A derivation of the Etherington's distance-duality equation

Yuri Heymann

3 rue Chandieu, 1202 Geneva, Switzerland.

E-mail: y.heymann@yahoo.com

(Dated: May 28, 2015)

Abstract

The Etherington's distance-duality equation is the relationship between the luminosity distance of standard candles and the angular-diameter distance. This relationship has been validated from astronomical observations based on the X-ray surface brightness and the Sunyaev-Zel'dovich effect of galactic clusters. In the present study, we propose a derivation of the Etherington's reciprocity relation in the dichotomous cosmology.

I. INTRODUCTION

The Etherington's distance-duality equation was introduced in 1933 [1]. Etherington mentioned this equation was proposed by Tolman as a way to test a cosmological model. Ellis proposed a proof of this equation in the context of Riemannian geometry [2, 3]. A quote from Ellis [3]: "The core of the reciprocity theorem is the fact that many geometric properties are invariant when the roles of the source and observer in astronomical observations are transposed". This statement is fundamental in the reciprocity theorem as shown here in the derivation of the theorem in the dichotomous cosmology. While the proof of the Etherington's distance duality in the context of Riemannian geometry is tedious, the derivation in the dichotomous cosmology is straightforward. As a reminder, the dichotomous cosmology [4, 5] consists of a static matter universe with an expanding luminous world. One needs to imagine a cube of light expanding, in a space where galaxies do not recede from each other.

Fortunately, the Etherington's distance-duality equation, which is a crucial relationship in cosmology, can be verified from astronomical observations. While the luminosity distance is measured from supernova observations, the angular-diameter distance is determined from the X-ray surface brightness and the Sunyaev-Zel'dovich effect [6] of galactic clusters [7]. In [8], the authors found that the ratio between the two distances D_L for the luminosity distance and D_A for the angular-diameter distance, defined as $\eta = \frac{D_L}{D_A(1+z)^2}$ is bound to be $\eta = 1.01 \pm 0.07$ at 68% c.l. Similar results were obtained in [9, 10], where no significant violation of the distance-duality relationship was found. In [11], the authors tested the cosmic distance duality for different galactic cluster samples. The study [12] is focused on analytical expressions for the deformation of the distance duality in terms of the cosmic absorption parameter. The reciprocity theorem is considered to be true when photon number is conserved, gravity is described by a metric theory with photons traveling on unique null geodesics [13]. Any violation of the distance duality would be attributed to exotic physics.

Following the introduction in section I, the distance measurements are derived in section II. To derive the Etherington's reciprocity theorem in the dichotomous cosmology, we first need the distance measurements, which may be derived from the tired-light paradigm (section II.A) or from expanding metrics (section II.B). Both derivations lead to the same equations as shown in II.A and II.B. In section II.C, we derive the Etherington's distance duality using our distance measurements. Finally, we offer our conclusion in section III.

II. DERIVATION OF THE DISTANCE MEASUREMENTS

A. Derivation from tired-light paradigm

When a photon loses energy during its travel in space, the wavelength of light is stretched, and because the number of cycles of the light wave is conserved, an expansion of the luminous world is produced. As a consequence of this stretching of light, the velocity of the light wavefront increases during its travel (Fig. 1). According to special relativity, the speed of light is invariable. Hence, in order to maintain the light wavefront at the speed of light, the model introduces a time contraction between the emission point and the observer. The study [4] mentions a time-dilation; in order to rectify this, the model is based on a time contraction in the arrow of time.

Considering that photons lose energy as light gets stretched, the following equation is obtained:

$$1 + z = \frac{E(z)}{E_0}\,, \qquad (1)$$

where $E(z)$ is the photon energy when emitted, E_0 is the photon energy at reception, and z is the redshift.

A simple decay law of the photon energy is considered:

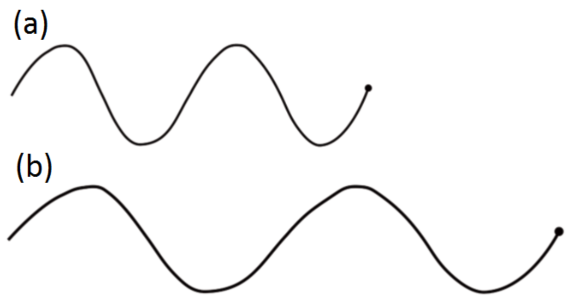

FIG. 1. Where (a) is the light wavefront without stretching, and (b) with stretching. We can see that in (b) the light wavefront is going faster than in (a).

$$\frac{\dot{E}}{E} = -H_0 \,, \tag{2}$$

where H_0 is the Hubble constant.

Therefore

$$E(t) = E_0 \exp(-H_0 t) \,, \tag{3}$$

and

$$E(T) = E_0 \exp(H_0 T) \,, \tag{4}$$

where t is the time which is equal to zero at the time of observation, and T the light travel time of the source from the observer.

A set of two transformations is considered: first a time-variable light wavefront to accomodate the expansion of the luminous world, and second a time contraction to maintain the light wavefront at the speed of light.

1. Light wavefront with respect to the emission point

The light wavefront velocity before time contraction is expressed as follows:

$$v(t) = c\frac{E_{emit}}{E(t)} \,. \tag{5}$$

To maintain the light wavefront at the speed of light, the following time contraction is applied:

$$\frac{\delta t'}{\delta t} = \frac{E_{emit}}{E(t)} \,. \tag{6}$$

Hence, the light travel time with respect to the source is:

$$T' = \int_{-T}^{0} \frac{\delta t'}{\delta t} dt = \int_{-T}^{0} \frac{E_{emit}}{E(t)} dt \,. \tag{7}$$

Introducing (3) in the previous equation and integrating, we get:

$$T' = \frac{E_{emit}}{E_0} \frac{1}{H_0} \left(1 - \frac{E_0}{E_{emit}}\right) \,. \tag{8}$$

Introducing (1) in the previous equation, we get:

$$T' = \frac{z}{H_0} \,, \tag{9}$$

which is the light travel time measurement for the luminosity distance.

2. *Light wavefront with respect to the observer*

The light wavefront velocity before time contraction is expressed as follows:

$$v(t) = c\frac{E_0}{E(t)} \,. \tag{10}$$

To maintain the light wavefront at the speed of light, the following time contraction is applied:

$$\frac{\delta t_0}{\delta t} = \frac{E_0}{E(t)} \,. \tag{11}$$

Hence, the light travel time with respect to the observer is:

$$T_0 = \int_{-T}^{0} \frac{\delta t_0}{\delta t} dt = \int_{-T}^{0} \frac{E_0}{E(t)} dt \,. \tag{12}$$

Introducing (3) in the previous equation and integrating, we get:

$$T_0 = \frac{1}{H_0} \left(1 - \exp(-H_0 T)\right) \,. \tag{13}$$

Introducing (4) in the previous equation, we get:

$$T_0 = \frac{1}{H_0} \left(1 - \frac{E_0}{E_{emit}}\right) \,. \tag{14}$$

Introducing (1) in the previous equation, we get:

$$T_0 = \frac{1}{H_0} \frac{z}{(1+z)} \,, \tag{15}$$

which is the light travel time measurement for the actual distance.

B. Derivation from expanding metrics

In the dichotomous cosmology, the luminous world is expanding; therefore, we can derive the distance measurements using expanding metrics.

1. *Luminosity distance*

The luminosity distance is the distance measured from the luminosity of standard candles. Supernovae Ia are considered stanard candles, meaning they all have the same absolute brightness when they explode. From their apparent brightness, we can deduce the luminosity distance, because the brightness diminishes proportionally to the inverse of the distance squared. The formula used to measure the luminosity distance is the distance modulus equation.

By considering a photon travelling away from the center of a supernova, the luminosity distance is calculated as follows:

$$\frac{dr_L}{dt} = c + H_0 r_L \,, \tag{16}$$

where r_L is the luminosity distance, H_0 the Hubble constant, and c the speed of light.

By integrating this equation between 0 and T, we get:

$$r_L = \frac{c}{H_0} \left(\exp(H_0 T) - 1\right) \,. \tag{17}$$

Because $\frac{da}{dt} = H_0 a$, we get $dt = \frac{da}{H_0 a}$, where a is the scale factor. In addition, the relationship between the scale factor and the redshift is given by the cosmological redshift equation $(1+z) = \frac{1}{a}$, where the scale factor is equal to one at present time.

Hence, the light travel time versus redshift is as follows:

$$T = \int_{1/(1+z)}^{1} \frac{da}{H_0 a} = \frac{1}{H_0} \ln(1+z)\,. \qquad (18)$$

Equations (17) and (18) yield:

$$r_L = \frac{c}{H_0} z\,. \qquad (19)$$

which is identical to (9) with $r_L = cT'$

2. *Euclidean distance*

A measurement of the distance is obtained by calculating the corresponding distance if there were no expansion, which we call the Euclidean distance. Let us introduce y to this distance measurement. By considering a photon moving towards the observer, we get:

$$\frac{dy}{dt} = -c + H_0 y\,. \qquad (20)$$

By setting time zero at a reference T_b in the past, we get: $t = T_b - T$; therefore, $dt = -dT$ (where T is the light travel time when looking at a source into the past). Hence:

$$\frac{dy}{dT} = c - H_0 y\,, \qquad (21)$$

with boundary condition $y(T=0) = 0$.

Integrating this equation between 0 and T, we get:

$$y = \frac{c}{H_0}\left(1 - \exp(-H_0 T)\right)\,. \qquad (22)$$

By substitution of (18) into (22), we get:

$$y = \frac{c}{H_0}\frac{z}{(1+z)}\,, \qquad (23)$$

which is identical to (15) with $y = cT_0$.

C. Etherington's distance duality

From (19) and (23), we get:

$$r_L = (1+z)y\,. \qquad (24)$$

The angular-diameter distance d_A of an object is defined in terms of x, the object's actual size, and θ, the angular size of the object as viewed from earth. The equation is as follows:

$$d_A = \frac{x}{\theta}\,. \qquad (25)$$

Because of the expansion of the luminous world, the apparent size of celestial objects is stretched by a factor (1+z), and the apparent angular size is increased by the same factor. Hence, the relationship between the actual distance y and the angular-diameter distance is as follows:

$$y = (1+z)d_A\,. \qquad (26)$$

Equations (24) and (26) yield:

$$r_L = (1+z)^2 d_A \,, \tag{27}$$

which is the Etherington's distance duality relationship. We have just derived the Etherington's reciprocity theorem.

III. CONCLUSION

The Etherington's distance-duality equation, which relates the luminosity distance of standard candles to the angular-diameter distance, is a crucial relationship in cosmology. Although the Etherington's reciprocity theorem is considered to be peculiar to cosmological models based on Riemannian geometry, in the present study we propose a new derivation of this relationship in the dichotomous cosmology. This derivation is straightforward and follows naturally from the dichotomous cosmology. Today, the Etherington's reciprocity theorem is considered established and has been verified using astronomical observations based on X-ray surface brightness and the Sunyaev-Zel'dovich effect of galactic clusters.

[1] I. M. H. Etherington, Philos. Mag. **15**, 761 (1933).

[2] G. F. R. Ellis, Proceedings of the 47th International School of Physics 'Enrico Fermi', edited by R. K. Sachs (Academic Press, New York and London) **15**, 104 (1971).

[3] G. F. R. Ellis, Gen. Relativ. Gravit. **39**, 1047 (2007).

[4] Y. Heymann, Progress in Physics **10(3)**, 217 (2014).

[5] Y. Heymann, Progress in Physics **10(4)**, 178 (2014).

[6] R. A. Sunyaev and Y. B. Zel'dovich, Comm. Astrophys. Space Phys. **4**, 173 (1972).

[7] J. Silk and S. D. M. White, The Astrophysical Journal Letters **226**, L103 (1978).

[8] F. Bernardis, E. Giusarma, and A. Melchiorri, International Journal of Modern Physics D **15**, 759 (2006).

[9] J.-P. Uzan, N. Aghanim, and Y. Mellier, Physical Review D **70**, 083533 (2004).

[10] R. Nair, S. Jhingan, and D. Jain, Preprint Archiv:1210.2642 [astro-ph.CO] (2013).

[11] R. S. Gonalves, R. F. L. Holanda, and J. S. Alcaniz, Monthly Notice Letters of the Royal Astronomical Society **420**, L43 (2012).

[12] J. A. S. Lima, J. V. Cunha, and V. T. Zanchin, The Astrophysical Journal Letters **742**, L26 (2011).

[13] B. A. Bassett and M. Kunz, Physical Review D **69**, 101305 (2004).