# Does Geometric Algebra provide a loophole to Bell's Theorem?

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14 April, 2015

**Abstract**

This paper describes the first and second versions of Joy Christian's model for the singlet correlations, working through the mathematical core of two of Christian's shortest, least technical, and most accessible works. The aim of the paper is to show that from the start, the model depended *both* on a conceptual error and on an algebraic error. For this purpose we start by giving an introduction to geometric algebra using the fact that the basic geometric algebra of 3D geometry is actually isomorphic to the algebra of the complex two-by-two matrices over the *real* numbers. Thus the reader who is already familiar with the Pauli spin matrices will find him- or herself in a completely familiar environment. This helps avoid the kind of beginner's errors which plague Christian's opus, and gives rapid access to (and understanding of) the so-called bivector algebra: the even subalgebra of $\mathcal{C}\ell_{3,0}(\mathbb{R})$, itself isomorphic to the quaternions.

Getting the basic facts of geometric algebra out front and crystal clear helps demystify Christian's project and hopefully is useful in its own right. We will see how Christian apparently realised, if only at a subconscious level, that there was a major gap in his first, 2007, paper, and attempted to patch this in 2011, making things, however, only worse.

Apart from providing a quick-start guide to geometric algebra, and a hopefully very accessible post-mortem analysis of Christian's project, the purpose of the paper is to discuss the psychology and sociology of Bell deniers: how can very clever people make such elementary mistakes, and persist so long in maintaining their illusion that they have created a major breakthrough?

# 1 Introduction

In 2007, Joy Christian surprised the world with his announcement that Bell's theorem was incorrect, because, according to Christian, Bell had unnecessarily restricted the co-domain of the measurement outcomes to be the traditional real numbers. Christian's first publication on arXiv spawned a host of rebuttals and these in turn spawned rebuttals of rebuttals by Christian. Moreover, Christian also published many sequels to his first paper, most of which became chapters of his book, of which a second edition came out in 2014.

His original paper Christian (2007) took the Clifford algebra $\mathcal{C}\ell_{3,0}(\mathbb{R})$ as outcome space. This is "the" basic geometric algebra for the geometry of $\mathbb{R}^3$, but geometric algebra goes far beyond this. Its roots are indeed with Clifford in the nineteenth century but the emphasis on geometry and the discovery of new geometric features is due to the pioneering work of David Hestenes who likes to promote geometric algebra as *the* language of physics, due to its seamless combination of algebra and geometry. The reader is referred to the standard texts Doran and Lasenby (2003) and Dorst, Fontijne and Mann (2007); the former focussing on applications in physics, the latter on applications in computer science, especially computer graphics. The wikipedia pages on geometric algebra, and on Clifford algebra, are two splendid mines of information, but the connections between the two sometimes hard to decode.

It seems that many of Christian's critics were not familiar enough with geometric algebra in order to work through Christian's papers in detail. Thus the criticism of his work was often based on general structural features of the model. One of the few authors who did go to the trouble to work through the mathematical details was Florin Moldoveanu. But this led to new problems: the mathematical details of many of Christian's papers seem to be different: the target is a moving target. So Christian claimed again and again that the critics had simply misunderstood him; he came up with more and more elaborate versions of his theory, whereby he also claimed to have countered all the objections which had been raised.

It seems to this author that it is better to focus on the shortest and most self-contained presentations by Christian since these are the ones which a newcomer has the best chance of actually "understanding" in the sense of being able to work through the *mathematics*, from beginning to end. In this way he or she can also get a feeling for whether the accompanying *words* (the English text) actually reflect the mathematics. Because of unfamiliarity with technical aspects and the sheer volume of knowledge which is encapsulated under the heading "geometric algebra", readers tend to follow the *words*, and just believe that the author is competent enough that the formulas match the words. If one criticises the words, Christian will respond with more words, and also with more mathematics, claiming that the mathematics show that the verbal criticism was unfounded. This leads to a never ending cycle of rebuttals and counter rebuttals. Of course, in the end everyone loses patience, but still: the ghost has not yet been laid to rest.

In order to work through the mathematics, some knowledge of the basics of geometric algebra is needed. That is difficult for the novice to come by, and let's be honest: almost everyone is a novice, despite the popularising efforts of David Hestenes and others, and the niche popularity of geometric algebra in computer graphics. Wikipedia has a page on "Clifford algebra" and another page on "geometric algebra" but it is hard to find out what is the connection between the two. There are different notations around and there is a plethora of different products: geometric product, wedge product, dot product, outer product ... when an author like Christian writes a "dot" or a "wedge" which product does he actually mean? The standard introductory texts on geometric algebra start with the familiar 3D vector products (dot product, cross product) and only come to an abstract or general definition of geometric product after several chapters of preliminaries. The algebraic literature jumps straight into abstract Clifford algebras, without geometry, and wastes little time on that one particular Clifford algebra which happens to have so many connections with the geometry of three dimensional Euclidean space. (Interestingly, David Hestenes also has ardently championed mathematical modelling as the study of stand-alone mathematical realities).

I believe that Christian's work can only be laid to rest in peace after one has actually checked the substance of the papers: the mathematics. The words, or perhaps the "spiel", should correspond to the formulas. The mathematics should stand on its own feet as "pure" mathematics. The words are there to help the reader get a feeling for the math, and to build a bridge between the math and the physical world. As a mathematical statistician, the present author is particularly aware of the distinction between mathematical model and reality – one has to admit that a mathematical model used in some statistical application, e.g., in psychology or economics, is really "just" a model, in the sense of being a tiny and oversimplified representation of the scientific phenomenon of interest. All models are wrong, but some are useful. It seems that in physics, the distinction between model and reality is not so commonly made. Physicists use mathematics as a language, the language of nature, to describe reality. They do not use it as a tool for creating artificial toy realities. For a mathematician, a mathematical model has its own reality in the world of mathematics.

The present author already published (on arXiv) a mathematical analysis, Gill (2012), of one of Christian's shortest papers: the so-called "one page paper", Christian (2011), which moreover contains the substance of the first chapter of Christian's book. In the present paper he analyses in the same spirit the paper Christian (2007), which was the foundation or starting shot in Christian's project. The present note will lay bare the same fundamental issues which are present in all of Christian's later works. It will show that the model is actually not the same in every publication, but constantly changes, so it seems as if Christian himself did become aware of short-comings in earlier publications, and attempted to "fix them" in later ones, though without ever explicitly saying that the earlier works had short-comings. In fact we will find out that Christian's (2007) model is not a model at all: the most important feature, a definition of local measurement functions, is omitted. However the author leaves little choice as to how to fill the gap. We will see that his later work did consist in a failed attempt to fill the gap in the first paper, so it is clear (to this author) that Christian was subconsciously aware of the short-coming of the first attempt; he never explicitly admitted it.

What was also present from the start was a sign error. Though one can say on general principles that the whole idea that Bell made a mistake in restricting the co-domain of his measurement functions is mistaken, it is amusing that right from the start Christian's work was also based on capitalising on a sign error, probably due to notational carelessness.

Why start with the shortest papers? Because if one sees that in these short papers Christian already makes appalling elementary *mathematical* errors *and* fundamental *conceptual* errors, there is no point at all in trying to see if the later, longer works have some how recovered from these mistakes. On the one hand, the conceptual errors mean that the research project itself is failed before it starts. However it might still have been the case that there was interesting mathematics there, even if it did not fit to the physical situation which the author had in mind. There might have been interesting mathematics which even had interesting implications for physics. But if the author cannot deal with the mathematics in a simple version of his model, and moreover vehemently denies that any of the works contain any errors at all, then we may safely assume that later elaborations are simply baroque constructions in which it is even more easy to hide both conceptual errors and mathematical errors. But I do not want to imply that the author hides these errors deliberately. On the contrary, I want to suggest that he makes (made) these errors accidentally, and does not see them.

The author's "model" is a verbal science fantasy. For a physicist like Christian, who only uses mathematics as an intuitive language in which to describe *physical* insights, if a particular mathematical formalism does not match his physical intuition, then the mathematics is at fault: not the intuition. For an outsider, though, that intuition was a fantasy, precisely because it cannot be grounded in a stand-alone mathematical model.

What is the use of digging up these, by now forgotten, utterly failed attempts to revolutionise quantum information theory and quantum computation?

First of all, because it is fun and profitable to learn something about geometric algebra! That's perhaps the main motivation of the author of this paper.

Secondly, to his mind, the history of the Christian model is fascinating from the point of view of psychology, and of sociology. On the one hand, how could a clearly highly gifted, highly intelligent person come to believe that he had discovered a flaw in a rather elementary piece of mathematics, which had stood up through more than 40 years of intense interest? Challenged again and again, but surviving all challenges till Christian came up with his model? Which is so simple that he can summarise it in one short page? This is the matter of individual psychology: how can a promising young scientist come to believe that they are a genius? Make no mistake: I do believe that Christian has always honestly believed in the correctness of his discovery, though at one level (subconscious) it seems to me that he has early on realised that he wasn't quite there yet. Something did not quite fit, more work was needed.

On the other hand, at the sociological level, how could Christian's theory have been taken seriously by quite a few serious researchers for quite a long time, if the flaws in it are as simple and devastating as I will try to explain?

I shall return to some kind of "post-mortem" at the end of this paper. First it is necessary to "remind" the reader of some basic theory of geometric algebra. What is the Clifford algebra $\mathcal{C}\ell_{3,0}(\mathbb{R})$ and what does it have to do with Euclidean (three-dimensional) geometry? After we have some basic familiarity with this wonderful field, we will be able to follow Christian all the way through the essential mathematics of his original model, and thereby clearly see that it really was dead from the start.

Which raises the sociological question: why was this not obvious to everyone, from the start, as well?

# 2 Geometric Algebra

I will first of all take a side-step and discuss the real algebra of the two-by-two complex matrices. We can add such matrices and obtain a new one; we can multiply two such matrices and obtain a new one. The zero matrix acts as a zero and the identity matrix acts as a unit. We can multiply a two-by-two complex matrix by a *real*. In this role, we talk about scalar multiplication. We can therefore take *real* linear combinations. Matrix multiplication is associative so we have a *real unital associative algebra*. Unital just means: with a unit; real means that the scalars in scalar multiplication are reals, and therefore, as a vector space (i.e., restricting attention to addition and scalar multiplication), our algebra is a vector space over the real numbers.

The four complex number entries in a two-by-two complex matrix can be separated into their real and imaginary parts. In this way, each two-by-two complex matrix can be expressed as a *real* linear combination of eight "basis" matrices: the matrices having either $+1$ or $i$ in just one position, and zeros in the other three positions. Obviously, if a real linear combination of those 8 matrices is zero, then all eight coefficients are zero. The vector-space dimension of our space is therefore eight.

I will now specify an alternative vector-space basis of our space. Define the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the identity

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and define

$$\mathbb{1} = I,\ e_1 = \sigma_x,\ e_2 = \sigma_y,\ e_3 = \sigma_z,\ \beta_1 = i\sigma_x,\ \beta_2 = i\sigma_y,\ \beta_3 = i\sigma_z,\ M = iI.$$

It is not difficult to check that this eight-tuple is also a vector space basis. Note the following:

$$e_1^2 = e_2^2 = e_3^2 = \mathbb{1},$$

$$e_1e_2 = -e_2e_1 = -\beta_3, e_2e_3 = -e_3e_2 = -\beta_2, e_3e_1 = -e_1e_3 = -\beta_1,$$

$$e_1e_2e_3 = M.$$

These relations show us that the two-by-two complex matrices, thought of as an algebra over the reals, are isomorphic to $\mathcal{C}\ell_{3,0}(\mathbb{R})$, or if you prefer, form a *representation* of this algebra. By definition, $\mathcal{C}\ell_{3,0}(\mathbb{R})$, is the associative unital algebra over the reals generated from $\mathbb{1}$, $e_1$, $e_2$ and $e_3$, with three of $e_1$, $e_2$ and $e_3$ squaring to $\mathbb{1}$ and none of them squaring to $-\mathbb{1}$, which is the meaning of the "3" and the "0" : three squares are positive, zero are negative. Moreover, the three basis elements $e_1$, $e_2$ and $e_3$ anti-commute. The largest possible algebra over the real numbers which can be created with these rules has vector space dimension $2^{3+0} = 8$ and as a real vector space, a basis can be taken to be $\mathbb{1}$, $e_1$, $e_2$, $e_3$, $e_1e_2 = -\beta_3$, $e_1e_3 = \beta_2$, $e_2e_3 = -\beta_1$, $e_1e_2e_3 = M$. The algebra is associative (like all Clifford algebras), but not commutative.

This description does not quite tell you the "official" definition of Clifford algebra in general, but is sufficient for our purposes. We are interested in just one particular Clifford algebra, $\mathcal{C}\ell_{3,0}(\mathbb{R})$, which is *the* Clifford algebra of three dimensional real geometry. It is called the (or a) *geometric algebra* and the product is called the geometric product, for reasons which will soon become clear.

I wrote $\mathbb{1}$ for the identity matrix and will later also write $\mathbb{0}$ for the zero matrix, as one step towards actually indentifying these with the scalars 1 and 0. However, for the time being, we should remember that we are not (at the outset) talking about numbers, but about elements of a complex matrix algebra over the reals. Complex matrices can be added, multiplied, and multiplied by scalars (reals).

Within this algebra, various famous and familiar structures are embedded.

To start with something famous, we can identify the quaternions. Look at just the four elements $\mathbb{1}, \beta_1, \beta_2, \beta_3$. Consider all real linear combinations of these four. Notice that

$$\beta_1^2 = \beta_2^2 = \beta_3^2 = -\mathbb{1},$$

$$\beta_1\beta_2 = -\beta_2\beta_1 = -\beta_3,$$

$$\beta_2\beta_3 = -\beta_3\beta_2 = -\beta_1,$$

$$\beta_3\beta_1 = -\beta_1\beta_3 = -\beta_2.$$

This is close to the conventional multiplication table of the quaternionic roots of minus one: the only thing that is wrong is the last minus sign in each of the last three rows. However, the sign can be fixed in many ways: for instance, by taking an odd permutation of $(\beta_1, \beta_2, \beta_3)$, or an odd number of sign changes of elements of $(\beta_1, \beta_2, \beta_3)$. In particular, changing the signs of all three does the job. But also the triple $e_1e_2 = -\beta_3, e_1e_3 = \beta_2, e_2e_3 = -\beta_1$ (the order is crucial) does the job too. Notice that we reversed the order of the three $\beta_j$s, which is an odd permutation, *and* changed two signs, an even number. We ended up with $e_1e_2, e_1e_3, e_2e_3$ in the conventional *lexicographic* ordering.

Switching some signs or permuting the order does not change the algebra generated by $\mathbb{1}, \beta_1, \beta_2, \beta_3$. These four elements do generate a sub-algebra of $\mathcal{C}\ell_{3,0}(\mathbb{R})$ which is isomorphic to the quaternions. The quaternion sub-algebra is called the "even sub-algebra" of our Clifford algebra since it is built from just the products of even numbers of $e_1$, $e_2$ and $e_3$.

Obviously, $M$ commutes with everything, and obviously $M^2 = -\mathbb{1}$. Also $Me_1 = \beta_1$, $Me_2 = \beta_2$, $Me_3 = \beta_3$ and, in duality with this, $e_1 = -M\beta_1$,

$e_2 = -M\beta_2$, $e_3 = -M\beta_3$. Thus any element of $\mathcal{C}\ell_{3,0}(\mathbb{R})$ can be expressed in the form $p + Mq$ where $p$ and $q$ are quaternions, $M$ commutes with the quaternions, and $M$ is itself another square root of minus one. However one must be careful with parametrisation and be aware of "left-handed" and "right-handed" ways to define and work with the quaternions.

Notice that $(\mathbb{1} - e_i)(\mathbb{1} + e_i) = \mathbb{0}$ (the zero two-by-two matrix). So the algebra $\mathcal{C}\ell_{3,0}(\mathbb{R})$ possesses zero divisors (in fact, very many!), and hence not all of its elements have inverses.

The real reason for calling this algebra a *geometric algebra* comes from locating real 3D space within it, and also recognising geometric operations and further geometric structures with the algebra. To start with, look at the linear span of $e_1$, $e_2$ and $e_3$. This can be identified with $\mathbb{R}^3$: from now on, real 3D vectors *are* real linear combinations of $e_1$, $e_2$ and $e_3$. So let us look at two elements $a$, $b$ of $\mathbb{R}^3$, thus simultaneously elements of the linear span of $e_1$, $e_2$ and $e_3$. In the latter incarnation (i.e., as elements of $\mathcal{C}\ell_{3,0}(\mathbb{R})$), we can multiply them; what do we find? The answer is easily seen to be the following:

$$ab = (a \cdot b)\,\mathbb{1} + M\,(a \times b).$$

Here, $a \cdot b$ stands for the (ordinary) inner product of two real (three dimensional) vectors, hence a real number, or a scalar; $a \times b$ stands for the (ordinary) cross product of two real three dimensional vectors, hence a real vector. As such, it is a real linear combination of $e_1$, $e_2$ and $e_3$. Multiplying by $M$ gives us the same real linear combination of $\beta_1$, $\beta_2$, $\beta_3$.

Thus the geometric algebra $\mathcal{C}\ell_{3,0}(\mathbb{R})$ contains as a linear subspace the real vectors of $\mathbb{R}^3$; the *geometric product* of two such elements encodes both their vector dot product and their vector cross product. The dot product and the cross product of real vectors can both be recovered from the geometric product, since these parts of the geometric product live in parts of the eight dimensional real linear space $\mathcal{C}\ell_{3,0}(\mathbb{R})$ spanned by disjoint sets of basis elements. The real linear subspace generated by $\beta_1$, $\beta_2$, $\beta_3$ is called the set of *bivectors*, just as the real linear subspace generated by $e_1$, $e_2$, $e_3$ is identified with the set of *vectors*. The real linear subspace generated by the single element $\mathbb{1}$ is identified with the real numbers or scalars and in particular, we identify the scalar 1 and the element $\mathbb{1}$, the scalar 0 and the element $\mathbb{0}$. The real linear subspace generated by the single element $M$ is called the set of *trivectors*, also known as *pseudo-scalars*.

So we have seen that $\mathcal{C}\ell_{3,0}(\mathbb{R})$ is a beautiful object, containing within itself the quaternions, the complex numbers, the two-by-two complex matrices, three dimensional real vectors; and its product, sometimes called "geometric product", contains within it both the inner and outer product of real vectors.

Moreover, everything comes in dual pairs: the vectors and the bivectors; the quaternions and the product of the quaternions with $M$, the scalars and the pseudo-scalars. Every element of the algebra can be split into four components: scalar, vector, bivector and pseudo-scalar. The algebra is called *graded.* The parts we just named constitute the zeroth, first, second and third grades of the algebra. The product of elements of grades $r$ and $s$ belong to the grade $r + s$ modulo 4.

The fact that $\mathcal{C}\ell_{3,0}(\mathbb{R})$ has as a representation the algebra of complex two-by-two matrices over the reals seems little noticed in the literature on geometric algebra. In the theory of Clifford algebra, it is just one tiny piece in the theory of classification of Clifford algebras in general. It seems to this author that this representation should help very much to "anchor" the theory for those coming from quantum theory and quantum foundations. The standard textbook on geometric algebra for physicists, Doran and Lasenby (2003), has two chapters on quantum mechanics, one of which is devoted specifically to quantum entanglement, using geometric algebra. However the reason why this works – the representation just mentioned – is not brought out explicitly.

## 2.1 Remarks on computation

Before continuing, I would like to make some remarks on computation. In order to test formulas, or in order to simulate models, it is convenient to have access to computer languages which can "do" computer algebra.

First of all, the fact that $\mathcal{C}\ell_{3,0}(\mathbb{R})$ can be identified with the two-by-two complex matrices means that one can implement geometric algebra as soon as one can "do" complex matrices. One needs to figure out how to get the eight coordinates out of the matrix. There are easy formulas but the result would be clumsy and involve a lot of programming at a low level. The relation with the quaternions is more promising: we can represent any element of $\mathcal{C}\ell_{3,0}(\mathbb{R})$ as a pair of quaternions, and the real and the imaginary parts of the two component quaternions immediately give us access to the scalar and bivector, respectively trivector (also called "pseudo scalar") and vector parts of the element of the algebra. For those who like to programme in the statistical language `R` it is good to know that there is an `R` package called "`onion`" which implements both the quaternions and the octonions.

Here a small digression starts. There exists a huge menagerie of somewhat related mathematical objects, and sometimes notations and nomenclature are confused and confusing. For instance, the octonions are another eight dimensional algebra which however is not associative. There are weird and wonderful and deep connections between these various objects and in particular, concerning the sequence 1, 2, 4, 8, which is where a particular construction taking us from the reals to the complex numbers to the quaternions and finally to the octonions stops. But this is not part of our story. Another Clifford algebra $\mathcal{C}\ell_{0,3}(\mathbb{R})$ is also related to pairs of quaternions, but also not part of our story. End of digression.

Nicest of all is to use a computer system for doing real geometric algebra, and for this purpose I highly recommend the programme `GAViewer` which accompanies the book Dorst, Fontijne and Mann (2007). It can be obtained from the authors' own book website `http://www.geometricalgebra.net`. It not only does geometric algebra computations, it also visualises them, i.e., connects to the associated geometry. Moreover the book and the programme are a nice starting point for two higher dimensional geometric algebras, of dimension 16 and 32 respectively, which encode more geometric objects (for instance, circles and affine subspaces), and more geometric operations, with the help of the further dimensions of the algebra.

# 3 Christian's first model

Christian (2007) takes two and a half pages of preliminaries before he starts describing his (allegedly) local realist model for the singlet (or EPR-B) correlations, obtained through the device of taking the co-domain of Bell's measurement functions to be elements of the geometric algebra $\mathcal{C}\ell_{3,0}(\mathbb{R})$ rather than the conventional (one dimensional) real line.

This insistence already reveals, to this writer's mind, that Christian does not know what he is talking about. Conventionally, a local hidden variables model for the singlet correlations consists of the following ingredients. First of all, there is a hidden variable $\lambda$ which is an element of some arbitrary space over which there is a probability distribution referred to in physicist's language sometimes as $\rho(\lambda)\mathrm{d}\lambda$, sometimes as $\mathrm{d}\rho(\lambda)$. This hidden variable is often thought to reside in the two particles sent to the two measurement devices in the two wings of the experiment, and therefore to come from the source; but one can also think of $\lambda$ as an assemblage of classical variables in the source and in both particles and in both measurement devices which together determine the outcomes of measurement at the two locations. Any "local stochastic" hidden variables model can also be re-written as a deterministic local hidden variables model. This rewriting (thinking of random variables as simply deterministic functions of some lower level random elements) might not correspond to physical intuition but as a mathematical device it is a legitimate and powerful simplifying agent.

Secondly, we have two functions $A(a, \lambda)$ and $B(b, \lambda)$ which take the values

$\pm 1$ only, and which denote the measurement outcomes at Alice's and Bob's locations, when Alice uses measurement setting $a$ and Bob uses measurement setting $b$. Here, $a$ and $b$ are 3D spatial directions conventionally represented by unit vectors in $\mathbb{R}^3$. The set of unit vectors is of course also known as the unit sphere $S^2$.

Bell's theorem states that there do not exist functions $A$ and $B$ and a probability distribution $\rho$, on any space of possible $\lambda$ whatever, such that

$$\int A(a,\lambda)B(b,\lambda)\mathrm{d}\rho(\lambda) = -a \cdot b$$

for all $a$ and $b$ in $S^2$.

Christian claims to have a counter-example and the first step in his claim is that Bell "unphysically" restricted the co-domain of the functions $A$ and $B$ to be the real line. Now this is a curious line to take: we are supposed to assume that $A$ and $B$ take values in the two-point set $\{-1, +1\}$. In fact, the correlation between $A$ and $B$ in such a context is merely the probability of getting equal (binary) outcomes minus the probability of getting different (binary) outcomes. In other words: Bell's theorem is about measurements which can only take on two different values, and it is merely by convention that we associate those values with the *numbers* $-1$ and $+1$. We could just as well have called them "spin up" and "spin down". In the language of probability theory, we can identify $\lambda$ with the element $\omega$ of an underlying probability space, and we have two families of random variables $A_a$ and $B_b$, taking values in a two point set, *without loss of generality* the set $\{-1, +1\}$, and Bell's claim is: it is impossible to have $\mathrm{Prob}(A_a = B_b) - \mathrm{Prob}(A_a \neq B_b) = a \cdot b$ for all $a$ and $b$.

However, let us bear with Christian, and allow that the functions $A$ and $B$ might just as well be thought of as taking values in a geometric algebra ... as long as we insist that they each only take on two different values.

Christian used the symbol $\mathbf{n}$ to denote an arbitrary unit vector (element of $S^2$) and in formulas (15) and (16) makes the following bold suggestion:

$$A_{\mathbf{n}} = B_{\mathbf{n}} = \boldsymbol{\mu} \cdot \mathbf{n} \cong \pm 1 \in S^2$$

where $\boldsymbol{\mu} \cdot \mathbf{n}$ has been previously defined to be $\pm M\mathbf{n}$ (I use the symbol $M$ instead of Christian's $I$). Christian talks about the dot here standing for a "projection" referring, presumably, to the duality between vectors and bivectors. Christian sees $\boldsymbol{\mu}$ as his local hidden variable, giving us the story that space itself picks at random a "handedness" for the trivector $M = e_1e_2e_3$, thought of as a *directed volume element.* This story is odd; after all, the "handedness" of $e_1e_2e_3$ is merely the expression of the sign of the evenness of the permutation 123. Of course, the multiplication rule of geometric algebra, bringing up the *cross product* does again involve a handedness convention: but this is nothing to do with physics, it is only to do with mathematical convention, i.e., with book-keeping.

But anyway, within Christian's "model" as we have it so far, we can just as well define $\lambda$ to be a completely random element of $\{-1,+1\}$, and then define $\boldsymbol{\mu} = \lambda M$. The resulting probability distribution of $\boldsymbol{\mu}$ is the same; we have merely changed some names, without changing the model.

So now we have the model

$$A(\mathbf{a},\lambda) = \lambda M\mathbf{a}, \quad B(\mathbf{b},\lambda) = \lambda M\mathbf{b}$$

which says that the two measurement functions have outcomes in the set of pure (unit length) bivectors. Now, those two sets are both isomorphic to $S^2$, and that is presumably the meaning intended by Christian when using the congruency symbol $\cong$: our measurements can be thought of as taking values in $S^2$. At the same time, each measurement takes on only one of two different values (given the measurement direction) hence we can also claim congruency with the two point set $\{-1,+1\} = \{\pm 1\}$. But of course, these are two different congruencies! And they still need to be sorted out. What is mapped to what, exactly ...

This is where things go badly wrong. On the one hand, the model is not yet specified, if Christian does not tell us how, exactly, he means to map the set of two possible values $\pm M\mathbf{a}$ onto $\{\pm 1\}$ and how he means to map the set of two possible values $\pm M\mathbf{b}$ onto $\{\pm 1\}$. On the other hand, Christain proceeds blithely to compute a correlation between bivector valued outcomes instead of between $\{\pm 1\}$ valued outcomes! No model, wrong correlation!

Let us take a look at each disaster separately. Regarding the first disaster, there are actually only two options, since Christian already essentially told us that the two values $\pm 1$ of (my) $\lambda$ are equally likely. Without loss of generality, his model (just for these two measurement directions) becomes *either*

$$A(\mathbf{a},\lambda) = B(\mathbf{b},\lambda) = \lambda = \pm 1$$

*or*

$$-A(\mathbf{a},\lambda) = B(\mathbf{b},\lambda) = \lambda = \pm 1$$

and hence his correlation is either $+1$ or $-1$, respectively.

Regarding the second disaster, Christian proceeds to compute the geometric product $(\boldsymbol{\mu}\cdot\mathbf{a})(\boldsymbol{\mu}\cdot\mathbf{b})$ (later, he averages this over the possible values of $\mu$). Now as we have seen this is equal to $(\lambda M\mathbf{a})(\lambda M\mathbf{b}) = \lambda^2 M^2 \mathbf{ab} = -\mathbf{a}\cdot\mathbf{b} - M(\mathbf{a}\times\mathbf{b})$ and therefore certainly *not* equal to $-\mathbf{a}\cdot\mathbf{b} - \lambda M(\mathbf{a}\times\mathbf{b}) = -\mathbf{a}\cdot\mathbf{b} - \boldsymbol{\mu}(\mathbf{a}\times\mathbf{b})$.

Looking at these few lines of Christian (2007), from equation (16) to equation (17), we see six lines of pure nonsense in the middle of a four page paper professing to revolutionise our understanding of Bell's theorem and thereby revolutionise our understanding of quantum information and quantum computation. I wonder if many of the readers of the paper actually read those half a dozen lines carefully (and knew enough geometric algebra to decode them)?

## 4 Christian's second model

I next would like to take the reader to Christian's one page paper, Christian (2011), simultaneously the main material of the first chapter of his book Christian (2014).

It seems clear to me that by 2011, Christian himself has realised that his "model" of 2007 was incomplete. There was no definition of the the measurement functions! So now he does come up with a model, and the model is astoundingly simple ... it is identical to my second model:

$$A(\mathbf{a},\lambda) = -B(\mathbf{b},\lambda) = \lambda = \pm 1.$$

However, he still needs to get the singlet correlation from this, and for that purpose, he daringly *redefines* correlation, by noting the following: associated with the unit vectors $\mathbf{a}$ and $\mathbf{b}$ are the unit bivectors (in my notation) $M\mathbf{a}$ and $M\mathbf{b}$. As purely imaginary elements of the bivector algebra or quaternions, these are square roots of minus one, and we write

$$A(\mathbf{a},\lambda) = (M\mathbf{a})(-\lambda M\mathbf{a}) = \lambda$$

$$A(\mathbf{b},\lambda) = (\lambda M\mathbf{b})(M\mathbf{b}) = -\lambda$$

where $\lambda$ is a "fair coin toss", i.e., equal to $\pm 1$ with equal probabilities $\frac{1}{2}$.

Now the cunning device of representing these two random variables as products of fixed bivectors and random bivectors allows Christian to make the brilliant move of computing the Pearson bivector correlation between $A$ and $B$ by dividing the mean value of the product, by the non random "scale quantities" $M\mathbf{a}$ and $M\mathbf{b}$.

Since the geometric product is non commutative, one must be careful about the order of these operations, but I will follow Christian in what does seem a natural choice.

Unfortunately, since $AB = -1$ with probability one, the Christian-Pearson correlation should be $-(M\mathbf{a})^{-1}(M\mathbf{b})^{-1} = -(M\mathbf{a})(M\mathbf{b}) = -\mathbf{a}\cdot\mathbf{b} - M(\mathbf{a}\times\mathbf{b})$, just as before! However, just as in the 2007 paper, Christian succeeds again in 2010 in getting a sign wrong, so as to "erase" the unwanted bivector cross-product term from the "correlation". I have elsewhere analysed where he went wrong and put forward an explanation why (ambiguous notation and sloppy terminology). It should be noted that he also hides the sign error somewhat deeper in complex computations, looking at the average of a large number of realisations and using the law of large numbers, rather than just computing the expectation "theoretically".

As I have made clear, the model was madness from the start, whether we would end up with the right or the wrong answer is almost quite irrelevant, but it is amusing to see that on the one hand, Christian seems to have felt that his 2007 model was incomplete; he came up with a "fantastic" original solution to the quandary, but still ran aground on the same algebraic errors.

I will not discuss the other versions of Christian's theory. He has elaborated more or less the same "theory" with the same repertoire of conceptual and algebraic errors in various papers with increasing levels of complexity. The adjective "Byzantine" is appropriate. This work shows a remarkable degree of dedication and persistance, and erudition too, as more and more complex mathematical constructions are brought into play. If only (for his own sake) the author had had the intelligence to recognise the mistakes he had made at the outset.

# 5 Post-mortem

As I outlined at the beginning, apart from the fun of learning geometric algebra, the fascinating aspects of the Christian story are surely the psychological and sociological aspects. How could a promising young researcher become so convinced that he was some kind of new Einstein? And why did anyone take any notice at all?

Regarding the first issue, it would be improper to discuss the personality of an author of a failed master-piece. What we can clearly see is however a lack of mathematical discipline. A young physicist specialising in the foundations of quantum mechanics can easily, it appears, lack any training in mathematical modelling, by which I refer to the analysis of mathematical structures inspired by structures perceived in nature, but carried out strictly on its own terms, and carried out with mathematical rigour (strictness, discipline). There is no need to be pedantic but there is a need to be precise. It seems that mathematical terms are used carelessly and vaguely; there is no real appreciation of the reality of the abstract structures of the mind which mathematics gives us.

Regarding the second issue, it seems that the same lack of familiarity with mathematical discipline is to blame, and the same reliance on the words; the formulas are seen as decoration. Anyone who could read the mathematical formulas of the 2007 paper could see that the paper was nonsense. No prior knowledge of geometric algebra was needed to spot the conceptual error at the heart of the paper. But one must be prepared to take a mathematical formula seriously. One must also share the point of view that the mathematics must stand on its own feet, without the verbiage around. One can spot the mathematical inconsistency at the heart of the 2007 paper on formal grounds, without knowing the mathematical definitions and properties of the various objects in those key formulas: $-1$ squared is equal to $+1$. There is no way after multiplying *twice* by $\mu$, which carries a random sign, that that random sign persists in one part of two terms added together on the right hand side of the equation.

The early critics of Christian (and there were many) did not spot these obvious conceptual and algebraic errors. In 2012, many people were dumbfounded when I pointed out in Gill (2012) that Christian's (2011) model started with the definition $-A = B = \pm 1$. But this was written clearly in the first lines of the 2011 paper. On the other hand, it seemed that no-one noticed that the "model" of 2007 was not a model at all, though the only way open to Christian to "complete" it was to add to it the definition $-A = B = \pm 1$. So Christian could keep active and feed a lot of interest for quite a few years.

If only it had been a new Sokal hoax: it would have been wonderful.

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