Stability of the Moons orbits in Solar system (especially of Earth's Moon) in the restricted three-body problem (R3BP, celestial mechanics)

Sergey V. Ershkov,

Institute for Time Nature Explorations,

M.V. Lomonosov's Moscow State University,

Leninskie gory, 1-12, Moscow 119991, Russia

e-mail: sergej-ershkov@yandex.ru

Abstract: We consider here equations of motion of three-body problem in a *Lagrange form* (which means a consideration of relative motions of 3-bodies in regard to each other). Analyzing such a system of equations, we consider in details the case of moon's motion of negligible mass around the 2-nd of two giant-bodies (*which are rotating around their common centre of masses on Kepler's trajectories*), the mass of which is assumed to be less than the mass of central body.

Under assumtion of R3BP, we derived a Hill-type equation for the motion of moon on the orbits around the 2-nd of two giant-bodies; if the the distance between the moon and the 2-nd of two giant-bodies is negligible in regard to the mutual distance between that two giant-bodies, Hill-type equation above could be reduced to the ordinary differential equation of "free-oscillations"-type, which also could be reduced in the most cases of motions of moons around their planets in Solar system to the simple ODE of 2-nd order, which is proved to have a solution of constant circle orbit or Archimedean spiral-type.

But the orbit of Earth's Moon differs from the constant circle orbit or Archimedean spiral-type's orbit, it could be associated only with a solutions of "free-oscillations"-type. Such a derivation explain and predict the perturbations of the Moon's orbit may be rather dangerously unstable at oscillating during the motion around the Earth in a future.

Key Words: restricted three-body problem, orbit of the Moon, relative motion

Introduction.

The stability of the motion of the Moon is the ancient problem which leading scientists have been trying to solve during last 400 years. A new derivation to estimate such a problem from a point of view of relative motions in restricted three-body problem (R3BP) is proposed here.

Systematic approach to the problem above was suggested earlier in KAM-(*Kolmogorov-Arnold-Moser*)-theory [1] in which the central KAM-theorem is known to be applied for researches of stability of Solar system in terms of *restricted* three-body problem [2-5], especially if we consider *photogravitational* restricted three-body problem [6-8] with additional influence of *Yarkovsky* effect of non-gravitational nature [9].

KAM is the theory of stability of dynamical systems [1] which should solve a very specific question in regard to the stability of orbits of so-called "small bodies" in Solar system, in terms of *restricted* three-body problem [3]: indeed, dynamics of all the planets is assumed to satisfy to restrictions of *restricted* three-body problem (*such as infinitesimal masses, negligible deviations of the main orbital elements, etc.*).

Nevertheless, KAM also is known to assume the appropriate Hamilton formalism in proof of the central KAM-theorem [1]: the dynamical system is assumed to be *Hamilton* system as well as all the mathematical operations over such a dynamical system are assumed to be associated with a proper Hamilton system.

According to the Bruns theorem [5], there is no other invariants except well-known 10 integrals for three-body problem (*including integral of energy, momentum, etc.*), this is a classical example of Hamilton system. But in case of *restricted* three-body problem, there is no other invariants except only one, Jacobian-type integral of motion [3].

Such a contradiction is the main paradox of KAM-theory: it adopts all the restrictions of *restricted* three-body problem, but nevertheless it proves to use the Hamilton formalism, which assumes the conservation of all other invariants (*the integral of energy, momentum, etc.*).

To avoid ambiguity, let us consider a relative motion in three-body problem [2].

1. Equations of motion.

Let us consider the system of ODE for restricted three-body problem in barycentric Cartesian co-ordinate system, at given initial conditions [2-3]:

$$m_1 \boldsymbol{q}_1'' = -\gamma \left\{ \frac{m_1 m_2 (\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} + \frac{m_1 m_3 (\boldsymbol{q}_1 - \boldsymbol{q}_3)}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3} \right\},$$

$$m_2 \boldsymbol{q}_2'' = -\gamma \left\{ \frac{m_2 m_1 (\boldsymbol{q}_2 - \boldsymbol{q}_1)}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3} + \frac{m_2 m_3 (\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\},$$

$$m_3 \boldsymbol{q}_3'' = -\gamma \left\{ \frac{m_3 m_1 (\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{m_3 m_2 (\boldsymbol{q}_3 - \boldsymbol{q}_2)}{|\boldsymbol{q}_3 - \boldsymbol{q}_2|^3} \right\} .$$

- here $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ - mean the radius-vectors of bodies m_1, m_2, m_3, accordingly; γ - is the gravitational constant.

System above could be represented for relative motion of three-bodies as shown below (by the proper linear transformations):

$$(\boldsymbol{q}_1 - \boldsymbol{q}_2)'' + \gamma (m_1 + m_2) \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} = \gamma m_3 \left\{ \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\},$$

$$(\boldsymbol{q}_2 - \boldsymbol{q}_3)'' + \gamma (m_2 + m_3) \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} = \gamma m_1 \left\{ \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} \right\},$$

$$(\boldsymbol{q}_3 - \boldsymbol{q}_1)'' + \gamma (m_1 + m_3) \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} = \gamma m_2 \left\{ \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} + \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\} .$$

Let us designate as below:

$$\boldsymbol{R}_{1,2} = (\boldsymbol{q}_1 - \boldsymbol{q}_2), \quad \boldsymbol{R}_{2,3} = (\boldsymbol{q}_2 - \boldsymbol{q}_3), \quad \boldsymbol{R}_{3,1} = (\boldsymbol{q}_3 - \boldsymbol{q}_1) \qquad (*)$$

Using of (*) above, let us transform the previous system to another form:

$$\boldsymbol{R}_{1,2}'' + \gamma (m_1 + m_2) \frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} = \gamma m_3 \left\{ \frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} + \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} \right\},$$

$$\boldsymbol{R}_{2,3}'' + \gamma (m_2 + m_3) \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} = \gamma m_1 \left\{ \frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} + \frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} \right\}, \qquad (1.1)$$

$$\boldsymbol{R}_{3,1}'' + \gamma (m_1 + m_3) \frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} = \gamma m_2 \left\{ \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} + \frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} \right\} .$$

Analysing the system (1.1) we should note that if we sum all the above equations one to each other it would lead us to the result below:

$$\boldsymbol{R}_{1,2}'' + \boldsymbol{R}_{2,3}'' + \boldsymbol{R}_{3,1}'' = 0 .$$

If we also sum all the equalities (*) one to each other, we should obtain

$$\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3} + \boldsymbol{R}_{3,1} = 0 \qquad (**)$$

Under assumption of restricted three-body problem, we assume that the mass of small 3-rd body $m_3 \ll m_1, m_2$, accordingly; besides, for the case of moving of small 3-rd body m_3 as a moon around the 2-nd body m_2, let us additionaly assume $|\boldsymbol{R}_{2,3}| \ll |\boldsymbol{R}_{1,2}|$.

So, taking into consideration (**), we obtain from the system (1.1) as below:

$$\boldsymbol{R}_{1,2}{}'' + \gamma(m_1 + m_2)\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} = 0,$$

$$\boldsymbol{R}_{2,3}{}'' + \gamma m_2 \frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3} = \gamma m_1 \left\{ \frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} - \frac{(\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3})}{|\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3}|^3} \right\}, \qquad (1.2)$$

$$\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3} + \boldsymbol{R}_{3,1} = 0 \ ,$$

- where the 1-st equation of (1.2) describes the relative motion of 2 massive bodies (*which are rotating around their common centre of masses on Kepler's trajectories*); the 2-nd describes the orbit of small 3-rd body m_3 (the Moon) relative to the 2-nd body m_2 (the Earth), for which we could obtain according to the trigonometric "Law of Cosines" [10]:

$$\boldsymbol{R}_{2,3}{}'' + \frac{\gamma m_1}{|\boldsymbol{R}_{1,2}|^3}\left(1 + 3\cos\alpha \frac{|\boldsymbol{R}_{2,3}|}{|\boldsymbol{R}_{1,2}|}\right)\boldsymbol{R}_{2,3} \cong -3\cos\alpha \left(\frac{\gamma m_1}{|\boldsymbol{R}_{1,2}|^3}\boldsymbol{R}_{1,2}\right)\frac{|\boldsymbol{R}_{2,3}|}{|\boldsymbol{R}_{1,2}|}\ , \qquad (1.3)$$

- here α – is the angle between the radius-vectors $\boldsymbol{R}_{2,3}$ and $\boldsymbol{R}_{1,2}$.

Equation (1.3) could be simplified under additional assumption $|\boldsymbol{R}_{2,3}| \ll |\boldsymbol{R}_{1,2}|$ for *restricted* mutual motions of bodies m_1, m_2 in R3BP [3] as below:

$$\boldsymbol{R}_{2,3}{}'' + \left(\frac{\gamma\, m_1}{\left|\boldsymbol{R}_{1,2}\right|^3}\right)\cdot \boldsymbol{R}_{2,3} = 0\,, \qquad (1.4)$$

- where Eq. (1.4) is known to be the ordinary differential equation of *Riccati* type, which has no solution in general case [10]; nevertheless, if $|\boldsymbol{R}_{1,2}|$ is a periodic function (in dependence on time-parameter), Eq. (1.4) is proved to be the *Hill* equation [10], which has appropriate class of periodic or quasi-periodic solutions (stable or unstable).

But if the meaning of $|\boldsymbol{R}_{1,2}|$ is varying slowly (in dependence on time-parameter), it has the approximate solutions; besides, let us assume $|\boldsymbol{R}_{1,2}| \cong$ const during a period of time which is assumed to be negligible in regard to the period of orbital motion of 2 massive bodies around each other (*on Kepler's trajectories*). In such a case, Eq. (1.4) describes *a periodic* motion of the Moon [10] relative to the 2-nd giant-body (Earth).

Moreover, if we present Eq. (1.4) as below

$$\boldsymbol{R}_{2,3}{}'' + \xi\cdot m_2 \cdot \left(\gamma \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3}\right) = 0\,, \qquad (1.5)$$

$$\xi = \left(\frac{m_1}{m_2}\cdot\frac{\left|\boldsymbol{R}_{2,3}\right|^3}{\left|\boldsymbol{R}_{1,2}\right|^3}\right)$$

- it should be classified as the equation of *free-falling* (see [10], case 6.188) of the infinitesimal mass ($\xi\cdot m_2$) relative to the 2-nd body m_2 (Earth); besides, if the dimensionless parameter $\xi \to 0$, equation (1.5) could be simplified as below:

$$\boldsymbol{R}_{2,3}{}'' \cong 0\,, \quad \Rightarrow \quad \boldsymbol{R}_{2,3} \cong \vec{V}_1 \cdot t + \boldsymbol{R}_0 \qquad (1.6)$$

- here $\boldsymbol{R}_0$ - is the constant radius-vector, given by the initial conditions, $\boldsymbol{V}_1$ - is the proper initial constant vector of velocity of moving (for example, it is well-known fact that the Moon is slowly moving away from the Earth with average velocity 4 cm/year). It means that the orbit of small 3-rd body m_3 (the Moon) relative to the 2-nd body m_2 (the Earth) should be the constant circle orbit or should be like Archimedean spiral [11], Fig.1:

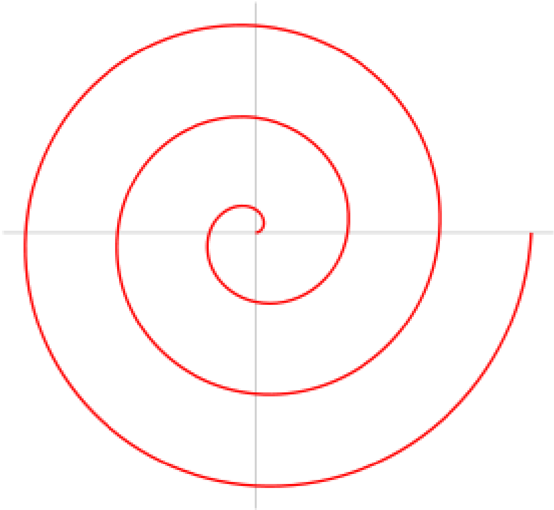

Fig.1. The orbit of the Moon relative to the Earth.

2. The comparison of the moons in Solar system.

As we can see from Eq. (1.5), ξ is the key parameter which determines the character of moving of the small 3-rd body m_3 (the Moon) relative to the 2-nd body m_2 (the Earth). Let us compare such a parameter for all considerable known cases of orbital moving of the moons in Solar system [12] (Tab.1):

Masses of the Planets (*Solar system*), kg	Ratio m_1 (Sun) to mass m_2 (Planet)	Distance $\lvert \boldsymbol{R}_{1,2} \rvert$ (*between Sun-Planet*), AU	Ratio m_3 (Moon) to mass m_2 (Planet)	Distance $\lvert \boldsymbol{R}_{2,3} \rvert$ (*between Moon-Planet*) in 10^3 km	Parameter $\xi = \left(\frac{m_1}{m_2} \cdot \frac{\lvert \boldsymbol{R}_{2,3} \rvert^3}{\lvert \boldsymbol{R}_{1,2} \rvert^3} \right)$
Mercury, $3.3 \cdot 10^{23}$	$\left(\frac{332'946}{0.055} \right)$	0.387 AU			
Venus, $4.87 \cdot 10^{24}$	$\left(\frac{332'946}{0.815} \right)$	0.723 AU			
Earth, $5.97 \cdot 10^{24}$	1 Earth = 332 946 kg	1 AU = 149 500 000 km	**12'300**$\cdot 10^{-6}$	383.4	Moon **5'532**$\cdot 10^{-6}$
Mars, $6.42 \cdot 10^{23}$	$\left(\frac{332'946}{0.107} \right)$	1.524 AU	1) Phobos $0.02 \cdot 10^{-6}$ 2) Deimos $0.003 \cdot 10^{-6}$	1) Phobos 9.38 2) Deimos 23.46	1) Phobos $0.217 \cdot 10^{-6}$ 2) Deimos $3.4 \cdot 10^{-6}$
Jupiter, $1.9 \cdot 10^{27}$	$\left(\frac{332'946}{317.8} \right)$	5.2 AU	1) Ganymede $79 \cdot 10^{-6}$ 2) Callisto $58 \cdot 10^{-6}$ 3) Io $47 \cdot 10^{-6}$ 4) Europa $25 \cdot 10^{-6}$	1) Ganymede **1 070** 2) Callisto **1 883** 3) Io 422 4) Europa 671	1) Ganymede $2.73 \cdot 10^{-6}$ 2) Callisto **14.89**$\cdot 10^{-6}$ 3) Io $0.168 \cdot 10^{-6}$ 4) Europa $0.674 \cdot 10^{-6}$

Saturn, $5.69 \cdot 10^{26}$	$\left(\frac{332'946}{95.16}\right)$	9.54 AU	1) Titan **240**$\cdot 10^{-6}$ 2) Rhea $4.1 \cdot 10^{-6}$ 3) Iapetus $3.4 \cdot 10^{-6}$ 4) Dione $1.9 \cdot 10^{-6}$	1) Titan **1 222** 2) Rhea 527 3) Iapetus **3 561** 4) Dione 377	1) Titan $2.2 \cdot 10^{-6}$ 2) Rhea $0.177 \cdot 10^{-6}$ 3) Iapetus **54.46**$\cdot 10^{-6}$ 4) Dione $0.065 \cdot 10^{-6}$
Uranus, $8.69 \cdot 10^{25}$	$\left(\frac{332'946}{14.37}\right)$	19.19 AU	1) Titania $40 \cdot 10^{-6}$ 2) Oberon $35 \cdot 10^{-6}$ 3) Ariel: $16 \cdot 10^{-6}$	1) Titania 436 2) Oberon 584 3) Ariel: 191	1) Titania $0.081 \cdot 10^{-6}$ 2) Oberon $0.195 \cdot 10^{-6}$ 3) Ariel: $0.007 \cdot 10^{-6}$
Neptune, $1.02 \cdot 10^{26}$	$\left(\frac{332'946}{17.15}\right)$	30.07 AU	1) Triton **210**$\cdot 10^{-6}$ 2) Proteus $0.48 \cdot 10^{-6}$ 3) Nereid $0.29 \cdot 10^{-6}$	1) Triton 355 2) Proteus 118 3) Nereid **5 513**	1) Triton $0.01 \cdot 10^{-6}$ 2) Proteus $0.0004 \cdot 10^{-6}$ 3) Nereid **35.81**$\cdot 10^{-6}$
Pluto, $1.3 \cdot 10^{22}$	$\left(\frac{332'946}{0.002}\right)$	39.48 AU	Charon **124’620**$\cdot 10^{-6}$	Charon 20	Charon $0.0062 \cdot 10^{-6}$

3. Discussion.

As we can see from the Tab.1 above, the dimensionless key parameter ξ, which determines the character of moving of the small 3-rd body m_3 (the Moon) relative to the 2-nd body m_2 (the Earth), is varying for all variety of the moons of the Planets (in Solar system) from the meaning $0.0004 \cdot 10^{-6}$ (for Proteus of Neptune) to the meaning $54.46 \cdot 10^{-6}$ (for Iapetus of Saturn); but it still remains to be negligible enough for adopting the stable moving of the (1.6)-type (constant circle orbit or Archimedean spiral).

But only in case of the Earth's Moon such a parameter increases to the crucial meaning $5'532 \cdot 10^{-6} = 0.0055$. It means that we should consider not (1.6)-type of orbit for relative motion of the Moon in regard to the Earth, but the type of motion (1.4) with $|\boldsymbol{R}_{1,2}| \cong$ const during a period of time which is assumed to be negligible in regard to the period of orbital "Sun-Earth" motion around each other (*on Kepler's trajectories*).

In such a case, Eq. (1.4) describes *a periodic* motion [10] of the Moon relative to the 2-nd giant-body (the Earth):

$$\boldsymbol{R}_{2,3}'' + \left(\frac{\gamma m_1}{\left|\boldsymbol{R}_{1,2}\right|^3} \right) \cdot \boldsymbol{R}_{2,3} = 0 \,, \qquad (3.1)$$

$$\Rightarrow \quad \boldsymbol{R}_{2,3} = \boldsymbol{R}_{(0)} \cdot \cos\left(t \cdot \sqrt{\frac{\gamma m_1}{\left|\boldsymbol{R}_{1,2}\right|^3}} \right) + \boldsymbol{R}_{(1)} \cdot \sin\left(t \cdot \sqrt{\frac{\gamma m_1}{\left|\boldsymbol{R}_{1,2}\right|^3}} \right),$$

- where $\boldsymbol{R}_{(0)}$, $\boldsymbol{R}_{(1)}$ - are the appropriate *constant* radius-vectors, given by the initial conditions.

Conclusion.

We consider here equations of motion of three-body problem in a *Lagrange form* (which means a consideration of relative motions of 3-bodies in regard to each other). Analyzing such a system of equations, we consider in details the case of moon's motion of negligible mass around the 2-nd of two giant-bodies (*which are rotating around their common centre of masses on Kepler's trajectories*), the mass of which is assumed to be less than the mass of central body.

Under assumtion of R3BP, we derived a Hill-type equation for the motion of moon on the orbits around the 2-nd of two giant-bodies; if the the distance between the moon and the 2-nd of two giant-bodies is negligible in regard to the mutual distance between that two giant-bodies, Hill-type equation above could be reduced to the ordinary differential equation of "free-oscillations"-type, which also could be reduced in the most cases of motions of moons around their planets in Solar system to the simple ODE of 2-nd order, which is proved to have a solution of constant circle orbit or Archimedean spiral-type.

But the orbit of Earth's Moon differs from the constant circle orbit or Archimedean spiral-type's orbit, it could be associated only with a solutions of "free-oscillations"-type. Such a derivation explain and predict the perturbations of the Moon's orbit may be rather dangerously unstable at oscillating during the motion around the Earth in a future.

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