Boolean Algebra and Propositional Logic

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ABSTRACT. In this article, we present yet another characterization of Boolean algebras and, using this characterization, establish a connection between propositional logic and Boolean algebras; in particular, we derive a deductive system for propositional logic starting with Boolean algebras.

Contents

1. Boolean Algebras

We adopt the following definition of a Boolean algebra for its closer affinity with propositional logic. By the series of lemmas that follow down to Theorem 1.10, the definition is shown to be equivalent to the usual definition of a Boolean algebra as a complemented distributive lattice.

Definition 1.1. A Boolean algebra $\mathbf{B} = (B, \vee, \neg)$ is a set B with a binary operation $\vee$ and a unary operation $\neg$ satisfying the following condition: there exists $u \in B$ such that

(1) the binary relation $\leq$ on B defined by

$$p \leq q \text{ iff } \neg p \vee q = u$$

is a partial order, and,

(2) in this partial order, $p \vee q$ is the join of p and q.

If such an element $u \in B$ exists, it will be the largest element of the poset $(B, \leq)$, because $p \leq \neg p \vee p$ and $\neg p \vee p = u$ (i.e. $p \leq p$) for any $p \in B$. We introduce a constant symbol 1 to denote this largest element and call it the unit. We have just seen

Lemma 1.2. $\neg p \vee p = 1$.

The following says that the pair $(\neg, \neg)$ forms a Galois connection from the poset $(B, \leq)$ to its dual.

Lemma 1.3. *$p \leq \neg q$ iff $q \leq \neg p$.*

Proof. Indeed,

$$
\begin{aligned}
p \leq \neg q &\iff \neg p \vee \neg q = 1 && \text{by the definition of } \leq. \\
&\iff \neg q \vee \neg p = 1 && \text{by the commutativity of join.} \\
&\iff q \leq \neg p && \text{by the definition of } \leq.
\end{aligned}
$$

□

Remark 1.4. As properties of a Galois connection, Lemma 1.3 implies

(1) $p \leq \neg\neg p$;

(2) if $p \leq q$, then $\neg q \leq \neg p$;

(3) $\neg p = \neg\neg\neg p$.

In fact, the following stronger conditions hold.

Lemma 1.5.

(1) $p = \neg\neg p$.

(2) $p \leq q$ iff $\neg q \leq \neg p$.

Proof.

(1) Since $p \leq \neg\neg p$ by Remark 1.4(1), it suffices to show that $\neg\neg p \leq p$ (i.e. $\neg\neg\neg p \vee p = 1$). But by Remark 1.4(3) and Lemma 1.2,

$$\neg\neg\neg p \vee p = \neg p \vee p = 1.$$

(2) The forward implication holds by Remark 1.4(2), and the reverse implication follows from the forward implication by virtue of the equation (1) above:

$$
\begin{aligned}
\neg q \leq \neg p &\Longrightarrow \neg\neg p \leq \neg\neg q \\
&\Longrightarrow p \leq q
\end{aligned}
$$

□

By Lemma 1.5, the mapping $p \mapsto \neg p$ is an order-reversing involution and provides an order isomorphism between the poset $(B, \leq)$ and its dual. A constant symbol 0 is introduced by

$$0 := \neg 1.$$

to denote the smallest element, and a binary operation $\wedge$ is defined by

$$p \wedge q := \neg(\neg p \vee \neg q)$$

to denote the meet of p and q. The order-reversing involution $p \mapsto \neg p$ turns a join into a meet and vice versa (de Morgan's law):

Lemma 1.6. $\neg(p \vee q) = \neg p \wedge \neg q$ *and* $\neg(p \wedge q) = \neg p \vee \neg q$.

As the dual of Lemma 1.2, we have

Lemma 1.7. $p \wedge \neg p = 0$.

A Boolean algebra $(B, \vee, \neg)$ thus yields a complemented lattice $(B, \vee, \wedge, \neg, 1, 0)$. The following says that $\neg p \vee q$ gives the pseudocomplement of p relative to q in this lattice.

Lemma 1.8. $r \wedge p \leq q$ *iff* $r \leq \neg p \vee q$.

Proof. Indeed,

$$\begin{aligned} r \wedge p \leq q &\iff \neg(r \wedge p) \vee q = 1 \\ &\iff \neg r \vee \neg p \vee q = 1 \qquad *^1 \\ &\iff r \leq \neg p \vee q \end{aligned}$$

($*^1$ by Lemma 1.6). □

The lattice $(B, \vee, \wedge)$ is thus Heyting, and hence distributive:

Lemma 1.9. $(r \vee s) \wedge p = (r \wedge p) \vee (s \wedge p)$.

Proof. By Lemma 1.8, the mappings $r \mapsto r \wedge p$ and $q \mapsto \neg p \vee q$ form a Galois connection. The mapping $r \mapsto r \wedge p$ thus preserves joins. □

We have completed the proof of the first part of the following theorem.

Theorem 1.10. *If* $(B, \vee, \neg)$ *is a Boolean algebra, then* $(B, \vee, \wedge, \neg, 1, 0)$ *is a complemented distributive lattice. Conversely, if* $(B, \vee, \wedge, \neg, 1, 0)$ *is a complemented distributive lattice, then* $(B, \vee, \neg)$ *is a Boolean algebra.*

Proof. It remains to prove the second assertion. But this is immediate because

$$p \leq q \text{ iff } \neg p \vee q = 1$$

holds in a complemented distributive lattice. □

Here we introduce another abbreviation:

$$p \to q := \neg p \vee q.$$

$p \to q$ denotes the pseudocomplement of p relative to q (recall Lemma 1.8). With this abbreviation, the condition in Definition 1.1(1) is written as

Lemma 1.11. *$p \leq q$ iff $p \to q = 1$.*

We use the following in the proof of Lemma 1.13.

Lemma 1.12. *The following hold in a Boolean algebra.*

(1) $p \to (q \to r) = (p \wedge q) \to r$.

(2) $(p \to r) \wedge (q \to r) = p \vee q \to r$.

(3) $p \wedge (p \to q) = p \wedge q$; in particular, $p \wedge (p \to q) \leq q$.

Proof. Indeed,

$$\begin{aligned} p \to (q \to r) &= \neg p \vee (\neg q \vee r) \\ &= (\neg p \vee \neg q) \vee r \\ &= \neg (p \wedge q) \vee r \\ &= (p \wedge q) \to r \end{aligned}$$

$$\begin{aligned} (p \to r) \wedge (q \to r) &= (\neg p \vee r) \wedge (\neg q \vee r) \\ &= (\neg p \wedge \neg q) \vee r \\ &= \neg (p \vee q) \vee r \\ &= p \vee q \to r \end{aligned}$$

$$\begin{aligned} p \wedge (p \to q) &= p \wedge (\neg p \vee q) \\ &= (p \wedge \neg p) \vee (p \wedge q) \\ &= p \wedge q \end{aligned}$$

□

The equations in the following lemma are used in the proof of Theorem 2.17.

Lemma 1.13. *The following hold in a Boolean algebra.*

(1) $p \to p = 1$.

(2) $(p \to q) \to ((q \to r) \to (p \to r)) = 1$.

(3) $p \to p \vee q = 1$ and $q \to p \vee q = 1$.

(4) $(p \to r) \to ((q \to r) \to (p \vee q \to r)) = 1$.

(5) $p \to (q \to (p \wedge q)) = 1$.

(6) $(p \wedge (p \to q)) \to q = 1$.

Proof.

(1) This is Lemma 1.2 rewritten in the abbreviated notation.

(2) Since we have

$$(p \to q) \to ((q \to r) \to (p \to r)) = (p \wedge (p \to q) \wedge (q \to r)) \to r$$

by the repeated application of Lemma 1.12(1), it suffices, thanks to Lemma 1.11, to show that

$$p \wedge (p \to q) \wedge (q \to r) \leq r.$$

But by Lemma 1.12(3),

$$p \wedge (p \to q) \wedge (q \to r) \leq q \wedge (q \to r) \leq r.$$

(3) Because $p \leq p \vee q$ and $q \leq p \vee q$, these equations follow from Lemma 1.11.

(4) By (1) and (2) in Lemma 1.12,

$$\begin{aligned}(p \to r) \to ((q \to r) \to (p \vee q \to r)) &= ((p \to r) \wedge (q \to r)) \to (p \vee q \to r) \\ &= (p \vee q \to r) \to (p \vee q \to r)\end{aligned}$$

But $(p \vee q \to r) \to (p \vee q \to r) = 1$ by (1).

(5) By Lemma 1.12(1),

$$p \to (q \to (p \wedge q)) = (p \wedge q) \to (p \wedge q)\,.$$

But $(p \wedge q) \to (p \wedge q) = 1$ by (1).

(6) Since $p \wedge (p \to q) \leq q$ by Lemma 1.12(3), the equation follows from Lemma 1.11.

□

2. Propositional Logic

2.1. The algebra of propositional logic

An L-algebra defined below provides an algebraic model for propositional logic.

Definition 2.1. An algebra of type $\mathrm{L} = (\vee, \neg)$, or an L-algebra, is an algebra consisting of a binary operation $\vee$ and a unary operation $\neg$. An L-homomorphism is a map between two L-algebras which preserves $\vee$ and $\neg$.

A Boolean algebra is an L-algebras. In fact, the class of Boolean algebras forms a variety (equationally defined class) of L-algebras; the following equations, due to Huntington, constitute one of the equational axiom systems for Boolean algebras.

$$p \vee (q \vee r) = (p \vee q) \vee r$$
$$p \vee q = q \vee p$$
$$\neg(\neg p \vee q) \vee \neg(\neg p \vee \neg q) = p$$

Just as we did for Boolean algebras in the previous section, we define the binary operation $\wedge$ by

$$p \wedge q := \neg(\neg p \vee \neg q),$$

and the binary operation $\to$ by

$$p \to q := \neg p \vee q.$$

Every L-homomorphism preserves the defined operations $\wedge$ and $\to$. If h is an L-homomorphism between two Boolean algebras, then h preserves 1; for,

$$h(1) = h(\neg p \vee p) = \neg h(p) \vee h(p) = 1.$$

Likewise, h preserves 0.

Definition 2.2. The category of Boolean algebras is denoted by **BA** and the category of L-algebras is denoted by **LA**.

The one-element Boolean algebra **1** is terminal in both **BA** and **LA**.

2.2. Booleanization

Definition 2.3. An L-homomorphism $h : \mathbf{A} \to \mathbf{B}$ is called a Boolean homomorphism if its codomain $\mathbf{B}$ is a Boolean algebra. A congruence θ on an L-algebra $\mathbf{A}$ is called a Boolean congruence if the quotient L-algebra $\mathbf{A}/\theta$ is a Boolean algebra.

The equivalence kernel of a Boolean homomorphism is a Boolean congruence. Conversely, if θ is a Boolean congruence on an L-algebra $\mathbf{A}$, the projection $[-] : \mathbf{A} \to \mathbf{A}/\theta$ is a Boolean homomorphism. For any L-algebra $\mathbf{A}$, there is a unique Boolean homomorphism from $\mathbf{A}$ to the one-element Boolean algebra **1**. Hence every L-algebra $\mathbf{A}$ has at least one Boolean congruence, the nonproper Boolean congruence on $\mathbf{A}$, consisting of a single equivalence class.

Definition 2.4. The kernel of a Boolean homomorphism $h : \mathbf{A} \to \mathbf{B}$ is the inverse image of the unit of the Boolean algebra $\mathbf{B}$, and the kernel of a Boolean congruence θ on an L-algebra $\mathbf{A}$ is the equivalence class which is the unit of the quotient Boolean algebra $\mathbf{A}/\theta$.

We will see later that a Boolean congruence is determined by its kernel.

Theorem 2.5. *Arbitrary intersection of Boolean congruences on an L-algebra is again a Boolean congruence.*

Proof. Let $\{\theta_i\}$ be a family of Boolean congruences on an L-algebra $\mathbf{A}$. Then the intersection $\bigcap_i \theta_i$ is given by the equivalence kernel of the direct product $\prod_i [-]_{\theta_i} : \mathbf{A} \to \prod_i \mathbf{A}/\theta_i$. Since each $\mathbf{A}/\theta_i$ is a Boolean algebra, $\prod_i \mathbf{A}/\theta_i$ is a Boolean algebra (the class of Boolean algebras is a variety and thus closed under direct products). Hence $\bigcap_i \theta_i$ is a Boolean congruence. □

Corollary 2.6. *The Boolean congruences on an L-algebra form a closure system and hence a complete lattice.*

Definition 2.7. Let $\mathbf{A}$ be an L-algebra.

(1) The least Boolean congruence on $\mathbf{A}$ is denoted by $\Delta_{\mathbf{A}}$ or just by Δ.

(2) The nonproper Boolean congruence on $\mathbf{A}$ is denoted by $\nabla_{\mathbf{A}}$ or just by ∇.

If $\mathbf{B}$ is a Boolean algebra, then $\Delta_{\mathbf{B}}$ is the trivial equivalence relation given by equality.

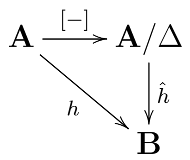
Theorem 2.8. *Let $\mathbf{A}$ be an L-algebra. If a congruence θ of $\mathbf{A}$ contains Δ, then θ is a Boolean congruence.*

Proof. Since $\mathbf{A}/\theta = (\mathbf{A}/\Delta) / (\theta/\Delta)$, $\mathbf{A}/\theta$ is a homomorphic image of $\mathbf{A}/\Delta$. Since $\mathbf{A}/\Delta$ is Boolean, so is $\mathbf{A}/\theta$. □

Corollary 2.9. *Let* $\mathbf{A}$ *be an L-algebra. The assignment* $\theta \mapsto \theta/\Delta$ *yields a lattice isomorphism from the lattice of Boolean congruences on* $\mathbf{A}$ *to the lattice of congruences on the Boolean algebra* $\mathbf{A}/\Delta$*.*

The projection $[-] : \mathbf{A} \to \mathbf{A}/\Delta$, or the Boolean algebra $\mathbf{A}/\Delta$ itself, is called the Booleanization of an L-algebra $\mathbf{A}$. The Booleanization is characterized by the following universal mapping property.

Theorem 2.10. *If* $h : \mathbf{A} \to \mathbf{B}$ *is a Boolean homomorphism from an L-algebra* $\mathbf{A}$ *to a Boolean algebra* $\mathbf{B}$*, then there is a unique homomorphism* $\hat{h} : \mathbf{A}/\Delta \to \mathbf{B}$ *such that the diagram*



commutes.

Proof. The homomorphism $\hat{h}$ is defined by

$$\hat{h}\,([a]) = h\,(a)$$

for $a \in A$. □

The Booleanization $\mathbf{A} \mapsto \mathbf{A}/\Delta$ thus yields a left adjoint (a reflector) of the inclusion $\mathbf{BA} \hookrightarrow \mathbf{LA}$, making $\mathbf{BA}$ a reflective subcategory of $\mathbf{LA}$. In fact, this is an instance of a general result of universal algebra: every variety forms a reflective subcategory.

2.3. Theories

A formal definition of a theory of an L-algebra is given below. We will soon see that a theory is nothing but the kernel of a Boolean congruence.

Definition 2.11. Let $\mathbf{A}$ be an L-algebra. A non-empty set $\Theta \subseteq A$ is called a theory of $\mathbf{A}$ if it satisfies the following conditions.

(1) The binary relation $\lesssim_\Theta$ on A defined by

$$p \lesssim_\Theta q \text{ iff } \neg p \vee q \in \Theta$$

is a preorder.

(2) In this preorder, $p \vee q$ is a join of p and q (see (4) in Appendix); that is,

  (a) $p \lesssim_\Theta p \vee q$ and $q \lesssim_\Theta p \vee q$;

  (b) if $p \lesssim_\Theta r$ and $q \lesssim_\Theta r$, then $p \vee q \lesssim_\Theta r$.

(3) Θ is upward closed with respect to $\lesssim_\Theta$; that is, if $p \in \Theta$ and $p \lesssim_\Theta q$, then $q \in \Theta$.

The equivalence relation induced by the preorder $\lesssim_\Theta$ is denoted by $\equiv_\Theta$. If $p \equiv_\Theta q$ (i.e. $p \lesssim_\Theta q$ and $q \lesssim_\Theta p$), p and q are said to be equivalent under the theory Θ.

Lemma 2.12. *Let Θ be a theory of an L-algebra* $\mathbf{A}$*. Then*

(1) if $q \in \Theta$, then $p \lesssim_\Theta q$ for any $p \in A$;

(2) if $p \equiv_\Theta p'$ and $q \equiv_\Theta q'$, then $p \vee q \equiv_\Theta p' \vee q'$;

(3) $p \lesssim_\Theta \neg q$ iff $q \lesssim_\Theta \neg p$;

(4) $p \lesssim_\Theta \neg\neg p$;

(5) if $p \lesssim_\Theta q$, then *$\neg q \lesssim_\Theta \neg p$.*

Proof.

(1) By Definition 2.11(2a), $q \lesssim_\Theta (\neg p \vee q)$. Hence, by Definition 2.11(3), $q \in \Theta$ implies $(\neg p \vee q) \in \Theta$, i.e. $p \lesssim_\Theta q$.

(2) See (5) in Appendix.

(3) The condition is rewritten as

$$(\neg p \vee \neg q) \in \Theta \text{ iff } (\neg q \vee \neg p) \in \Theta.$$

Since $(\neg p \vee \neg q) \equiv_\Theta (\neg q \vee \neg p)$ (see (5) in Appendix), the assertion follows from Definition 2.11(3).

(4, 5) The condition (3) above says that the pair $(\neg, \neg)$ is a Galois connection from the preordered set $\langle A, \lesssim_\Theta \rangle$ to its dual, and is thus equivalent to the conjunction of the conditions (4) and (5).

□

The affinity between the conditions (1), (2) in Definition 2.11 and the defining conditions of a Boolean algebra (Definition 1.1) suggests the correspondence between theories and Boolean congruences, and indeed this is the case as we see below in Theorem 2.13 and Theorem 2.14.

Theorem 2.13. *If Θ is a theory of an L-algebra* $\mathbf{A}$*, then the equivalence relation $\equiv_\Theta$ is a Boolean congruence on* $\mathbf{A}$ *and Θ is the kernel of $\equiv_\Theta$.*

Proof. By (2) and (5) in Lemma 2.12, $\equiv_\Theta$ is a congruence. We now show that Θ is an equivalence class of $\equiv_\Theta$. Let $p \in \Theta$. We need to see that $q \equiv_\Theta p$ iff $q \in \Theta$ for any $q \in A$. But by Definition 2.11(3), $q \equiv_\Theta p$ and $p \in \Theta$ imply $q \in \Theta$, and by Lemma 2.12(1), $q \in \Theta$ and $p \in \Theta$ imply $q \equiv_\Theta p$. It remains to prove that the quotient L-algebra $A/{\equiv_\Theta}$ is a Boolean algebra with Θ being the unit. For this it suffices to show that Θ satisfies the conditions (1) and (2) in Definition 1.1. Denote the equivalence class of $p \in A$ under $\equiv_\Theta$ by $[p]$, and define the binary relation $\leq$ on $A/{\equiv_\Theta}$ by

$$[p] \leq [q] \text{ iff } \neg[p] \vee [q] = \Theta.$$

Since $\neg[p] \vee [q] = \Theta$ iff $\neg p \vee q \in \Theta$, we have

$$[p] \leq [q] \text{ iff } p \lesssim_\Theta q.$$

The relation $\leq$ is thus nothing but the partial order induced by the preorder $\lesssim_\Theta$ (see (1) in Appendix). Since the projection $[-] : A \to A/{\equiv_\Theta}$ preserves joins (see (6) in Appendix), $[p] \vee [q] = [p \vee q]$ is the join of $[p]$ and $[q]$. □

Theorem 2.14. *If θ is a Boolean congruence on an L-algebra $\mathbf{A}$, then the kernel Θ of θ is a theory of $\mathbf{A}$ and θ is determined by Θ.*

Proof. Denote the equivalence class of $p \in A$ under θ by $[p]$, and denote the partial order of the Boolean algebra A/θ by $\leq$; that is,

$$[p] \leq [q] \text{ iff } \neg[p] \vee [q] = \Theta.$$

Now define the binary relation $\lesssim_\Theta$ on A by

$$p \lesssim_\Theta q \text{ iff } \neg p \vee q \in \Theta.$$

Since $\neg[p] \vee [q] = \Theta$ iff $\neg p \vee q \in \Theta$, we have

$$p \lesssim_\Theta q \text{ iff } [p] \leq [q].$$

The relation $\lesssim_\Theta$ is thus nothing but the preorder induced by the partial order $\leq$ on A/θ, and θ coincides with the equivalence relation $\equiv_\Theta$ induced by the preorder $\lesssim_\Theta$ (see (3) in Appendix). θ is thus determined by Θ. It remains to prove that Θ and $\lesssim_\Theta$ satisfy the conditions in Definition 2.11. We have already seen that $\lesssim_\Theta$ is a preorder. Since the projection $[-] : A \to A/\theta$ reflects joins (see (6) in Appendix) and $[p \vee q] = [p] \vee [q]$ is the join of $[p]$ and $[q]$ in $(A/\theta, \leq)$, $p \vee q$ is a join of p and q in $(A, \lesssim_\Theta)$. Finally, Θ is upward closed in $(A, \lesssim_\Theta)$ since Θ is the largest element of $(A/\theta, \leq)$. □

Corollary 2.6 and the bijective correspondence between theories and Boolean congruences we have just seen yield the following.

Theorem 2.15. *The theories of an L-algebra $\mathbf{A}$ form a closure system and thus a complete lattice. There is a canonical isomorphism between the lattice of Boolean congruences on $\mathbf{A}$ and the lattice of theories of $\mathbf{A}$.*

By this isomorphism, a theory and the corresponding Boolean congruence are identified with each other and often denoted by the same symbol.

Definition 2.16. Let $\mathbf{A}$ be an L-algebra.

(1) The closure system of theories of $\mathbf{A}$ is denoted by $\mathcal{T}_\mathbf{A}$ or just by $\mathcal{T}$.

(2) The closure operator associated with the closure system $\mathcal{T}_\mathbf{A}$ is also denoted by $\mathcal{T}_\mathbf{A}$ or $\mathcal{T}$. Given a subset S of A, $\mathcal{T}(S)$ is the smallest theory of $\mathbf{A}$ containing S and called the theory generated, or axiomatized, by S.

(3) The smallest theory of $\mathbf{A}$, $\mathcal{T}_\mathbf{A}(\varnothing)$, is denoted by $\Delta_\mathbf{A}$ or just by Δ.

(4) The set A is also denoted by $\nabla_\mathbf{A}$ or just by ∇ and called the inconsistent theory of $\mathbf{A}$. A theory is called consistent if it is not inconsistent. We also call a subset S of A consistent (resp. inconsistent) if the theory axiomatized by S is consistent (resp. inconsistent).

(5) A consistent theory of $\mathbf{A}$ is called complete if it is not properly included in any consistent theory of $\mathbf{A}$.

Theorem 2.17. *For a non-empty subset Θ of an L-algebra $\mathbf{A}$, the following are equivalent.*

(1) Θ *is a theory.*

(2) Θ *satisfies the following:*

(a) $(p \to p) \in \Theta$;

(b) if $(p \to q), (q \to r) \in \Theta$, *then* $(p \to r) \in \Theta$;

(c) $(p \to p \vee q), (q \to p \vee q) \in \Theta$;

(d) if $(p \to r), (q \to r) \in \Theta$, *then* $(p \vee q \to r) \in \Theta$;

(e) if $p, (p \to q) \in \Theta$, *then* $q \in \Theta$.

(3) Θ *satisfies the following:*

(a) $(p \to p) \in \Theta$;

(b) $((p \to q) \to ((q \to r) \to (p \to r))) \in \Theta$;

(c) $(p \to p \vee q), (q \to p \vee q) \in \Theta$;

(d) $((p \to r) \to ((q \to r) \to (p \vee q \to r))) \in \Theta$;

(e) if $p, (p \to q) \in \Theta$, *then* $q \in \Theta$.

(4) Θ *satisfies the following:*

(a) $\Delta \subseteq \Theta$;

(b) if $p, (p \to q) \in \Theta$, *then* $q \in \Theta$.

(5) Θ *satisfies the following:*

(a) if $p, q \in \Theta$, *then* $(p \wedge q) \in \Theta$;

(b) if $p \in \Theta$ *and* $p \lesssim_\Delta q$ *(i.e.* $(p \to q) \in \Delta$*), then* $q \in \Theta$.

Proof.

(1) $\Leftrightarrow$ (2) The conditions in (2) are just those in Definition 2.11 written using the abbreviation $p \to q := \neg p \vee q$.

(1) , (2) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (3) Since $p \to p = 1$ in any Boolean algebra (Lemma 1.13(1)), $(p \to p) \in \Delta$. Hence (a) holds because $\Delta \subseteq \Theta$. (b), (c), and (d) are shown to hold in the same way using the equations (2), (3), and (4) in Lemma 1.13.

(3) $\Rightarrow$ (2) To see that (b) holds, assume that $(p \to q), (q \to r) \in \Theta$. Then (b) in (3) yields $(p \to r) \in \Theta$ by the repeated application of (e). Similarly (d) follows from (d) in (3) by virtue of (e).

(4) $\Rightarrow$ (5) Clearly, (4) implies (b) in (5). To see that (a) holds, let $p, q \in \Theta$. Since $p \to (q \to (p \wedge q)) = 1$ in any Boolean algebra (Lemma 1.13(5)), $(p \to (q \to (p \wedge q))) \in \Delta$, and hence $(p \to (q \to (p \wedge q))) \in \Theta$ by (a) in (4). Now we have $(p \wedge q) \in \Theta$ by the repeated application of (b) in (4).

(5) $\Rightarrow$ (4) To see that (a) holds, let $q \in \Delta$. By Lemma 2.12(1), $p \lesssim_\Delta q$ for any $p \in \Theta$. Hence $q \in \Theta$ by (b) in (5). To see that (b) holds, assume that $p, (p \to q) \in \Theta$. By (a) in (5), $(p \wedge (p \to q)) \in \Theta$. Since $(p \wedge (p \to q)) \to q = 1$ in any Boolean algebra (Lemma 1.13(6)), $((p \wedge (p \to q)) \to q) \in \Delta$. Now we have $q \in \Theta$ by (b) in (5).

□

If **B** is a Boolean algebra, then Δ is the trivial congruence and the preorder $\lesssim_\Delta$ coincides with the intrinsic partial order $\leq$ of **B**, and the conditions in Theorem 2.17(5) are written as

(1) if $p, q \in \Theta$, then $(p \wedge q) \in \Theta$;

(2) if $p \in \Theta$ and $p \leq q$, then $q \in \Theta$.

A non-empty subset Θ of a Boolean algebra satisfying these conditions is called a filter. The notions of a theory and a filter thus coincide in a Boolean algebra. For a general L-algebra, the following theorem holds.

Theorem 2.18. *Let* **A** *be an L-algebra. The assignment* $\Theta \mapsto \Theta/\Delta$ *yields a lattice isomorphism from the lattice of theories of* **A** *to the lattice of filters of the Boolean algebra* $\mathbf{A}/\Delta$*.*

Proof. Immediate from Corollary 2.9 and Theorem 2.15. □

Theorem 2.19 and Theorem 2.20 below are derived from each other.

Theorem 2.19. *Let* S *be a non-empty subset of a Boolean algebra* **B**. *An element* $q \in B$ *is in the filter generated by* S *if and only if* $p_1 \wedge \cdots \wedge p_n \leq q$ *for some* $p_1, \ldots, p_n \in S$.

Proof. See the proof of Theorem 2.20. □

Theorem 2.20. *(Deduction theorem). Let* S *be a non-empty subset of an L-algebra* **A**. *An element* $q \in A$ *is in the theory axiomatized by* S *if and only if* $p_1 \wedge \cdots \wedge p_n \lesssim_\Delta q$ *(i.e.* $(p_1 \wedge \cdots \wedge p_n \to q) \in \Delta$*) for some* $p_1, \ldots, p_n \in S$.

Proof. Define $\overline{S}$ by

$$\overline{S} = \{q \in A : p_1 \wedge \cdots \wedge p_n \lesssim_\Delta q \text{ for some } p_1, \ldots, p_n \in S\}.$$

We must prove that $\overline{S} = \mathcal{T}(S)$, and for this it suffices to show that

(1) $S \subseteq \overline{S}$;

(2) $\overline{S}$ satisfies the conditions in Theorem 2.17(5);

(3) if a set $\Theta \subseteq A$ contains S and satisfies the conditions in Theorem 2.17(5), then Θ contains $\overline{S}$.

But, these are easily verified. □

Corollary 2.21. *(Compactness theorem). The closure system of theories of an L-algebra* **A** *is algebraic; that is, for any subset* S *of* A,

$$\mathcal{T}(S) = \bigcup \{\mathcal{T}(C) : C \text{ is a finite subset of } S\}.$$

2.4. Formal deductions

The defining conditions of a theory such as (2) and (3) in Theorem 2.17 give rise to a deductive system. The following deductive system is derived from the conditions in Theorem 2.17 (2).

- Logical axioms:
  (1) $p \to p$
  (2) $p \to p \vee q$
  (3) $q \to p \vee q$
- Inference rules:
  (1) $p \to q, q \to r \vdash p \to r$
  (2) $p \to r, q \to r \vdash p \vee q \to r$
  (3) $p, p \to q \vdash q$

Except for the last inference rule (modus ponens), the deductive system has its origin in the defining conditions of a Boolean algebra (the conditions in Definition 1.1).

Definition 2.22. Let $\mathbf{A}$ be an L-algebra, $S \subseteq A$, and $p \in A$. A finite sequence $(p_1, \ldots, p_n)$ of elements of A such that $p = p_n$ is called a deduction (or proof) of p from S if for each $i \leq n$ one of the following holds:

(1) p_i is a logical axiom;

(2) $p_i \in S$;

(3) there are $j, k < i$ such that $p_j, p_k \vdash p_i$ is an inference rule.

S is said to syntactically entail p, written $S \vdash p$, if there is a deduction of p from S.

Theorem 2.23. *Let* $\mathbf{A}$ *be an L-algebra,* $S \subseteq A$, *and* $p \in A$. *Then* $p \in \mathcal{T}(S)$ *if and only if* $S \vdash p$.

Proof. Define $\overline{S}$ by

$$\overline{S} = \{p : S \vdash p\}.$$

We must prove that $\mathcal{T}(S) = \overline{S}$, and for this it suffices to show that

(1) $S \subseteq \overline{S}$;

(2) $\overline{S}$ satisfies the conditions in Theorem 2.17 (2);

(3) if a set $\Theta \subseteq A$ contains S and satisfies the conditions in Theorem 2.17 (2), then Θ contains $\overline{S}$.

But, these are easily verified. □

Only a finite number of elements in S appear in a deduction. This fact gives another proof of Corollary 2.21.

2.5. Interpretations

A filter of a Boolean algebra $\mathbf{B}$ is called an ultrafilter if it is not properly included in any proper filter of $\mathbf{B}$.

Theorem 2.24. *A theory Φ of an L-algebra $\mathbf{A}$ is complete if and only if Φ/Δ is an ultrafilter of the Boolean algebra $\mathbf{A}/\Delta$.*

Proof. Immediate from Theorem 2.18. □

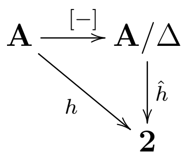
The following characterization of ultrafilters is easily proved.

Fact 2.25. *A filter Φ of a Boolean algebra $\mathbf{B}$ is an ultrafilter if and only if Φ is the kernel of some homomorphism from $\mathbf{B}$ to the two-element Boolean algebra $\mathbf{2}$.*

A complete theory in a general L-algebra is characterized in the same say.

Theorem 2.26. *A theory Φ of an L-algebra $\mathbf{A}$ is complete if and only if Φ is the kernel of some Boolean homomorphism from $\mathbf{A}$ to the two-element Boolean algebra $\mathbf{2}$.*

Proof. By Theorem 2.10, Boolean homomorphisms $\mathbf{A} \to \mathbf{2}$ and homomorphisms $\mathbf{A}/\Delta \to \mathbf{2}$ correspond one-to-one via the following commutative diagram:



The assertion thus follows from Theorem 2.24 and Fact 2.25. □

The following fact is a consequence of Zorn's lemma.

Fact 2.27. *(Ultrafilter theorem). Every proper filter Θ of a Boolean algebra is included in an ultrafilter; in fact, Θ is given as the intersection of the ultrafilters that include it.*

In a general L-algebra, the ultrafilter theorem is rephrased as follows.

Theorem 2.28. *Every consistent theory Θ of an L-algebra is included in a complete theory; in fact, Θ is given as the intersection of the complete theories that include it.*

Proof. Immediate from Fact 2.27 on noting Theorem 2.18 and Theorem 2.24. □

Definition 2.29. Let $\mathbf{A}$ be an L-algebra. A Boolean homomorphism from $\mathbf{A}$ to the two-element Boolean algebra $\mathbf{2}$ is called an interpretation of $\mathbf{A}$. Given $S \subseteq A$, an interpretation $h : \mathbf{A} \to \mathbf{2}$ is called a model of S if S is included in the kernel of h; that is, if h assigns 1 to every element of S. Given $S \subseteq A$ and $p \in A$, S is said to semantically entail p, written $S \Vdash p$, if every model of S is a model of $\{p\}$.

Theorem 2.30. *Let $\mathbf{A}$ be an L-algebra. A subset S of A has a model if and only if S is consistent.*

Proof. If an interpretation $h : \mathbf{A} \to \mathbf{2}$ is a model of S, then the kernel of h is a complete theory of $\mathbf{A}$ and contains $\mathcal{T}(S)$. Hence S is consistent. If S is consistent (i.e. $\mathcal{T}(S)$ is a consistent theory), then by Theorem 2.28, $\mathcal{T}(S)$ is included in a complete theory Φ. By Theorem 2.26, Φ is the kernel of some Boolean homomorphism $h : \mathbf{A} \to \mathbf{2}$. Clearly, h is a model of S. □

Theorem 2.31. *Let* $\mathbf{A}$ *be an L-algebra,* $S \subseteq A$*, and* $p \in A$*. Then* $p \in \mathcal{T}(S)$ *if and only if* $S \Vdash p$*.*

Proof. If S is inconsistent (i.e. $\mathcal{T}(S) = A$), then by Theorem 2.30, S has no models and $S \Vdash p$ holds vacuously for any $p \in A$. Hence the assertion is true in this case. Now suppose that S is consistent (i.e. $\mathcal{T}(S)$ is a consistent theory). Since $\mathcal{T}(S)$ is the intersection of all the theories containing S, $p \in \mathcal{T}(S)$ iff p is in every consistent theory containing S. By Theorem 2.26, $S \Vdash p$ iff p is in every complete theory containing S. Hence the proof is complete if we show that p is in every consistent theory containing S iff p is in every complete theory containing S. But this is immediate from Theorem 2.28. □

Theorem 2.32. *Let* $\mathbf{A}$ *be an L-algebra,* $S \subseteq A$*, and* $p \in A$*.*

(1) (Completeness theorem). If $S \Vdash p$*, then* $S \vdash p$*.*

(2) (Soundness theorem). If $S \vdash p$*, then* $S \Vdash p$*.*

Proof. Immediate from Theorem 2.31 and Theorem 2.23. □

2.6. The formal language of propositional logic

The formal language of propositional logic is given by the term algebra $\mathrm{L}(V)$ of type $\mathrm{L} = (\vee, \neg)$ generated by a set V of propositional variables. The elements of $\mathrm{L}(V)$ constitute the well-formed formulas of propositional logic.

By the freeness of $\mathrm{L}(V)$, every map from V to an L-algebra $\mathbf{A}$ extends uniquely to an L-homomorphism from $\mathrm{L}(V)$ to $\mathbf{A}$. In particular, a truth assignment (i.e. a map from V to the two-element Boolean algebra $\mathbf{2}$) extends uniquely to an interpretation of $\mathrm{L}(V)$. Semantic entailment can thus be defined in terms of truth assignments.

If Θ is a theory of $\mathrm{L}(V)$, then the quotient Boolean algebra $\mathrm{L}(V)/\Theta$ is called the Lindenbaum algebra of Θ. The Booleanization $\mathrm{L}(V)/\Delta$ of $\mathrm{L}(V)$ yields a free Boolean algebra over V. Since every algebra is a homomorphic image of a free algebra, every Boolean algebra is isomorphic to a Lindenbaum algebra.

A. Appendix. Preorder

Provided below are some basic facts on preorders.

(1) Every preordered set $(A, \lesssim)$ induces a poset $(A/{\equiv}, \leq)$ by the equivalence relation $\equiv$ defined on A by

$$x \equiv y \text{ iff } x \lesssim y \text{ and } y \lesssim x$$

and the partial order $\leq$ defined on the quotient set $A/{\equiv}$ by

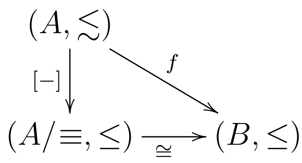
$$[x] \leq [y] \text{ iff } x \lesssim y$$

, where $[x]$ and $[y]$ denote the equivalence classes containing x and y.

(2) Let f be a surjective map from a set A onto a poset $(B, \leq)$. Then f induces a preorder $\lesssim$ on A by

$$x \lesssim y \text{ iff } f(x) \leq f(y).$$

The map f then becomes a preorder morphism $(A, \lesssim) \to (B, \leq)$. Moreover, the poset $(A/{\equiv}, \leq)$ induced by the preordered set $(A, \lesssim)$ is isomorphic to $(B, \leq)$; in fact, there is a canonical isomorphism $(A/{\equiv}, \leq) \cong (B, \leq)$ making the diagram



commute.

(3) As a special case of (2) above, consider an equivalence relation θ on a set A and suppose that a partial order $\leq$ is defined on the quotient set A/θ. Then the projection $[-] : A \to A/\theta$ induces a preorder $\lesssim$ on A by

$$x \lesssim y \text{ iff } [x] \leq [y].$$

The projection $[-]$ then becomes a preorder morphism $(A, \lesssim) \to (A/\theta, \leq)$, and the poset induced by the preordered set $(A, \lesssim)$ coincides with the original poset $(A/\theta, \leq)$.

(4) If $(A, \lesssim)$ is a preordered set, a join (least upper bound) of any two elements $a, b \in A$ is defined in the same way as in a poset and denoted by $a \vee b$. A join $a \vee b$, if exists, is characterized by the following properties:

(a) $a \lesssim a \vee b$ and $b \lesssim a \vee b$;

(b) for all $c \in A$, if $a \lesssim c$ and $b \lesssim c$, then $a \vee b \lesssim c$.

(5) A preordered set $(A, \lesssim)$ may be viewed as a thin category. Any two elements $x, y \in A$ are isomorphic (in the sense of category theory) precisely when $x \equiv y$, and a join $a \vee b$ is the same thing as a coproduct of $a, b \in A$. Joins in a preordered set thus enjoy the properties of coproducts in a category, in particular:

(a) $a \vee b \equiv b \vee a$;

(b) if $a \equiv a'$ and $b \equiv b'$, then $a \vee b \equiv a' \vee b'$.

(6) Let $(A/{\equiv}, \leq)$ be the poset induced by a preordered set $(A, \lesssim)$. If $(A, \lesssim)$ and $(A/{\equiv}, \leq)$ are viewed as thin categories, the projection $[-] : A \to A/{\equiv}$ will be an equivalence functor. The projection thus preserves and reflects joins (i.e. coproducts) as well as all other colimits and limits.

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