Boolean Algebra and Propositional Logic

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This article provides yet another characterization of Boolean algebras and, using this characterization, establishes a more direct connection between propositional logic and Boolean algebras.

Contents

1. Boolean Algebras

A definition of a Boolean algebra is given below. We will show that the definition is equivalent to the usual definition of a Boolean algebra as a complemented distributive lattice.

Definition 1.1. A Boolean algebra $\mathbf{B} = (B, \vee, \neg)$ is a set B with a binary operation $\vee$ and a unary operation $\neg$ satisfying the following condition: there exists $u \in B$ such that

1. the relation $p \leq q$ on B defined by $\neg p \vee q = u$ is a partial order, and,
2. in this partial order, $p \vee q$ is the join of p and q.

If such an element $u \in B$ exists, it is unique, because $u = \neg p \vee p$ (i.e. $p \leq p$) for any $p \in B$. This unique element is called the unit and a constant symbol 1 is introduced to denote it. The following lemma lists some elementary properties of a Boolean algebra immediately derived from the definition.

Lemma 1.2.

1. $\neg p \vee p = 1$.
2. $p \leq 1$; *that is,* 1 *is the top element.*
3. $p \leq \neg q$ *iff* $q \leq \neg p$.

Proof.

1. Immediate by definition.
2. We need to show that $\neg p \vee 1 = 1$. But

$$\begin{aligned} \neg p \vee 1 &= \neg p \vee (\neg p \vee p) && \text{by (1).} \\ &= (\neg p \vee \neg p) \vee p && \text{by the associativity of join.} \\ &= \neg p \vee p && \text{by the idempotency of join.} \\ &= 1 && \text{by (1).} \end{aligned}$$

3. Indeed,

$$\begin{aligned} p \leq \neg q &\iff \neg p \vee \neg q = 1 && \text{by the definition of } \leq. \\ &\iff \neg q \vee \neg p = 1 && \text{by the commutativity of join.} \\ &\iff q \leq \neg p && \text{by the definition of } \leq. \end{aligned}$$

□

Remark 1.3. Lemma 1.2 (3) says that the pair $(\neg, \neg)$ is a Galois connection from the poset $(B, \leq)$ to itself, and is thus equivalent to the conjunction of the following two conditions:

1. $p \leq \neg\neg p$.
2. if $p \leq q$, then $\neg q \leq \neg p$.

In fact the following stronger conditions hold.

Lemma 1.4.

1. *$p = \neg\neg p$.*
2. *$p \leq q$ iff $\neg q \leq \neg p$.*

Proof.

1. Since $p \leq \neg\neg p$ by Remark 1.3 (1), it suffices to show that $\neg\neg p \leq p$ (i.e. $\neg\neg\neg p \vee p = 1$). By Lemma 1.2 (1) and Remark 1.3 (1),

$$1 = \neg p \vee p \leq \neg\neg\neg p \vee p.$$

   Hence, by Lemma 1.2 (2), $\neg\neg\neg p \vee p = 1$.

2. The forward implication holds by Remark 1.3 (2), and the equation in (1) above turns the forward implication into the reverse implication:

$$\begin{aligned} \neg q \leq \neg p &\implies \neg\neg p \leq \neg\neg q \\ &\implies p \leq q \end{aligned}$$

□

By Lemma 1.4, the mapping $p \mapsto \neg p$ is an order-reversing involution and provides an order isomorphism between the poset $(B, \leq)$ and its dual $(B, \geq)$. A constant symbol 0 is defined by

$$0 := \neg 1.$$

to denote the bottom element, and a binary operation $\wedge$ is introduced by

$$p \wedge q := \neg(\neg p \vee \neg q)$$

to denotes the meet of p and q. The order-reversing involution $p \mapsto \neg p$ turns a join into a meet and vice versa (de Morgan's law):

Lemma 1.5. $\neg(p \vee q) = \neg p \wedge \neg q$ *and* $\neg(p \wedge q) = \neg p \vee \neg q$.

As a dual of Lemma 1.2 (1), we have

Lemma 1.6. $p \wedge \neg p = 0$.

A Boolean algebra $(B, \vee, \neg)$ thus yields a complemented lattice $(B, \vee, \wedge, \neg, 1, 0)$.
We now introduce another abbreviation:

$$p \to q := \neg p \vee q.$$

The following is immediate by definition.

Lemma 1.7. $p \leq q$ *iff* $p \to q = 1$.

The following lemma says that the operation → gives an exponential in a Boolean algebra.

Lemma 1.8. $p \wedge q \leq r$ *iff* $p \leq (q \to r)$.

Proof. Indeed,

$$\begin{aligned} p \wedge q \leq r &\iff \neg(p \wedge q) \vee r = 1 \\ &\iff \neg p \vee \neg q \vee r = 1 && 1^* \\ &\iff p \leq \neg q \vee r \\ &\iff p \leq (q \to r) \end{aligned}$$

(1^* by Lemma 1.5). □

The algebra $(B, \vee, \wedge, \to, 1, 0)$ is thus a Heyting algebra. The distributivity of $(B, \vee, \wedge)$ now follows from Lemma 1.8. In fact any Heyting algebra is a distributive lattice. A proof is given below.

Lemma 1.9. $(r \vee s) \wedge q = (r \wedge q) \vee (s \wedge q)$.

Proof. Regard the poset $(B, \leq)$ as a thin category. Then the mapping $p \mapsto p \wedge q$ is order-preserving and thus forms a functor. This functor has a right adjoint by Lemma 1.8, and thus preserves colimits, in particular, joins. □

We have completed the proof of the first part of the following theorem.

Theorem 1.10. *If* $(B, \vee, \neg)$ *is a Boolean algebra, then* $(B, \vee, \wedge, \neg, 1, 0)$ *is a complemented distributive lattice. Conversely, if* $(B, \vee, \wedge, \neg, 1, 0)$ *is a complemented distributive lattice, then* $(B, \vee, \neg)$ *is a Boolean algebra.*

Proof. Since

$$p \leq q \text{ iff } \neg p \vee q = 1$$

holds in a complemented distributive lattice, the second assertion is immediate. □

We state the following fact without proof.

Fact 1.11. *(Huntington). The class of Boolean algebras is axiomatized by the following equations:*

1. $p \vee (q \vee r) = (p \vee q) \vee r$.
2. $p \vee q = q \vee p$.
3. $\neg(\neg p \vee q) \vee \neg(\neg p \vee \neg q) = p$.

The lemma below is used in the next section.

Lemma 1.12. *The following equations hold in a Boolean algebra.*

1. $p \to (q \to r) = (p \wedge q) \to r$.
2. $(p \to r) \wedge (q \to r) = (p \vee q \to r)$.
3. $(p \to p \vee q) = 1$.
4. $(p \to ((p \to q) \to q)) = 1$.
5. $(p \to q) \to ((q \to r) \to (p \to r)) = 1$.

Proof. Indeed,

1.

$$\begin{aligned} p \to (q \to r) &= \neg p \vee (\neg q \vee r) \\ &= (\neg p \vee \neg q) \vee r \\ &= \neg\left(p \wedge q\right) \vee r \\ &= (p \wedge q) \to r \end{aligned}$$

2.

$$\begin{aligned} (p \to r) \wedge (q \to r) &= (\neg p \vee r) \wedge (\neg q \vee r) \\ &= (\neg p \vee \neg q) \vee r \\ &= \neg\left(p \wedge q\right) \vee r \\ &= (p \vee q \to r) \end{aligned}$$

3.

$$\begin{aligned} p \to p \vee q &= \neg p \vee (p \vee q) \\ &= (\neg p \vee p) \vee q \\ &= 1 \vee q \\ &= 1 \end{aligned}$$

4.

$$\begin{aligned} p \to ((p \to q) \to q) &= \neg p \vee (\neg\left(\neg p \vee q\right) \vee q) \\ &= (\neg p \vee q) \vee \neg(\neg p \vee q)) \\ &= 1 \vee 1 \\ &= 1 \end{aligned}$$

5.

$$\begin{aligned} (p \to q) \to ((q \to r) \to (p \to r)) &= (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r) \\ &= (\neg p \vee (p \wedge \neg q)) \vee ((q \wedge \neg r) \vee r) \\ &\geq \neg q \vee q \\ &= 1 \end{aligned}$$

□

2. Propositional Logic

2.1. Algebra of propositional logic

An L-algebras defined below provides an algebraic model for propositional logic.

Definition 2.1. An algebra of type $\mathrm{L} = (\vee, \neg)$, or an L-algebra, is an algebra consisting of a binary operation $\vee$ and a unary operation $\neg$. An L-homomorphism is a map between two L-algebras which preserves $\vee$ and $\neg$.

A Boolean algebra is an L-algebras. In fact, by Fact 1.11, the class of Boolean algebras forms a variety (equationally defined class) of L-algebras. Just as we did for Boolean algebras, the binary operations $a \to b$ and $a \wedge b$ are defined by $\neg a \vee b$ and $\neg(\neg a \vee \neg b)$ respectively. Every L-homomorphism preserves the defined operation $\to$ and $\wedge$. If h is an L-homomorphism between two Boolean algebras, then h preserves 1; for,

$$h(1) = h(p \vee \neg p) = h(p) \vee \neg h(p) = 1.$$

Likewise, h preserves 0.

Definition 2.2. The category of Boolean algebras is denoted by $\mathbf{BA}$ and the category of L-algebras is denoted by $\mathbf{LA}$.

The one-element Boolean algebra $\mathbf{1}$ is terminal in both $\mathbf{BA}$ and $\mathbf{LA}$.

2.2. Boolean congruences

Definition 2.3. An L-homomorphism $h : \mathbf{A} \to \mathbf{B}$ is called a Boolean homomorphism if its codomain $\mathbf{B}$ is a Boolean algebra. A congruence θ on an L-algebra $\mathbf{A}$ is called a Boolean congruence if the quotient L-algebra $\mathbf{A}/\theta$ is a Boolean algebra.

The equivalence kernel of a Boolean homomorphism is a Boolean congruence. Conversely, if θ is a Boolean congruence on an L-algebra $\mathbf{A}$, the projection $[-] : \mathbf{A} \to \mathbf{A}/\theta$ is a Boolean homomorphism. For any L-algebra $\mathbf{A}$, there is a unique Boolean homomorphism from $\mathbf{A}$ to the one-element Boolean algebra $\mathbf{1}$. Hence every L-algebra $\mathbf{A}$ has at least one Boolean congruence, the nonproper Boolean congruence on $\mathbf{A}$, consisting of a single equivalence class.

Definition 2.4. The kernel of a Boolean homomorphism $h : \mathbf{A} \to \mathbf{B}$ is the inverse image of the unit of the Boolean algebra $\mathbf{B}$, and the kernel of a Boolean congruence θ on an L-algebra $\mathbf{A}$ is the equivalence class constituting the unit of the quotient Boolean algebra $\mathbf{A}/\theta$.

We will see later that a Boolean congruence is determined by its kernel.

Theorem 2.5. *Arbitrary intersection of Boolean congruences on an L-algebra is again a Boolean congruence.*

Proof. Let $\{\theta_i\}$ be a family of Boolean congruences on an L-algebra $\mathbf{A}$. Then the intersection $\bigcap_i \theta_i$ is given by the equivalence kernel of the direct product $\prod_i [-]_{\theta_i} : \mathbf{A} \to \prod_i \mathbf{A}/\theta_i$. Since each $\mathbf{A}/\theta_i$ is a Boolean algebra, $\prod_i \mathbf{A}/\theta_i$ is a Boolean algebra (the class of Boolean algebras is a variety and thus closed under direct products). Hence $\bigcap_i \theta_i$ is a Boolean congruence. $\square$

Corollary 2.6. *The Boolean congruences on an L-algebra form a closure system and hence a complete lattice.*

Definition 2.7. Let $\mathbf{A}$ be an L-algebra.

1. The least Boolean congruence on $\mathbf{A}$ is denoted by $\Delta_{\mathbf{A}}$ or just by Δ.

2. The nonproper Boolean congruence on $\mathbf{A}$ is denoted by $\nabla_{\mathbf{A}}$ or just by ∇.

If $\mathbf{B}$ is a Boolean algebra, then $\Delta_{\mathbf{B}}$ is the trivial equivalence relation given by equality.

Theorem 2.8. *Let $\mathbf{A}$ be an L-algebra. If a congruence θ of $\mathbf{A}$ contains Δ, then θ is a Boolean congruence.*

Proof. Since $\mathbf{A}/\theta = (\mathbf{A}/\Delta)\,/\,(\theta/\Delta)$, $\mathbf{A}/\theta$ is a homomorphic image of $\mathbf{A}/\Delta$. Since $\mathbf{A}/\Delta$ is Boolean, so is $\mathbf{A}/\theta$. ☐

Corollary 2.9. *Let $\mathbf{A}$ be an L-algebra. The assignment $\theta \mapsto \theta/\Delta$ yields a lattice isomorphism from the lattice of Boolean congruences on $\mathbf{A}$ to the lattice of congruences on the Boolean algebra $\mathbf{A}/\Delta$.*

The projection $[-] : \mathbf{A} \to \mathbf{A}/\Delta$, or the Boolean algebra $\mathbf{A}/\Delta$ itself, is called the Booleanization of $\mathbf{A}$. The Booleanization is characterized by the following universal mapping property.

Theorem 2.10. *If $h : \mathbf{A} \to \mathbf{B}$ is a Boolean homomorphism from an L-algebra $\mathbf{A}$ to a Boolean algebra $\mathbf{B}$, then there is a unique homomorphism $\hat{h} : \mathbf{A}/\Delta \to \mathbf{B}$ such that the diagram*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "{[-]}"] \arrow[dr, "h"'] & \mathbf{A}/\Delta \arrow[d, "\hat{h}"] \\
& \mathbf{B}
\end{tikzcd}$$

commutes.

Proof. The homomorphism $\hat{h}$ is defined by

$$\hat{h}\,([a]) = h\,(a)$$

for $a \in A$. ☐

The Booleanization $A \mapsto A/\Delta$ thus yields a left adjoint (a reflector) of the inclusion $\mathbf{BA} \hookrightarrow \mathbf{LA}$. $\mathbf{BA}$ is thus a reflective subcategory of $\mathbf{LA}$. In fact, this is an instance of a general result of universal algebra: every variety forms a reflective subcategory.

2.3. Theories

A formal definition of a theory of an L-algebra is given below. We will soon see that a theory is nothing but the kernel of a Boolean congruence.

Definition 2.11. Let $\mathbf{A}$ be an L-algebra. A non-empty set $\Theta \subseteq A$ is called a theory of $\mathbf{A}$ if it satisfies the following conditions.

1. The relation $p \lesssim_\Theta q$ on A defined by $\neg p \vee q \in \Theta$ is a preorder.
2. In this preorder, $p \vee q$ is a join of p and q (see (4) in Appendix); that is,
   a) $p \lesssim_\Theta p \vee q$, $q \lesssim_\Theta p \vee q$
   b) if $p \lesssim_\Theta r$ and $q \lesssim_\Theta r$, then $p \vee q \lesssim_\Theta r$
3. Θ is upward closed with respect to $\lesssim_\Theta$; that is, if $p \in \Theta$ and $p \lesssim_\Theta q$, then $q \in \Theta$.

The equivalence class induced by the preorder $\lesssim_\Theta$ is denoted by $\equiv_\Theta$. If $p \equiv_\Theta q$ (i.e. $p \lesssim_\Theta q$ and $q \lesssim_\Theta p$), p and q are said to be equivalent under the theory Θ.

Lemma 2.12. *Let Θ be a theory of an L-algebra $\mathbf{A}$. Then*

1. *if $q \in \Theta$, then $p \lesssim_\Theta q$ for any $p \in A$.*
2. *if $p \equiv_\Theta p'$ and $q \equiv_\Theta q'$, then $p \vee q \equiv_\Theta p' \vee q'$.*
3. *$p \lesssim_\Theta \neg q$ iff $q \lesssim_\Theta \neg p$.*

Proof.

1. By Definition 2.11 (2a), $q \lesssim_\Theta (\neg p \vee q)$. Hence, by Definition 2.11 (3), $q \in \Theta$ implies $(\neg p \vee q) \in \Theta$, i.e. $p \lesssim_\Theta q$.
2. See (5) in Appendix.
3. The condition is rewritten as

$$(\neg p \vee \neg q) \in \Theta \text{ iff } (\neg q \vee \neg p) \in \Theta$$

. Since $(\neg p \vee \neg q) \equiv_\Theta (\neg q \vee \neg p)$ (see (5) in Appendix), the assertion follows from Definition 2.11 (3).

□

Remark 2.13. Lemma 2.12 (3) says that the pair $(\neg, \neg)$ is a Galois connection from the preordered set $\langle A, \lesssim_\Theta \rangle$ to itself, and is thus equivalent to the conjunction of the following two conditions:

1. $p \lesssim_\Theta \neg\neg p$.
2. if $p \lesssim_\Theta q$, then $\neg q \lesssim_\Theta \neg p$.

The affinity between the conditions (1), (2) in Definition 2.11 and the defining conditions of a Boolean algebra (Definition 1.1) suggests the correspondence between theories and Boolean congruences, and indeed this is the case as we see below in Theorem 2.14 and Theorem 2.15.

Theorem 2.14. *If Θ is a theory of an L-algebra $\mathbf{A}$, then the equivalence relation $\equiv_\Theta$ is a Boolean congruence on $\mathbf{A}$ and Θ is the kernel of $\equiv_\Theta$.*

Proof. We first show that Θ is an equivalence class of $\equiv_\Theta$. Let $p \in \Theta$. We need to see that $q \equiv_\Theta p$ iff $q \in \Theta$ for any any $q \in A$. By Definition 2.11 (3), $q \equiv_\Theta p$ and $p \in \Theta$ implies $q \in \Theta$, and, by Lemma 2.12 (1), $q \in \Theta$ and $p \in \Theta$ implies $q \equiv_\Theta p$. By Lemma 2.12 (2) and Remark 2.13 (2), $\equiv_\Theta$ is a congruence. It remains to prove that the quotient L-algebra $A/{\equiv_\Theta}$ is a Boolean algebra with Θ being the unit. For this it suffices to show that Θ satisfies the conditions (1) and (2) in Definition 1.1. Denote the equivalence class of $p \in A$ under $\equiv_\Theta$ by $[p]$, and define the relation $\leq$ on $A/{\equiv_\Theta}$ by

$$[p] \leq [q] \text{ iff } \neg[p] \vee [q] = \Theta.$$

Since $\neg[p] \vee [q] = \Theta$ iff $\neg p \vee q \in \Theta$, we have

$$[p] \leq [q] \text{ iff } p \lesssim_\Theta q.$$

The relation $\leq$ is thus nothing but the partial order induced by the preorder $\lesssim_\Theta$ (see (1) in Appendix). Since the projection $[-] : A \to A/{\equiv_\Theta}$ preserves joins (see (6) in Appendix), $[p] \vee [q] = [p \vee q]$ is the join of $[p]$ and $[q]$. □

Theorem 2.15. *If θ is a Boolean congruence on an L-algebra $\mathbf{A}$, then the kernel Θ of θ is a theory of $\mathbf{A}$ and θ is determined by Θ.*

Proof. Denote the equivalence class of $p \in A$ under θ by $[p]$, and denote the partial order of A/θ by $\leq$; that is,

$$[p] \leq [q] \text{ iff } \neg[p] \vee [q] = \Theta.$$

Now define the relation $\lesssim_\Theta$ on A by

$$p \lesssim_\Theta \text{ iff } \neg p \vee q \in \Theta.$$

Since $\neg[p] \vee [q] = \Theta$ iff $\neg p \vee q \in \Theta$, we have

$$p \lesssim_\Theta q \text{ iff } [p] \leq [q].$$

The relation $\lesssim_\Theta$ is thus nothing but the preorder induced by the partial order $\leq$ on A/θ, and θ coincides with the equivalence relation $\equiv_\Theta$ induced by the preorder $\lesssim_\Theta$ (see (3) in Appendix). θ is thus determined by Θ. It remains to prove that Θ and $\lesssim_\Theta$ satisfy the conditions in Definition 2.11. We have already seen that $\lesssim_\Theta$ is a preorder. Since the projection $[-] : A \to A/\theta$ reflects joins (see (6) in Appendix) and $[p \vee q] = [p] \vee [q]$ is the join of $[p]$ and $[q]$ in $(A/\theta, \leq)$, $p \vee q$ is a join of p and q in $(A, \lesssim_\Theta)$. Finally, Θ is upward closed in $(A, \lesssim_\Theta)$ since $\{\Theta\}$ is in $(A/\theta, \leq)$. ☐

Corollary 2.6 and the bijective correspondence between theories and Boolean congruences we have just seen yield the following.

Theorem 2.16. *The theories of an L-algebra* $\mathbf{A}$ *form a closure system and thus a complete lattice. There is a canonical isomorphism between the lattice of Boolean congruences on* $\mathbf{A}$ *and the lattice of theories of* $\mathbf{A}$.

By this isomorphism, a theory and the corresponding Boolean congruence are identified with each other and often denoted by the same symbol.

Definition 2.17. Let $\mathbf{A}$ be an L-algebra.

1. The closure system of theories of $\mathbf{A}$ is denoted by $\mathcal{T}_{\mathbf{A}}$ or just by $\mathcal{T}$.
2. The closure operator associated with the closure system $\mathcal{T}_{\mathbf{A}}$ is also denoted by $\mathcal{T}_{\mathbf{A}}$ or $\mathcal{T}$. Given a subset S of A, $\mathcal{T}(S)$ is the smallest theory of $\mathbf{A}$ containing S and called the theory generated, or axiomatized, by S.
3. The smallest theory of $\mathbf{A}$, $\mathcal{T}_{\mathbf{A}}(\varnothing)$, is denoted by $\Delta_{\mathbf{A}}$ or just by Δ.
4. The set A is also denoted by $\nabla_{\mathbf{A}}$ or just by ∇ and called the inconsistent theory of $\mathbf{A}$. A theory is called consistent if it is not inconsistent.
5. A theory Θ of $\mathbf{A}$ is called complete if the only theory properly containing Θ is ∇.

Theorem 2.18. *For a non-empty subset* Θ *of an L-algebra* $\mathbf{A}$, *the following are equivalent:*

1. Θ *is a theory.*
2. Θ *satisfies the following:*
   a) $(p \to p) \in \Theta$.
   b) *if* $(p \to q), (q \to r) \in \Theta$, *then* $p \to r \in \Theta$.
   c) $(p \to p \vee q), (q \to p \vee q) \in \Theta$.
   d) *if* $(p \to r), (q \to r) \in \Theta$, *then* $p \vee q \to r \in \Theta$.
   e) *if* $p, (p \to q) \in \Theta$, *then* $q \in \Theta$.
3. Θ *satisfies the following:*

  - *a)* $(p \to p) \in \Theta$.
  - *b)* $(p \to q) \to ((q \to r) \to (p \to r)) \in \Theta$.
  - *c)* $(p \to p \vee q), (q \to p \vee q) \in \Theta$.
  - *d)* $((p \to r) \to ((q \to r) \to (p \vee q \to r))) \in \Theta$.
  - *e)* *if* $p, (p \to q) \in \Theta$, *then* $q \in \Theta$.

4. Θ *satisfies the following:*
  - *a)* $\Delta \subseteq \Theta$.
  - *b)* *if* $p, (p \to q) \in \Theta$, *then* $q \in \Theta$.

5. Θ *satisfies the following:*
  - *a)* *if* $p, q \in \Theta$, *then* $p \wedge q \in \Theta$.
  - *b)* *if* $p \in \Theta$ *and* $p \lesssim_\Delta q$ *(i.e.* $(p \to q) \in \Delta$*), then* $q \in \Theta$.

Proof.

$(1 \Leftrightarrow 2)$ Since $p \to q$ iff $\neg p \vee q$ by definition, the conditions in (2) are just literal translations of the conditions in Definition 2.11.

$(1, 2 \Rightarrow 4)$ Obvious.

$(4 \Rightarrow 3)$ Since $(p \to p) = 1$ holds in a Boolean algebra, $(p \to p) \in \Delta \subseteq \Theta$. Hence (a) holds. (b), (c), and (d) are shown to hold in the same way using the equations in Lemma 1.12.

$(3 \Rightarrow 2)$ (b), (e) in (3) imply (b) in (2), and (d), (e) in (3) imply (d) in (2).

$(4 \Rightarrow 5)$ Clearly, (4) implies (b) in (5). Since $p \to (q \to (p \wedge q)) = 1$ hold in a Boolean algebra (see Lemma 1.12), $p \to (q \to (p \wedge q)) \in \Delta \subseteq \Theta$. Now (b) in (4) implies (a) in (5).

$(5 \Rightarrow 4)$ To see that (a) holds, let $q \in \Delta$. By Lemma 2.12 (1), $p \lesssim_\Delta q$ for any $p \in \Theta$. Hence (b) in (5) implies $q \in \Theta$. To see that (b) holds, assume that $p, (p \to q) \in \Theta$. By (a) in (5), $(p \wedge (p \to q)) \in \Theta$. Since $((p \wedge (p \to q)) \to q) = 1$ hold in a Boolean algebra (see Lemma 1.12), $((p \wedge (p \to q)) \to q) \in \Delta$. Now (b) in (5) implies (b) in (4).

□

If **B** is a Boolean algebra, then Δ is the trivial congruence and the preorder $\lesssim_\Delta$ coincides with the intrinsic partial order $\leq$ of **B**, and the conditions in Theorem 2.18 (5) are written as

1. if $p, q \in \Theta$, then $p \wedge q \in \Theta$.
2. if $p \in \Theta$ and $p \leq q$, then $q \in \Theta$.

A subset Θ of a Boolean algebra satisfying these conditions is called a filter. The notions of theories and filters thus coincide in a Boolean algebra. For a general L-algebra, the following theorem holds.

Theorem 2.19. *Let* **A** *be an L-algebra. The assignment* $\Theta \mapsto \Theta/\Delta$ *yields the lattice isomorphism from the lattice of theories of* **A** *to the lattice of filters of the Boolean algebra* $\mathbf{A}/\Delta$.

Proof. Immediate from Corollary 2.9 and Theorem 2.16. □

Theorem 2.20 and Theorem 2.21 below are derived from each other.

Theorem 2.20. *Let* S *be a non-empty subset of a Boolean algebra* **B**. *An element* $q \in \mathbf{B}$ *is in the filter generated by* S *if and only if* $p_1 \wedge \cdots \wedge p_n \leq q$ *for some* $p_1, \ldots, p_n \in S$.

Proof. See the proof of Theorem 2.21. $\square$

Theorem 2.21. *(Deduction theorem). Let S be a non-empty subset of an L-algebra $\mathbf{A}$. An element $q \in \mathbf{A}$ is in the theory axiomatized by S if and only if $p_1 \wedge \cdots \wedge p_n \lesssim_\Delta q$ (i.e. $(p_1 \wedge \cdots \wedge p_n \to q) \in \Delta$) for some $p_1, \ldots, p_n \in S$.*

Proof. Define $\overline{S}$ by

$$\overline{S} = \{q \in A : p_1 \wedge \cdots \wedge p_n \lesssim_\Delta q \text{ for some } p_1, \ldots, p_n \in S\}.$$

We must prove that $\overline{S} = \mathcal{T}(S)$, and for this it suffices to show that

1. $S \subseteq \overline{S}$.
2. $\overline{S}$ satisfies the conditions in Theorem 2.18 (5).
3. If a set $\Theta \subseteq A$ contains S and satisfies the conditions in Theorem 2.18 (5), then Θ contains $\overline{S}$.

But, these are easily verified. $\square$

Corollary 2.22. *(Compactness theorem). The closure system $\mathcal{T}$ of an L-algebra $\mathbf{A}$ is algebraic; that is, for any subset S of A,*

$$\mathcal{T}(S) = \bigcup\{\mathcal{T}(C) : C \textit{ is a finite subset of } S\}.$$

2.4. Formal deductions

The defining conditions of a theory such as (2) and (3) in Theorem 2.18 give rise to a deductive system. The following deductive system is derived from the conditions in Theorem 2.18 (2).

- Logical axioms:
  1. $p \to p$
  2. $p \to p \vee q$
  3. $q \to p \vee q$
- Inference rules:
  1. $((p \to q), (q \to r), (p \to r))$
  2. $((p \to r), (q \to r), (p \vee q \to r))$
  3. $(p, (p \to q), q)$

So, a deductive system is obtained from theories, not the other way around. Except for the last inference rule (modus ponens), the deductive system has its origin in the defining conditions of a Boolean algebra (the conditions in Definition 1.1).

Definition 2.23. Let $\mathbf{A}$ be an L-algebra, $S \subseteq A$, and $p \in A$. A finite sequence $(p_1, \ldots, p_n)$ of elements of A such that $p = p_n$ is called a deduction (or proof) of p from S if for each $i \leq n$ one of the following holds:

1. p_i is a logical axiom.
2. $p_i \in S$.
3. There are $j, k < i$ such that (p_j, p_k, p_i) is an inference rule.

S is said to syntactically entail p, written $S \vdash p$, if there is a deduction of p from S.

Theorem 2.24. *Let* $\mathbf{A}$ *be an L-algebra,* $S \subseteq A$*, and* $p \in A$*. Then,* $p \in \mathcal{T}(S)$ *if and only if* $S \vdash p$*.*

Proof. Define $\overline{S}$ by

$$\overline{S} = \{p : S \vdash p\}.$$

We must prove that $\mathcal{T}(S) = \overline{S}$, and for this it suffices to show that

1. $S \subseteq \overline{S}$.
2. $\overline{S}$ satisfies the conditions in Theorem 2.18 (2).
3. If a set $\Theta \subseteq A$ contains S and satisfies the conditions in Theorem 2.18 (2), then Θ contains $\overline{S}$.

But, these are easily verified. ☐

Only a finite number of elements in S appear in a deduction. This fact gives another proof of Corollary 2.22.

2.5. Interpretations

A filter of a Boolean algebra $\mathbf{B}$ is called an ultrafilter if is not included in any proper filter of $\mathbf{B}$.

Theorem 2.25. *A theory* Φ *of an L-algebra* $\mathbf{A}$ *is complete if and only if* Φ/Δ *is a ultrafilter of the Boolean algebra* $\mathbf{A}/\Delta$*.*

Proof. Immediate from Theorem 2.19. ☐

The following characterization of ultrafilters is easily proved.

Fact 2.26. *A filter* Φ *of a Boolean algebra* $\mathbf{B}$ *is an ultrafilter if and only if* Φ *is the the kernel of some homomorphism from* $\mathbf{B}$ *to the two-element Boolean algebra* $\mathbf{2}$*.*

A complete theory in a general L-algebra is characterized in the same say.

Theorem 2.27. *A theory* Φ *of an L-algebra* $\mathbf{A}$ *is complete if and only if* Φ *is the the kernel of some Boolean homomorphism from* $\mathbf{A}$ *to the two-element Boolean algebra* $\mathbf{2}$*.*

Proof. By Theorem 2.10, Boolean homomorphisms $\mathbf{A} \to \mathbf{2}$ and Boolean homomorphisms $\mathbf{A}/\Delta \to \mathbf{2}$ correspond one-to-one via the following commutative diagram:

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "{[-]}"] \arrow[dr, "h"'] & \mathbf{A}/\Delta \arrow[d, "\hat{h}"] \\
 & \mathbf{2}
\end{tikzcd}$$

The assertion thus follows from Theorem 2.25 and Fact 2.26. ☐

The following fact is a consequence of Zorn's lemma.

Fact 2.28. *(Ultrafilter theorem). Every proper filter in a Boolean algebra is included in an ultrafilter.*

In a general L-algebra, the ultrafilter theorem is rephrased as follows.

Theorem 2.29. *Every consistent theory of an L-algebra is included in a complete theory.*

Proof. Immediate from Fact 2.28 on noting Theorem 2.19 and Theorem 2.25. ☐

Definition 2.30. Let $\mathbf{A}$ be an L-algebra. A Boolean homomorphism from $\mathbf{A}$ to the two-element Boolean algebra $\mathbf{2}$ is called an interpretation of $\mathbf{A}$. If S is a subset of A and $p \in A$, then S is said to semantically entail p, written $S \Vdash p$, if every interpretation of A that sends S to 1 sends p to 1.

Theorem 2.31. *Let $\mathbf{A}$ be an L-algebra, $S \subseteq A$, and $p \in A$. Then the following are equivalent:*

1. $p \in \mathcal{T}(S)$.
2. *p is in every theory containing S.*
3. *p is in every complete theory containing S.*
4. $S \Vdash p$.

Proof.

$(1 \Leftrightarrow 2)$ Obvious since $\mathcal{T}(S)$ is the intersection of all the theories containing S.

$(3 \Leftrightarrow 4)$ Immediate from Theorem 2.27.

$(2 \Leftrightarrow 3)$ By Theorem 2.19, the conditions (2) and (3) are equivalent to

(2′) $[p]_\Delta$ is in every filter of $\mathbf{A}/\Delta$ containing S/Δ.

(3′) $[p]_\Delta$ is in every ultrafilter of $\mathbf{A}/\Delta$ containing S/Δ.

The equivalence of (2) and (3) is thus reduced to the following fact of Boolean Algebra, which is a consequence of the ultrafilter theorem.

□

Fact 2.32. *Every filter in a Boolean algebra is the intersection of the ultrafilters that include it.*

Theorem 2.33. *Let $\mathbf{A}$ be an L-algebra, $S \subseteq A$, and $p \in A$. Then*

1. *(Completeness theorem). If $S \Vdash p$, then $S \vdash p$.*
2. *(Soundness theorem). If $S \vdash p$, then $S \Vdash p$.*

Proof. Immediate from Theorem 2.31 and Theorem 2.24. □

2.6. Propositional calculus

The language and operation of propositional calculus is given by the term algebra $\mathrm{L}(V)$ of type $\mathrm{L} = (\vee, \neg)$ generated by a set $V$of propositional variables. The elements of $\mathrm{L}(V)$ constitute the sentences of propositional calculus.

By the freeness of $\mathrm{L}(V)$, every map from V to an L-algebra $\mathbf{A}$ extends uniquely to an L-homomorphism from $\mathrm{L}(V)$ to $\mathbf{A}$. In particular, a truth assignment (i.e. a map from V to the two-element Boolean algebra $\mathbf{2}$) extends uniquely to an interpretation of $\mathrm{L}(V)$. Semantic entailment can thus be defined in terms of truth assignments.

If Θ is a theory of $\mathrm{L}(V)$, then the quotient Boolean algebra $\mathrm{L}(V)/\Theta$ is called a Lindenbaum algebra of Θ. The Booleanization $\mathrm{L}(V)/\Delta$ of $\mathrm{L}(V)$ yields a free Boolean algebra over V. Since every algebra is a homomorphic image of a free algebra, every Boolean algebra is isomorphic to a Lindenbaum algebra.

A. Appendix. Preorder

Provided below are some basic facts on preorders.

1. Every preordered set $(A, \lesssim)$ induces a poset $(A/{\equiv}, \leq)$ by the equivalence relation $\equiv$ defined on A by
$$x \equiv y \text{ iff } x \lesssim y \text{ and } x \lesssim y$$
and the partial order $\leq$ defined on the quotient set $A/{\equiv}$ by
$$[x] \leq [y] \text{ iff } x \lesssim y$$
, where $[x]$ and $[y]$ denote the blocks containing $x, y \in A$.

2. Let f be a surjective map from a set A onto a poset $(B, \leq)$. Then f induces a preorder $\lesssim$ on A by
$$x \lesssim y \text{ iff } f(x) \leq f(y) .$$
The map f then becomes a preorder morphism $(A, \lesssim) \to (B, \leq)$. Moreover, the poset $(A/{\equiv}, \leq)$ induced by the preordered set $(A, \lesssim)$ is isomorphic to $(B, \leq)$; in fact, there is a canonical isomorphism $(A/{\equiv}, \leq) \cong (B, \leq)$ making the diagram
$$\begin{tikzcd}
(A, \lesssim) \arrow[d, "{[-]}"'] \arrow[dr, "f"] & \\
(A/{\equiv}, \leq) \arrow[r, "\cong"'] & (B, \leq)
\end{tikzcd}$$
commute.

3. As a special case of (2) above, consider an equivalence relation $\equiv$ on a set A and suppose that a partial order $\leq$ is defined on $A/{\equiv}$. Then the projection $[-] : A \to B$ induces a preorder $\lesssim$ on A by
$$x \lesssim y \text{ iff } [x] \leq [y] .$$
The projection $[-]$ then becomes a preorder morphism $(A, \lesssim) \to (A/{\equiv}, \leq)$, and the poset induced by the preordered set $(A, \lesssim)$ coincides with the original poset $(A/{\equiv}, \leq)$.

4. If $(A, \lesssim)$ is a preordered set, a join (least upper bound) of any two elements $a, b \in A$ is defined in the same way as in a poset and denoted by $a \vee b$. A join $a \vee b$ is characterized by the following properties:
   a) $a \lesssim a \vee b$ and $b \lesssim a \vee b$.
   b) for all $c \in A$, if $a \lesssim c$ and $b \lesssim c$, then $a \vee b \lesssim c$.

5. A preordered set $(A, \lesssim)$ may be viewed as a thin category. Any two elements $x, y \in A$ are isomorphic if and only if $x \equiv y$, and a join $a \vee b$ of $a, b \in A$ is the same thing as a coproduct of a and b. Hence in particular the following hold.
   a) $a \vee b$ is unique up to isomorphism.
   b) $a \vee b \equiv b \vee a$.
   c) if $a \equiv a'$ and $b \equiv b'$, then $a \vee b \equiv a$’ $\vee$ b’.

6. Let $(A/{\equiv}, \leq)$ be the poset induced by a preordered set $(A, \lesssim)$. If $(A, \lesssim)$ and $(A/{\equiv}, \leq)$ are viewed as thin categories, the projection $[-] : A \to A/{\equiv}$ forms an equivalence functor and thus preserves and reflects colimits (and limits), in particular, the projection preserves and reflects joins (i.e. coproducts).

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