Two results on ZFC:

(1) If ZFC is consistent then it is deductively incomplete,

(2) ZFC is inconsistent

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June 18 & 19 2015

Abstract

The Zermelo-Fraenkel-Axiom-of-Choice (ZFC) axioms for set theory appear to be inconsistent. A step in developing this proof is the observation that ZFC would be deductively incomplete if it were consistent. Both points are proven by means of the singleton. The axioms are still too lax on the notion of a 'well-defined set'.

Keywords: Paul of Venice • Russell's Paradox • Cantor's Theorem • ZFC • naive set theory • well-defined set • set of all sets • diagonal argument • transfinites

MSC2010: 03E30 Axiomatics of classical set theory and its fragments

03E70 Nonclassical and second-order set theories

97E60 Mathematics education: Sets, relations, set theory

Contents

1. Introduction

1.1. The problem

The Zermelo-Fraenkel-Axiom-of-Choice (ZFC) axioms for set theory are studied here with a focus on the singleton. Section 2 defines the case. Section 3 shows deductive incompleteness, i.e. there is a truth that cannot be derived. Section 4 derives this truth and thus shows inconsistency. Section 5 discusses the results. Section 6 concludes.

This introduction proceeds with basic definitions and theorems.

1.2. Definition of ZFC

We take our definitions from a matricola course in set theory at Leiden and Delft.

Definition (Coplakova et al. (2011:18), I.4.7): Let *A* be a set. The power set of *A* is the set of all subsets of *A.* Notation: *P*[*A*]. Another notation is 2^A, whence its name.

Definition (Coplakova et al. (2011:144-145)): ZFC.

Comment: This includes the axiom that each set has a power set. (POW)

Definition of the *Axiom of Separation* (Coplakova et al. (2011:145), inserting here a by-line on freedom): If *A* is a set and $\gamma[x]$ is a formula with variable *x,* while *B* is not free in $\gamma[x]$, then there exists a set *B* that consists of the elements of *A* that satisfy $\gamma[x]$:

$$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \ \Leftrightarrow\ ((x \in A)\ \&\ \gamma[x])) \qquad \text{(SEP)}$$

Comment: This is also called an axiom-schema since there is no quantifier on γ.

1.3. Cantorian sets in ZFC

Definition of a Cantorian set. Let *A* be a set, *P*[*A*] its power set. Consider function $f : A \to P[A]$. A strictly Cantorian set is $\Psi = \{x \in A \mid x \notin f[x]\}$. Observe that the latter depends upon f, i.e. $\Psi = \Psi[f]$. A generalized Cantorian set has $x \notin f[x]$ as part of its definition. The meaning of 'Cantorian set' without qualification depends upon the context.

Theorem 1.3.A. Existence of a strictly Cantorian set. Let *A* be a set, *P*[*A*] its power set. For every function $f : A \to P[A]$ there is a strictly Cantorian set.

Proof: (a) *P*[*A*] exists because of the Axiom of the Power set. (b) f can be regarded as a subset of *A* x *P*[*A*], and f exists because of Axiom of Pairing. (c) Ψ exists because of the Axiom of Separation. Find $\Psi \subseteq A$, thus $\Psi \in P[A]$. Q.E.D.

Comment: When $\alpha \in A$ then we can use $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$.

Theorem 1.3.B. Weakest Conjecture on strictly Cantorian sets. Let *A* be a set. For every $f : A \to P[A]$ there is a $\Psi \in P[A]$ such that for all $\alpha \in A$ it holds that $\Psi \neq f[\alpha]$.

Proof: Define $\Psi = \{x \in A \mid x \notin f[x]\}$. Take $\alpha \in A$. Check the two possibilities.

Case 1: $\alpha \in \Psi$. In this case $\alpha \notin f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in \Psi \setminus f[\alpha]$.)

Case 2: $\alpha \notin \Psi$. In this case $\alpha \in f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in f[\alpha] \setminus \Psi$.) Q.E.D.

Comment: This theorem-conjecture combines various issues: the definition of strictly Cantorian sets, the existence proof and an identification of their key property. It is essentially a rewrite of the definition $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$ for $\alpha \in A$.

1.4. Appendices

The author has some experience in logic, see Colignatus (1981, 2007, 2011) "*A Logic of Exceptions"* (**ALOE**), and see for background Colignatus (2013). This present paper is derived from Colignatus (2014b, 2015) (PV-RP-CDA-ZFC) on a condition by Paul of Venice (1369-1429). It must be observed that this author is no expert on ZFC. Thus this paper is essentially based upon logic. ALOE in 1981 applied the Paul of Venice consistency condition to the Russell set (p129), and applied it in 2007 (p239) also to Cantor's (diagonal) argument (in Russell's version for the power set). ALOE does not develop the formal ZFC system of axioms for set theory. ALOE's discussion may be seen as intermediate between naive set theory and this present paper. **Appendix A** discusses the versions of ALOE, for proper reference. **Appendix B** has more on the genesis of this paper.

2. The singleton

2.1. The singleton with a nutshell link between Russell and Cantor

Let A be a set with a single element, $A = \{\alpha\}$. Thus $P[A] = \{\emptyset, A\}$. Let $f: A \to P[A]$.

If $f[\alpha] = \emptyset$ then $\alpha \notin f[\alpha]$. If $f[\alpha] = A$ then $\alpha \in f[\alpha]$. Thus $f[\alpha] = \emptyset \Leftrightarrow \alpha \notin f[\alpha]$. Consider:

(1) In steps: define $\Psi = \{x \in A \mid x \notin f[x]\}$, find $\Psi \in P[A]$, then try $f[\alpha] = \Psi$.

(2) Directly: $f[\alpha] = \{x \in A \mid x \notin f[x]\}$

(3) Either directly or indirectly via (1) or (2): $\Psi = \{x \in A \mid x \notin \Psi\}$.

The latter is a variant of Russell's paradox: $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin \Psi)$.

Thus (1) - (3) are only consistent when $\Psi \neq f[\alpha]$. This is an instance of **Theorem 1.3.B**.

Choosing $f[\alpha] = \Psi$ in (1) assumes freedom that conflicts with the other properties.

We have liberty to choose $f[\alpha] = \emptyset$ or $f[\alpha] = A$. This choice defines f and we should write $\Psi = \Psi[f]$. This shows why (2) with $f[\alpha] = \Psi[f]$ is tricky. If (2) is an implicit definition of f then it doesn't exist. If it exists then this $f[\alpha]$ will not be in its definition.

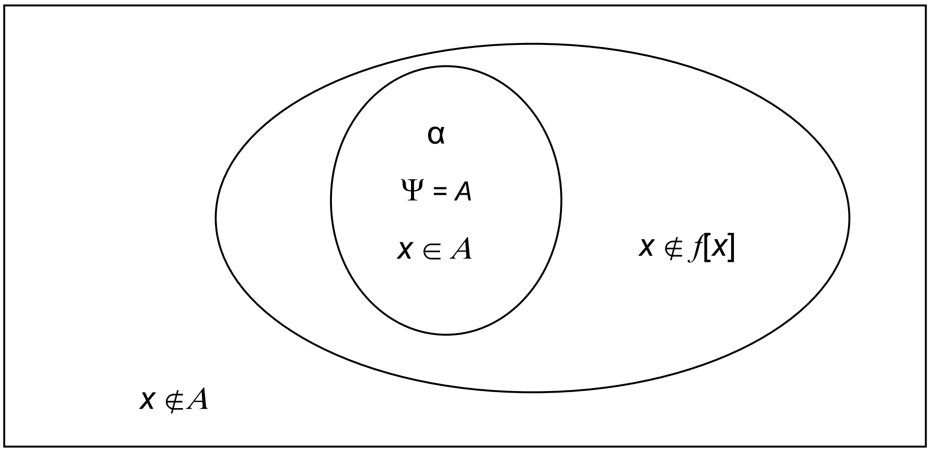
2.2. Possibilities for the singleton

Checking all possibilities in the former subsection gives Table 1. The cells are labeled with Δ-*case-numbers*. The Δ refers to a *difference analysis* when a set is extended with single element. Because of $\alpha \notin f[\alpha]$, the row $f[\alpha] = \emptyset$ is important for us. The case of Δ2 is depicted in a Venn-diagram in Figure 1.

Table 1. Test of the singleton: $\alpha \in \Psi$? **via** $\alpha \in \Psi \Leftrightarrow \alpha \notin f[\alpha]$

For all cases: $\alpha \in A$	$\Psi = \emptyset$, $\alpha \notin \Psi$	$\Psi = A$, $\alpha \in \Psi$
$f[\alpha] = \emptyset$ $\alpha \notin f[\alpha]$	**Δ1**: $\alpha \in \emptyset \Leftrightarrow \alpha \notin \emptyset$ $f[\alpha] = \Psi$, **impossible**	**Δ2**: $\alpha \in A \Leftrightarrow \alpha \notin \emptyset$ $f[\alpha] \neq \Psi$, possible
$f[\alpha] = A$ $\alpha \in f[\alpha]$	**Δ3**: $\alpha \in \emptyset \Leftrightarrow \alpha \notin A$ $f[\alpha] \neq \Psi$, possible	**Δ4**: $\alpha \in A \Leftrightarrow \alpha \notin A$ $f[\alpha] = \Psi$, **impossible**

Figure 1. Diagram of the Cantorian set for the singleton, case Δ2: $f[\alpha] = \varnothing \neq \Psi$



3. Deductive incompleteness

3.1. Existence of Δ1

An idea is that Ψ in Theorems 1.3.AB or Table 1 covers all $\alpha \notin f[\alpha]$. This appears to be false: it doesn't cover Δ1. The cell is declared *impossible*. Let us first verify that it exists as a truth (outside of ZFC), and then accept deductive incompleteness.

Theorem 3.1. Case Δ1 exists as a possibility with ($\alpha \notin f[\alpha]$).

Proof: We consider the case $f[\alpha] = \varnothing$, so that ($\alpha \notin f[\alpha]$).

Take *q* = ($\alpha \notin f[\alpha]$) and use tautology T1: $q \Rightarrow (p \Leftrightarrow (q \,\&\, p))$ for any *p*, see Table 2.

Table 2. Truthtable for a singleton Cantorian set, with $q \Rightarrow (p \Leftrightarrow (q \,\&\, p))$

Case	$\alpha \notin f[\alpha]$	$\Rightarrow$	(*p*	$\Leftrightarrow$	($\alpha \notin f[\alpha]$	&	*p*))
Δ2	1	1	1	**1**	1	1	1
Δ4	0	1	1	**0**	0	0	1
Δ1	1	1	0	**1**	1	0	0
Δ3	0	1	0	**1**	0	0	0

We are free to take *p* = ($\alpha \in A$) or not-*p* = ($\alpha \in \varnothing$). Take the latter, apply modus ponens on *q* and tautology T1, and find $\alpha \in \varnothing \Leftrightarrow (\alpha \notin \varnothing \,\&\, \alpha \in \varnothing)$. The equivalence reduces into $\alpha \notin \varnothing$ or $\alpha \in A$. The description is consistent, so that it is possible. Taking both $f[\alpha] = \varnothing$ and $\Psi = \varnothing$, which is case Δ1, fits this relation: $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \,\&\, \alpha \in \Psi)$. Q.E.D.

Comment: The case pops up by requiring that *all* properties of the case hold simultaneously, thus jointly $(\alpha \notin \varnothing \,\&\, \alpha \in \varnothing)$ and not only $(\alpha \notin \varnothing)$. See Figure 2.

3.2. Definition, theorem and proof

Definition (DeLong (1971:132)): "A formal system is *deductively complete* if under the intended interpretation there is no truth which is not also a theorem."

Theorem 3.2. If ZFC is consistent then it is deductively incomplete.

Proof: Let *A* = {α} have a single element. Thus *P*[*A*] = {∅, *A*}.

Let $f: A \to P[A]$ with $f[\alpha] = \varnothing$. Then $\alpha \notin \varnothing$ and $\alpha \notin f[\alpha]$.

Under the intended interpretation, there is the case Δ1 that has $\alpha \notin f[\alpha]$.

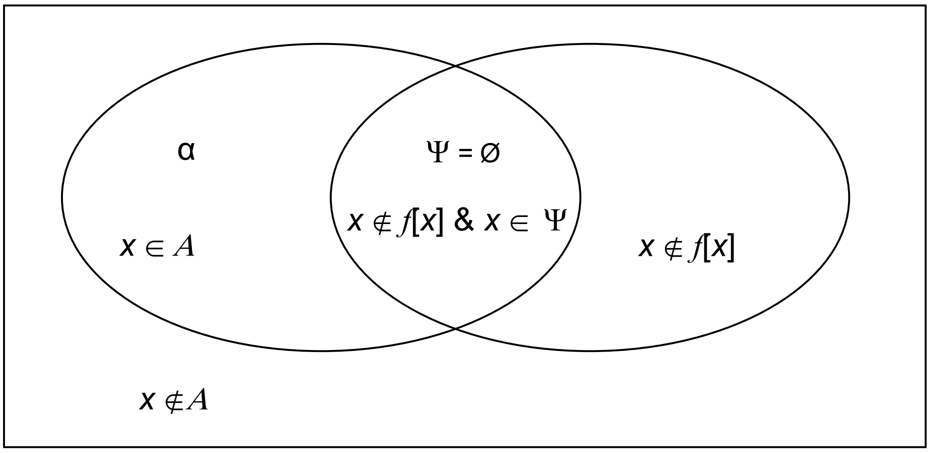
Ψ is formulated such that it should contain *all cases* with $\alpha \notin f[\alpha]$.

However, trying to that prove Δ1 fits Ψ, causes a shift to Δ2 or $\Psi = A$ (**Theorem 1.3.B**).

If ZFC is consistent then there is no path to reach Δ1. Q.E.D.

Comment: If there is such a path then ZFC becomes inconsistent.

Figure 2. Diagram of the Cantorian set for the singleton, case Δ1: $f[\alpha] = \varnothing = \Psi$



4. Inconsistency

4.1. An implication for the singleton Cantorian set

Consider the definition $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$. It is possible to weaken this by means of another tautology T2: $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$. The truthtable for the singleton Cantorian set is in Table 3. The truthtable holds for every f while $\Psi = \Psi[f]$.

Theorems 1.3.AB establish the LHS. Modus ponens with T2 gives the RHS as a *separate* expression - provided that we maintain the original $\Psi = \Psi[f]$:

$$\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \ \&\ \alpha \in \Psi).$$

Consider again $f[\alpha] = \varnothing$. The equivalence on the LHS only allows a solution $\Psi = A$. Look at row Δ1. On the LHS we have Δ1 with $(\alpha \notin \Psi)$ & $(\alpha \notin f[\alpha])$, and the equivalence would declare this combination impossible. However, there is also the relaxed condition on the RHS, that we already encountered in **Theorem 3.1.**

Table 3. Truthtable for a singleton Cantorian set, with $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$

Case	$(\alpha \in \Psi$	$\Leftrightarrow$	$\alpha \notin f[\alpha])$	$\Rightarrow$	$(\alpha \in \Psi$	$\Leftrightarrow$	$(\alpha \notin f[\alpha]$	&	$\alpha \in \Psi))$
Δ2	1	**1**	1	1	1	**1**	1	1	1
Δ4	1	**0**	0	1	1	**0**	0	0	1
Δ1	0	**0**	1	1	0	**1**	1	0	0
Δ3	0	**1**	0	1	0	**1**	0	0	0

For the RHS, that exists separately, we get Table 4. The same Δ-*case-numbers* apply.

Now Δ1 is allowed too: a possible $f[\alpha] = \Psi$ rather than an impossible $f[\alpha] = \Psi$.

Table 4. Test of the singleton: $\alpha \in \Psi$ **? via** $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \ \& \ \alpha \in \Psi)$

For all cases: $\alpha \in A$	$\Psi = \varnothing$, $\alpha \notin \Psi$	$\Psi = A$, $\alpha \in \Psi$
$f[\alpha] = \varnothing$ $\alpha \notin f[\alpha]$	$\alpha \in \varnothing \Leftrightarrow (\alpha \notin \varnothing \ \& \ \alpha \in \varnothing)$ $f[\alpha] = \Psi$, **possible**: $\alpha \notin \varnothing$	$\alpha \in A \Leftrightarrow (\alpha \notin \varnothing \ \& \ \alpha \in A)$ $f[\alpha] \neq \Psi$, possible
$f[\alpha] = A$ $\alpha \in f[\alpha]$	$\alpha \in \varnothing \Leftrightarrow (\alpha \notin A \ \& \ \alpha \in \varnothing)$ $f[\alpha] \neq \Psi$, possible	$\alpha \in A \Leftrightarrow (\alpha \notin A \ \& \ \alpha \in A)$ $f[\alpha] = \Psi$, **impossible**: $\alpha \notin A$

Note that $f[\alpha] = \varnothing$ doesn't give a unique Ψ now. Both $\Psi = \varnothing$ (Figure 2) and $\Psi = A$ Figure 1) are possible. Note that f is still a function and no correspondence.

PM. We can also gain access to Δ4 by another relaxing condition but we are interested in the $\alpha \notin f[\alpha]$ case.

4.2. A counterexample for Theorem 1.3.B

Let us make the latter observations formal. The discovery of Δ1 and tautology T2 gives a contradiction to Theorem 1.3.B. While Theorem 3.2 did not see a path towards Δ1, we now found that path, namely tautology T2, which gives Theorem 4.2.A. When Δ1 not merely exists as a truth outside of ZFC but also can been proven from Ψ, it becomes a counterexample for Theorem 1.3.B.

Theorem 4.2.A. For the singleton Cantorian case there are a f and Ψ with $f[\alpha] = \Psi$.

Proof: Let $A = \{\alpha\}$ have a single element. Thus $P[A] = \{\varnothing, A\}$.

Let $f: A \rightarrow P[A]$ with $f[\alpha] = \varnothing$. Then $\alpha \notin \varnothing$ and $\alpha \notin f[\alpha]$.

Consider $\Psi = \{x = \alpha \mid x \notin f[x]\}$ or for convenience $\Psi = \{x \in A \mid x \notin f[x]\}$. Look at Table 3. Use $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$ and tautology $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$, and apply modus ponens to find $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \ \& \ \alpha \in \Psi)$. In this deduction we have maintained the definition of Ψ. The modus ponens is independent of the possibility that also $\Psi = A$ might be derived via another route. The formula now *stands as a separate relation* for Ψ.

For this relation (also on the RHS of Table 3), we find that $f[\alpha] = \varnothing = \Psi$ can remain consistent. We have the same case $\alpha \in \varnothing \Leftrightarrow (\alpha \notin \varnothing \ \& \ \alpha \in \varnothing)$ that we saw above, which reduces to $\alpha \notin \varnothing$, or $\alpha \in A$. Thus we have $f[\alpha] = \varnothing = \Psi$ not only as a truth outside of ZFC (tautology T1) but also as provable from the definition of Ψ (tautology T2).

Q.E.D.

Theorem 4.2.B. ZFC is inconsistent.

Proof: For the singleton, **Theorems 1.3.AB** generate that $\Psi = A$, and **Theorem 4.2.A** generates the possibility that $\Psi = \varnothing$. Thus it is possible that $A = \varnothing$. This is a clear contradiction. Q.E.D.

5. Discussion

5.1. Nominalism versus realism

This paper deals with self-reference and derives a contradiction. It may thus be difficult to follow. The reader can maintain clarity by holding on to the key notion of *freedom of definition.* When a restriction on that freedom generates a consistent framework, while release of that restriction generates confusion, then the restriction is to be preferred above too much freedom. Amendment of ZFC thus will tend to reduce the freedom of definition, unless one allows for a three-valued logic that is strong enough to recognise nonsense.

We can look at Table 1 and Table 4 in horizontal or vertical direction. This reflects the schism in philosophy between *nominalism* and *realism*. (See William of Ockham.)

(1) The horizontal view gives the **realists** who take predicates as 'real': $\alpha \notin f[\alpha]$ versus $\alpha \in f[\alpha]$. They are also sequentialist: Δ1 & Δ2 versus Δ3 & Δ4.

(2) The vertical view gives the **nominalists** (Occam) who regard the horizontal properties as mere stickers, and who more realistically look at Ψ = Ø versus $\Psi = A$. They see the table in *even* versus *uneven* fashion: Δ2 & Δ4 versus Δ1 & Δ3.

The nominalist reasoning is: The sets Ø and A exist, as above tables show. We are merely discussing how they are referred to. The expression for Ψ is not a *defining* statement but a *derivative* observation. Once the functions have been mapped out, the criteria can be used to see whether the underlying sets may get also another sticker Ψ. We are discussing *'consistent referring'* and not existence.

At issue is now whether ZFC has sufficient logical strength to block nonsensical situations. ZFC has a realist bend. It translates predicates into sets (their extensions). Instead it can be better to *only test* whether a predicate is useful. Merely cataloguing differently what already exists should not be confused with existence itself. The freedom of definition can be a mere illusion and then should not be abused to create nonsense.

A definition of Ψ on the LHS results via the tautology into a weaker relation on the RHS that contradicts that definition.

The problem with Theorems 1.3.AB is that they ***impose*** the equivalence. This assumes a freedom of definition, whence it assumes that the truthtable on the LHS is true, whence Δ1 is forbidden. But that freedom of definition does not exist. Something exists, that is infringed upon by the definition. When Ψ is the empty set, as in the singleton possibility of Figure 2, then one no longer has the freedom to switch from Ø to *A,* see Figure 1.

The discussion is not without consequence, see PV-RP-CDA-ZFC:

> The logical construction $x \notin f[x]$ and only a single problematic element, in badly understood self-reference, should not be abused to draw conclusions on the infinite. There are ample reasons to look for ways how this can be avoided.

5.2. Diagnosis, and an axiom for a solution set

The diagnosis is that Ψ is rather a variable (name) than a constant. There is a solution set Ψ^* = {Ø, A}, and Ψ is a variable that runs over Ψ^*. Compare to algebra, when one uses a variable x with value $x = 2$ in one case and $x = 4$ in another case: then one might derive $2 = x = 4$, but this goes against the notion of a variable. The inconsistency in ZFC is caused by that it does not allow for that Ψ is such a variable.

The following is not in ZFC but will help to understand ZFC.

Definition of an *Axiom of a Solution Set* (this paper):

$$(\forall A)\ (\exists Z)\ (\exists B)\ ((B \in Z) \Leftrightarrow\ (\forall x)\ ((x \in B)\ \Leftrightarrow\ ((x \in A)\ \&\ \gamma[x]))) \qquad \text{(SOL)}$$

This SOL could reduce to the Axiom of Separation (SEP). A way is to eliminate $B \in Z$ as superfluous and self-evident, which it apparently isn't. Another way is to replace $B \in Z$ by $B = Z$. This imposes uniqueness. When $\gamma[x]$ has more solutions then a contradiction arises when SEP requires that a single solution B is also the whole set Z.

For the singleton $A = \{\alpha\}$ and $f[\alpha] = Ø$, **Theorem 1.3.B** finds $B = \Psi = A$ but we find $Z = \{Ø, A\} = P[A]$. In itself it is true that $\Psi \in P[A]$, but when $Z = P[A]$ then it is erroneous to require $Z \in P[A]$.

It is not just an issue of notation. It is not sufficient to suggest to read **Theorem 1.3.B** now as generating a value for the variable, rather than restricting the solution set to that value: then one reads something into SEP which it does not do, for it really restricts that solution set.

In ZFC Ψ creates the illusion of a unique set, and one needs amendment of ZFC to correct that. One might hold that **Theorems 1.3.AB** are not necessarily wrong, since one can find for any f a $\Psi[f]$ such that for all α: $f[\alpha] \neq \Psi$. (For the singleton $f[\alpha] = Ø$ gives $\Psi = A$.) But the formula of that Ψ allows **Theorem 4.2.A** to also find another $f[\alpha] = \Psi$. (For the singleton $f[\alpha] = Ø$ also gives $\Psi = Ø$.) One can conceive that the two options co-exist, but ZFC does not allow for that. Thus **Theorem 4.2.A** is a real counterexample for **Theoreom 1.3.B.** The freedom of definition used in **Theorem 1.3.B** depends upon the existence **Theorem 1.3.A**. Then something is wrong with Theorem 1.3.A, that proved the existence of what 1.3.B uses. The theorems were derived in ZFC. Thus ZFC has a counterexample and thus is inconsistent.

Again, consider $f[\alpha] = Ø = \Psi$ (Δ1). This is consistent, but cannot be seen directly by Ψ, even though it is covered in Table 3 by the falsehood of $\alpha \in \Psi$. In a realist mode of thought, we deduce from $f[\alpha] = Ø$ that $\Psi = A$, which is the only possibility on the LHS for $\alpha \notin f[\alpha]$ that Ψ recognises (row Δ2). This is not necessarily the proper response. The problem with ZFC is that it focuses on the LHS and neglects the RHS. We can derive a relaxed condition and then Theorem 4.2.A allows to recover Δ1. The latter deductions are actually within ZFC and thus there is scope to argue that Theorem 1.3.B presents only part of the picture. However, that part is formulated in such manner that it causes the contradiction in **Theorem 4.2.B.** We must switch to a better axiomatic system that covers the *intended interpretation* and that blocks the paradoxical Ψ. The better system blocks the LHS and allows only the RHS.

While this paper has a destructive flavour on ZFC, it is actually constructive since it indicates what the improvement will be. See PV-RP-CDA-ZFC for this development.

5.3. Logical structure of this paper

The inconsistency shows itself in Table 3 with two cases on the LHS for **Theorem 1.3.B** and three cases on the RHS for **Theorem 4.2.A**. In itself it might be possible to use only this table and forget about deductive incompleteness. However, it is useful to build up understanding with first explaining such existence by the use of tautology T1.

The idea that Ψ in Theorems 1.3.AB or Table 1 covers all $\alpha \notin f[\alpha]$ appears to be false: it doesn't cover Δ1. Thus when $\alpha \notin \Psi$, there still exists a case of $\alpha \notin f[\alpha]$. Now, isn't Ψ *supposed* to cover *all* such latter cases ? The conclusion is: Theorems 1.3.AB do not cover the *intended interpretation* (DeLong (1971)). However: since Theorem 4.2.A deducts this neglected truth, and still is in ZFC, ZFC becomes inconsistent, can prove everything, and the notion of deductive incompleteness loses meaning. Looking at only consistency would cause us to lose sight of Theorems 3.1 and 3.2

Since **Theorems 1.3.AB** are well accepted in the literature and **Theorem 4.2.A** is new, there is great inducement to find error in **Theorem 4.2.A.** Indeed, **Theorem 4.2.A** allows the deduction of a contradiction in **Theorem 4.2.B,** and thus one might hold that it should go. However, its steps are correct. It is more productive for the reader to accept the inconsistency of ZFC. A discussion about self-reference that identifies a contradiction is always difficult to follow. The problem lies not in the identification of the logical structure but in the inconsistency of ZFC. Potentially the distinction between constant and variable has most effect for clarity. But it also helps to see the distinct role of the two taulogies T1 and T2.

5.4. Further reading

This paper is a rewrite of sections of PV-RP-CDA-ZFC version June 17 2015. See there for a longer discussion and a link to Cantor' Theorem and the transfinites, and suggestions for new axioms for set theory. It would not be a solution to repair ZFC in such a way that the transfinites would be saved, since they are a figment of $x \notin f[x]$ confusions, and there is no intended interpretation for them outside of those Cantorian confusions.

6. Conclusion

1. If ZFC is consistent then it is deductively incomplete, via tautology T1.

2. ZFC is inconsistent, via tautology T2. See PV-RP-CDA-ZFC for alternatives.

3. Users of ZFC who do not accept inconsistency are advised a course in elementary logic (ALOE) and the longer discussion in CCPO-PCWA and PV-RP-CDA-ZFC.

If one holds that ZFC is consistent, against all logic, then one also accepts the transfinites - which makes one wonder what ZFC is a model for. We can agree with Cantor that the essence of mathematics lies in its freedom, but the freedom to create nonsense would no longer be mathematics.

Acknowledgements

Let me repeat my gratitude stated in the other papers CCPO-PCWA and PV-RP-CDA-ZFC. For this paper, I thank Richard Gill (Leiden) for various discussions, and Klaas Pieter Hart (Delft) over 2011-2015 and Bas Edixhoven (Leiden) in 2014 for some comments and for causing me to look closer at ZFC. Jan Bergstra (Amsterdam) gave the final inducement to select only ZFC from PV-RP-CDA-ZFC. Hart and Edixhoven apparently have missed the full analysis and take a 'Cantorian position'. I am sorry to have to report a breach in scientific integrity, see Colignatus (2015e). All errors remain mine.

Appendix A: Versions of ALOE

The following comments are relevant for accurate reference.

(1) Colignatus (1981, 2007, 2011) (**ALOE**) existed first unpublished in 1981 as *In memoriam Philetas of Cos,* then in 2007 rebaptised and self-published. It was both retyped and programmed in the computer-algebra environment of *Mathematica* to allow ease of use of three-valued logic. In 2011 it was marginally adapted with a new version of *Mathematica.* At that moment it could also refer to a new rejection of Cantor's particular argument for the natural and real numbers, using the notion of *bijection by abstraction* - in 2011 still called *bijection in the limit* - now developed in Colignatus (2012, 2013) (CCPO-PCWA), and see Colignatus (2015af) on abstraction.

(2) Gill (2008) reviewed the 1^{st} edition of ALOE of 2007. That edition refers to Cantor's standard set-theoretic argument and rejects it.

(3) Gill (2008) did not review the 2^{nd} edition of ALOE of 2011. That edition also refers to Cantor's original argument on the natural and real numbers in particular. That edition of ALOE mentions the suggestion that $\aleph \sim \Re$. The discussion itself is not in ALOE but is now in CCPO-PCWA, using the notion of *bijection by abstraction.*

(4) ALOE is a book on logic and not a book on set theory. It presents the standard notions of naive set theory (membership, intersection, union) and the standard axioms for first order predicate logic that of course are relevant for set theory. But I have always felt that discussing *axiomatic* set theory (with ZFC) was beyond the scope of the book and my actual interest and developed expertise. This present paper is in my sentiment rather exploratory.

Appendix B: On the genesis of this paper

Colignatus (2013) explains my background and **Appendix A** explains about ALOE. It is joy to see that the application of basic propositional logic still pops up to resolve the issue of this paper. It is quite conceivable that ZFC theorists simply don't have this affinity with logic that I can advise to every student.

Theorem 1.3.A is a reformulation of the addendum provided by B. Edixhoven, statement in Colignatus (2014a), its appendix D.

Theorem 1.3.B was given by K.P. Hart (TU Delft), 2012, in Colignatus (2015b).

A visit to a restaurant in October 27 2014 and subsequent e-mail exchange with Edixhoven (Leiden), co-author of Coplakova et al. (2011), led to the memos Colignatus (2014ab), and the inspiration to write about ZFC. Originally I asked Edixhoven the question on the relation between Cantorian Ψ and Pauline Φ (see PV-RP-CDA-ZFC). Edixhoven agreed that the Pauline consistency condition should have no effect, and I asked him to explain that it could have an effect. Since November 2014, see Colignatus (2014ab), I have not received a response even though the question was clear and articulate. Hart (Delft), who has invested deeply into the transfinites, apparently rejects the usefulness of these questions. Having seen ZFC more often in the course of these exchanges, I decided on the morning of Wednesday May 27 2015 to provide for the answers myself, and established the singleton case before noon. The rest is basically didactics. Advised reading is Colignatus (2015e).

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