$\vec{r}_{AB} - \vec{K} = \vec{f}(t,a,b)$ (28)

[a and b are constants of integration to be evaluated from the initial conditions]

With the other equations we assume the existence of solutions in the transformed state and that such solutions can be inverse transformed to the original variables. With these equations we solve equations similar to (27) and then transform to obtain two body motion for each particular equation like (23.2) and solve it for the two body equation obtained: for the other equations e again assume that solutions do exist in the transformed state and that the solutions may be inverse transformed to the original variables.

Each time we are transforming we are moving into a different space. Six such spaces have been called up. On inverse transforming we are returning to the same original space having the same variables. The solution set in the original set will be the same for all inverse transformations.

We have six equations of the type (28)

$\vec{K}$ will be of the same form for four equations[from 23.1 to 23.4] but its form will be different for 23.5 and 23.6. The function $\vec{f}$ will have the same form for all six equations of type (28)with respect to t and the concerned Ks. But the Ks will have different form for 23.5 and 23.6. The constants of integration a and b will be different for each equation. We have three independent vectors and three independent equations equivalent to six consistent equations from (22.1) through (22.6) or six consistent equations from 23.1 to 23.6 . So we should be able to obtain consistent solutions at the final stage. That we finally have equations of identical form is logically correct if the relative orientation or symmetry is considered. The Each particle should view the others in a similar manner in relation to the final formulas…..

For the appropriate solution of $\vec{K}$ against equation (23.1) we have the transformed value of $Gm_C\vec{X} + Gm_D\vec{Y}$ equal to zero

We may write equation (23.1) as: $\frac{d^2\vec{r}_{AB}}{dt^2} = -GM\frac{\vec{r}_{AB}}{{r_{AB}}^2} + \vec{J}$

Where $\vec{J} = Gm_C\vec{X} + Gm_D\vec{Y}$ s

Transferring the origin to the tip of $\vec{J}$ by using some dimension factor like D and later making it unity as we did in the three body situation, the process of solution may be simplified.

This type of treatment may be extended to the many body system; the "n" body system may be resolved into C_2^n equations of two body motion. We may also think in terms of force laws other than the inverse squared law. Similar techniques may be applied.

Impulse Considerations in the Many Body Problem

We are considering "n" bodies which are interacting between themselves.

Force on the ith body due to the jth one ;$\vec{F}_{ij}$

Net force on the ith body: $\vec{F}_i = \sum_j \vec{F}_{ij}$

Or,

$$\frac{d\vec{p}_i}{dt} = \sum_j \vec{F}_{ij}$$

Multiplying both sides of the above by dt we have:

$$\frac{d\vec{p}_i}{dt}dt = \sum_j \vec{F}_{ij}\,dt$$

Or,

$d\vec{p}_i = \sum_j \vec{F}_{ij}\,dt$ (29)

Given the initial configuration we may calculate the right hand side of the above to obtain the infinitesimal change of the momentum vector(hence the change in the velocity vector) for each particle at any point of time starting from the initial configuration. From the existing velocities at any instant of time, the velocities for the next infinitesimal dt may be obtained for each particle. Over the said dt each particle is displaced through:$(\vec{v}_i + \Delta\vec{v}_i)dt$.Proceeding in such a manner we may determine the configuration of the system at any point of time starting from a known initial configuration in relation to position and velocities.

Finer Points from Two Body Motion

From the conservation of linear momentum in a two body problem

$\vec{p}_1 + \vec{p}_2 = \vec{K}_1$

From the conservation of angular momentum it follows:

$\vec{L}_1 + \vec{L}_2 = \vec{K}_2$

[$\vec{K}_1$ and $\vec{K}_2$ are constant vectors(independent of time)]

Now, we have from the strong form of Newton's third law:

$$(\vec{r}_1 - \vec{r}_2) \times \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} = 0$$

Now,

$$\frac{d}{dt}\left[(\vec{r}_1 - \vec{r}_2) \times \frac{d(\vec{r}_1 - \vec{r}_2)}{dt}\right] = \frac{d(\vec{r}_1 - \vec{r}_2)}{dt} \times \frac{d(\vec{r}_1 - \vec{r}_2)}{dt} + (\vec{r}_1 - \vec{r}_2) \times \frac{d(\vec{r}_1 - \vec{r}_2)}{dt}$$

$$= 0 + (\vec{r}_1 - \vec{r}_2) \times \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} = 0$$

Or,

$$(\vec{r}_1 - \vec{r}_2) \times \frac{d(\vec{r}_1 - \vec{r}_2)}{dt} = \vec{K}_3$$

[$\vec{K}_3$:Constant vector]

$$(\vec{r}_1 - \vec{r}_2) \times (\vec{v}_1 - \vec{v}_2) = \vec{K}_3$$

$$(\vec{r}_1 \times \vec{v}_1) + (\vec{r}_2 \times \vec{v}_2) - (\vec{r}_1 \times \vec{v}_2) - (\vec{r}_2 \times \vec{v}_1) = \vec{K}_3$$

$$(\vec{r}_1 \times \frac{\vec{p}_1}{m_1}) + (\vec{r}_2 \times \frac{\vec{p}_2}{m_2}) - (\vec{r}_1 \times \frac{\vec{p}_2}{m_2}) - (\vec{r}_2 \times \frac{\vec{p}_1}{m_1}) = \vec{K}_3$$

$$\frac{\vec{L}_1}{m_1} + \frac{\vec{L}_2}{m_2} - (\vec{r}_1 \times \frac{(\vec{K}_1 - \vec{p}_1)}{m_2}) - (\vec{r}_2 \times \frac{\vec{K}_1 \times \vec{p}_2}{m_1}) = \vec{K}_3$$

$$or,$$

$$\frac{\vec{L}_1}{m_1} + \frac{\vec{L}_2}{m_2} + \frac{\vec{L}_1}{m_2} + \frac{\vec{L}_2}{m_1} - \left(\frac{\vec{r}_1}{m_2} + \frac{\vec{r}_2}{m_1}\right) \times \vec{K}_1 = \vec{K}_3$$

$$\vec{L}_1\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \vec{L}_1\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 m_2} \times \vec{K}_1 = \vec{K}_3$$

$$\frac{\vec{L}_1}{\mu} + \frac{\vec{L}_2}{\mu} - \frac{M\vec{r}_{cm}}{m_1 m_2} \times \vec{K}_1 = \vec{K}_3$$

Reduced mass, $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$\frac{\vec{L}_1}{\mu}+\frac{\vec{L}_2}{\mu}-\frac{\vec{r}_{cm}}{\mu}\times\vec{K}_1=\vec{K}_3$$
$$\vec{L}_1+\vec{L}_2-\vec{r}_{cm}\times\vec{K}_1=\mu\vec{K}_3$$

$\vec{K}_2-\vec{r}_{cm}\times\vec{K}_1=\mu\vec{K}_3$ (30)
$\vec{r}_{cm}$ is the only variable in the above equation, $\vec{K}_1$and $\vec{K}_2$ have been taken with respect to an arbitrary inertial origin, $\vec{K}_3$ is the relative angular momentum
Let us check is the problem gets diluted at the three body or at the multi-body interaction level. In the three body or the many body systems we do not have a simple relation like $(\vec{r}_1-\vec{r}_2)\times\frac{d^2(\vec{r}_1-\vec{r}_2)}{dt^2}=0$ even if Newton's third law is considered in the strong form. The quantity $\frac{d^2(\vec{r}_1-\vec{r}_2)}{dt^2}$ is decided not only by the interaction between the bodies at $\vec{r}_1$ and $\vec{r}_2$but also by interaction from other bodies. There is a possibility of resolution to the problem.

In an n-body interacting(isolated) system we consider the following two points:

1) Center of mass
2) Neutral point

The center of mass is at rest or moves with uniform velocity with respect to inertial frames

At the neutral point the net force is zero. A body[test mass] kept on it will move uniformly or will be at rest

So relative acceleration between the two mentioned points should be zero

*Le*t us test this with a two body system the bodies being at a separation of L units of length

Center of mass coordinates

$x_1=\frac{m_2}{m_1+m_2}L(t)$

$x_1=\frac{m_1}{m_1+m_2}L(t)$

x_1 and x_2 and are distances of the center of mass from the masses m_1 and m_2 respectively.

Calculation Neutral point coordinates:

$$G\frac{m_1m}{{X_1}^2}=G\frac{m_2m}{{X_2}^2}$$

X_1 and X_2 are the distances of $m_1$and m_2 respectively.

$\sqrt{m_1}X_2=\sqrt{m_2}X_1$
But $X_1+X_2=L$
Therefore

$$X_1=\frac{m_2}{m_1+m_2}L$$
$$X_2=\frac{\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}}L$$

Distance of the neural point and the center of mass is given by $\left|\frac{m_2}{m_1+m_2}-\frac{\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}}\right|L(t)$ or by $\left|\frac{m_1}{m_1+m_2}-\frac{\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}}\right|L(t)$.Both are identical in absolute value.

But there can always be a non zero acceleration between the two points if $\frac{d^2L}{dt^2}\neq 0$.

For most planetary orbits eccentricity is close to zero $\frac{d^2L}{dt^2}$ small.That makesBut there are exceptions also. Theoretically the point remains. But the situation gets diluted when we consider three body or many body motion.

For the 3-body situation we have two vector equations from (3.1) to (3.3) and an extra equation so that there is no acceleration between the neutral point and the center of mass. In total there are three vector equations and two vector unknowns[six scalar quantities and nine scalar equations.

For the n-body system, there are n-1 vector quantities to be determined while the number of independent vector equations is "n". These "n" equations include the one asserting there is no acceleration between the center of mass and the neutral point: the situation definitely gets eased off. I view of the fact that we have 3n scalar unknowns and 3n+3 scalar equations. The ratio between the number of scalar unknowns or variables and the number of independent equations tends to unity as we increase the number of bodies in the interaction scenario. The last equation-----the extra one is of global significance especially when you consider galaxies separated by huge distances and that they are in relative motion. The other equations are locally powerful. The "extra" equation does not have any meaning in the local context. We may assume its validity in the global context as mentioned leaving it in the hands of nature to fix it up consistently with the other ones which are locally powerful, Perfect isolation has to be shunned.

With the hind sight of Friedman's concept of homogenous and isotropic universe we consider that the center of mass and the neutral point could be any where as two coincident point [or two coming points at relative rest in terms of coordinate labels.

Considering an infinite number of objects as we have in the universe and with the comprehension that gravitational interaction has an infinite range, there is reason to admire nature's skill in disposing the situation. We are of course thinking of gravitation here and not the other fundamental forces that exist in nature.

Conclusion

The article delineates simple methods to solve the three body, four body and in general the many body problem using the technique of differential equations. A simple type of a numerical method has also been suggested. Certain peculiarities in the context of the two body problem have been notified. The possibility of these peculiarities being diluted in the context of the many body problem has been indicated at.

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3. Collection of Remarkable Three Body Motions

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4. Inertial Frames and Newtonian Mechanics

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