Electrodynamics on the threshold of the fourth stage of its development

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Abstract

Development of electrodynamics during last one and half a century is discussed and it is shown that every half a century its contents changes drastically. In fact, during each of the three past stages three distinct doctrines were reigning in the field, which can safely be visioned as three different theories. Each of these three theories is critically analyzed and it is shown that the third stage is over and electrodynamics has reached the threshold of the fourth stage of its development.

1 Introduction

Discovery of the Maxwell's equations circa one and half a century ago was the birth of electrodynamics. Before that, electric and magnetic phenomena were not put together into one general phenomenon of electromagnetism. After that, this branch of theoretical physics passed three stages, within which which it took various forms, due to changes of generally-accepted views upon space, time and corresponding mathematical frameworks. Evolution of the study had the form of modifications of the currently accepted theory so that a new one was created, perhaps, the same way as the very first theory which was Maxwell's electrodynamics, also was created as a modification of electrostatics and magnetostatics by introducing time-varying fields.

Further modifications occurred approximately every half a century. Each of them signified that the current stage of evolution of the study is over and a new one is ahead. During the next stage Maxwell's electrodynamics in its original form, was being replaced with a new theory. The new theory has new mathematical structure which employs another mathematical language. So, after three half-a century stages, classical electrodynamics took the form of a consequence of theories which are presented in the literature in chronological order.

In this review we outline theories which constitute the subject of classical electrodynamics and expose modifications which each of them underwent at the end of its stage. A special attention will be paid to mathematical structures of the theories, differences between them and the reasons why a new theory should replace the previous one. All these consideration led to the current state of classical electrodynamics, in which all signs are seen that the current stage is over and a new one is ahead.

2 Maxwell's electrodynamics

The passage from static to time-varying fields requires, first of all, that time appears in the scope as the fourth independent variable. Since, till the moment, all independent variables were spatial coordinates, a question arose, whether the fourth independent variable should represent the fourth dimension. If so, a certain geometry of the 4-dimensional continuum should be specified. Another opportunity was to leave everything as it was in the Newtonian theory. This theory reads that the space is absolute and immobile and time is just an additional independent variable, hence, also absolute. So, it happened that Maxwell's electrodynamics was built within context of the Newtonian theory.

Usually, mathematical structure of a physical theory defines two kind of objects which are the domain and functions on it. In case of Maxwell's electrodynamics, the domain is presented by space and time, thus as a 4-dimensional continuum. This domain splits into Euclidean 3-space $\mathbb{E}^3$ and time. The earlier possesses the well-known geometric and algebraic structures and the latter has no geometric meaning. Geometric and algebraic structures of $\mathbb{E}^3$ contain operations of scalar and vector multiplications that underlie operations of vector analysis which employ the operator ∇ and which exist only in three dimensions.

This domain serves as a support of functions which represent the field and its sources. The fields are represented by the vectors of electric $\vec{E}$ and magnetic $\vec{H}$ strengths and the sources are scalar charge density and vector current density $\vec{J}$. These strengths and densities satisfy Maxwell equations [1]

$$\begin{aligned} \nabla \cdot \vec{E} &= 4\pi\rho, & \nabla \cdot \vec{H} &= 0 \\ \nabla \times \vec{H} &= 4\pi\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} & \nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \end{aligned} \tag{1}$$

where c is speed of light.

3 Post-Maxwellian electrodynamics

The main reason to include time as the fourth dimension is motion. Motion of a charge density ρ with velocity $\vec{v}$ transforms it such a way that a current density $\vec{J} = \rho\vec{v}$ appears. Since in Maxwell's electrodynamics ρ produces only $\vec{E}$ and $\vec{J}$ does only $\vec{H}$, this transformation touches the strengths. These considerations, among others, lead to the idea of a grand unification under which the notions of rest and motion become relational, space and time join together into space-time, electric and magnetic fields do into a new entity called electromagnetic field. All this means that Maxwell's electrodynamics as it was presented above was to be replaced with a new theory based on the idea of pseudo-Euclidean space-time of dimension 4, in which charge density ρ is the time component of the vector of current density. The strength $\vec{E}$ and $\vec{H}$ are not longer vectors, but components of an object which was not known before. The commonplace vector algebra which works in 3 dimensions, is useless under new conditions as well as the operator ∇, therefore, the field equations cannot longer be expressed in the form (1).

The new mathematical structure is needed which is manifestly Lorentz-invariant. Since Lorentz transformations have been conducted only in Cartesian coordinates $\{t, x, y, z\}$ (here and below the speed of light is put equal to unity; the index '0' will denote the time component), only this kind of coordinate systems was employed on this stage. Thus, mathematical structure of post-Maxwellian electrodynamics is organized as follows. The domain is Minkowski space-time endowed with Cartesian coordinates $\{t, x, y, z\}$. The field and its source are represented by the vectors of potential and current density specified by their components A^i and J^i where A^0 and J^0 stand for electrostatic potential and charge density. Maxwell equations are replaced with manifestly Lorentz-invariant equations

$$F_{ij} = \partial_i A_j - \partial_j A_i, \quad \partial_i F^{ij} = 4\pi J^i, \tag{2}$$

where antisymmetric tensor F_{ij} consists of components of the field strengths. All operations over the functions are defined in the well-known tensor analysis.

All solutions which can be obtained in Cartesian coordinates, are plane waves and their combinations. The only approach to the field equations (2) suggested by the theory, consists in the following.

1. It was accepted without any satisfactory foundation that vector potential of electromagnetic field can be treated as quartet of massless scalar fields. Then, the source-free equations (2) can be replaced with the similar scalar equation

$$\Box A^i = 0. \tag{3}$$

2. It was accepted without any foundation that a vector of current density taken in each space-time point produces its own contribution to the entire field as a point-like source and that this contribution is strictly parallel to the source. Then, the entire field is vector sum of contributions from all points of the source. These assumptions were used as a formal justification of the assumption that Green's function for the equations (2) exists.

3. It was accepted that Green's functions for the equations (2,3) have the form

$$G^i_j = \delta^i_j G$$

where G is Green's function for the equation (3) and δ^i_j is Kronecker's delta. It must be pointed out that any Green's function for a vector equation is referred to vector frames established in two distinct points, which, in general, are different. The definition of the Green function presented above requires, first of all, that the space-time is flat, and second, that only Cartesian coordinates can be used in the theory [1, 2].

4. The scalar Green's function has been represented in the form of Fourier integrals over plane wave solutions of the D'Alembert equation (3).

4 On the Green's function for the D'Alembert equation

Since in post-Maxwellian electrodynamics D'Alembert equation is used as that for the vector potential, it is natural to consider some properties of the Green's function for it. It is natural to expect, from physical point of view, that the Green's function $G(x, y)$ is unique. There is theorem which states uniqueness of solution of equations like (3) that also signifies that the Green's function in question is unique. Contrary to it, standard texts on quantum field theory expose a special theory of Green's functions which reads that there exist various Green's functions for this equation. Below we discuss some points of this theory.

In pseudo-Euclidean space-time, a Green's function becomes unique only after the role of causality is established. Causality determines in which of three zones – inside of one of two light cones or beyond of both of them the field can be produced from a given point of the space-time. Ordinary causality requires that the field must only be produced in the light cone of the future. However, there exists another point of view, according to which a charge and its field constitute an integral whole, so that a charge does not produce its field, they exist only together. Whenever "another" field acts on a charge, it first changes the field around, so that change of the field predates perturbation of charge motion. From this point of view, the Green's function must be non-zero only between the cones of the past and the future.

Explicit form of the Green's function can be obtained the same way as Coulomb potential was obtained as a relevant solution of the Laplace equation. By analogy to it, one obtains from the D'Alembert equation that the desired Green's function is equal to inverse square of the interval from the given point. Besides, there exist retarded and advanced Green's functions used by Lienard and Wiechert in XIX century when constructing the field of moving charge. So even if one has made his choice and decided which one of domains, interior of the light cone of the future or the space-time beyond the cones should serve as the support of the Green's function, there are two functions which satisfy the choice whereas there must be only one. The complete list of Green's functions is:

$$\begin{aligned} G &= \frac{1}{t^2 - r^2} \\ G_{ret} &= \frac{\delta(t-r)}{t+r} \\ G_{adv} &= \frac{\delta(t+r)}{t-r} \end{aligned} \tag{4}$$

and one needs to chose only one in accord with the assumed role of causality. If the field is produced by a charge, the Green's function is non-zero only in the light cone of the future, otherwise its support lies beyond the cones. In both cases one has inverse square taken for the corresponding domain and either retarded or the sum of retarded and advanced functions. Thus, totally two functions in both cases. However, according to the uniqueness theorem, only one of them is valid. The retarded function was used in classical electrodynamics, whereas the inverse square have never been used. Therefore it is natural to look what results the latter gives in the simplest case of the field of a rest point-like charge for which the retarded Green's function yields the expected Coulomb's law. This result comes out for both electromagnetic and scalar fields. Below we show what the inverse square yields in this case.

As usual, calculation will be completed in spherical coordinates $\{t, r, \theta, \varphi\}$ with unit charge which produces scalar field is placed on the world line $r = 0$. Consider the field in a point with coordinates t and r with arbitrary θ and φ. The field in question is equal to

$$\int_{-\infty}^{t-r} \frac{\mathrm{d}s}{(t-s)^2 - r^2} = \int_{-\infty}^{-r} \frac{\mathrm{d}u}{u^2 - r^2},$$

where we have substituted $u = s - t$. This integral is logarithmically divergent, so, this calculation gives infinite value for the field in all space-time points. Evidently, the same will happen for any other source of the field because each its point contributes this way. Thus, application of the inverse square as the Green's function yields an apparently unreasonable result, consequently inverse square of interval is not Green's function for massless scalar field. Thereby uniqueness of the Green's function for this field is proved because $G_{ret}(x, y)$ is the only one for it. It must be pointed out that Green's function used in quantum electrodynamics differs from this one, consequently whenever it is used, the result should be infinite. In the next section we analyze consequences of usage of this function for quantum electrodynamics.

Any Green's function for the D'Alembert equation which does not break causality and which is not exactly G_{ret}, contains a part proportional to inverse square of the interval and hence, whenever it is used, the result obtained is infinite. In classical electrodynamics the field obtained by the method of Green's function is strictly finite almost everywhere because the right Green's function was used. As for quantum electrodynamics, it is known that for some reason there another Green's function is used, hence, the used one inevitably turns any amplitude where it is used, into infinity. This phenomenon is well-known. Physicists use to explain it by punctuality of the electron. In this section we analyze these explanations.

In quantum field theory, Green's functions are being represented only in the form of their Fourier transforms, so that their shapes in the space-time is not seen. In particular, what cannot be seen in this form, is inevitability of divergences whenever photon propagator is used. Nevertheless, for some reason, ultraviolet divergences are encountered only in loop diagrams. If no other ultraviolet diagrams really exist, the encountered ones can be justified by the theory of renormalizations. However, it seems to be strange that, on one hand, divergences should appear whenever photon propagator was used and on the other hand, they did not in case of *ee*-scattering. So, a question arises, how it happened.

5 Uniqueness of the Green's function for the D'Alembert equation

By definition, the Green's function for the D'Alembert equation is solution of the equation

$$\Box G(P,Q) = \delta(P,Q) \tag{5}$$

where P and Q stand for two space-time points and the right-hand side is the well-known Dirac's δ-function on it. Here the equation is represented not in the generally-accepted coordinate form because this form is well-known only for Cartesian coordinates which are not used below. The problem of Green's function arose before in the theory of gravitational potential. Then it was obtained as the relevant solution of the Laplace equation, which is nothing but the Coulomb potential $\Phi = 1/r$ in spherical coordinates $\{r, \theta, \varphi\}$. It can easily be shown that δ-function in the right-hand side appears due to constant flux of gradient of Φ through any closed surface about the origin of coordinates.

It is natural to use this approach to the equation (4). For this end, introduce pseudo-spherical coordinates $\{\zeta, \eta, \theta, \varphi\}$ for the space-time, where

$$\zeta = \sqrt{t^2 - r^2}, \qquad \eta = arctanh\frac{r}{t}; \tag{6}$$

$$\zeta = \sqrt{r^2 - t^2}, \qquad \eta = arctanh\frac{t}{r} \tag{7}$$

for the interior of the light cone of the future and for the domain beyond the cones correspondingly. Inside the cone ζ is the time coordinate, and beyond this role belongs to the coordinate η. In both cases the most relevant solution of the D'Alembert equation depends only on ζ has the form ζ^{-2} that is inverse square of the interval. Hence, this approach yields the function G as it is given in the equation (4). Though everything was done exactly as in the previous case, the function obtained is not the desired Green's function for the D'Alembert equation. The point is that coordinate surfaces $\zeta = \text{const}$ are not closed and the origin of coordinates is not one of them. Though the flux of gradient through any of these surfaces is same, they do not reduce to a point, therefore δ-function does not appear in the right-hand side of the equation (4) and as a result, solution of the D'Alembert equation is not the Green's function for it. At the same time, another function of single variable ζ, which is $\delta(\zeta^2)$, was actually accepted for the role of the Green's function for this equation. Indeed, this function can be represented as the sum of G_{ret} and G_{adv}, which have distinct supports, hence can be employed independently. Though it is unclear, whether or not this function satisfies the equation (5), this one was widely used in this capacity in classical electrodynamics.

6 Green's functions and plane waves

Another opportunity to obtain the Green's function is Fourier transform. Represent the desired function in Cartesian coordinates as a superposition of plane waves

$$G(t,\vec{r}) = \int \mathrm{d}^4x e^{\imath(\omega t - \vec{k}\cdot\vec{r})} g(\omega,\vec{k}). \tag{8}$$

Then, according to the equation (5) we have

$$(\omega^2 - \vec{k}^2) g(\omega,\vec{k}) = 1,$$

hence, the Green's function can be represented in the form

$$G(t,\vec{r}) = \int \frac{\mathrm{d}\omega \mathrm{d}\vec{k}}{\omega^2 - \vec{k}^2}\, e^{\imath(\omega t - \vec{k}\cdot\vec{r})} \tag{9}$$

and it remains to take this integral. One can expect that the result will be equal to the inverse square of the interval. Then Fourier transform of the inverse square of the interval would yield the inverse square of the wave 4-vector so that these two inverse squares would be Fourier transforms of each other. However, it is not so because the integrals diverge. In other words, the integral (9) does not exist as a certain function of t and $\vec{r}$ and vice versa, Fourier transform of the inverse square of the interval does not exist as a certain function of ω and $\vec{k}$ because it is the same diverging integral. In both cases the integrand is an oscillating function with growing amplitude. To see this, let us calculate the integral

$$g(\omega,\vec{k}) = \int \frac{\mathrm{d}t \mathrm{d}\vec{r}}{\omega^2 - \vec{k}^2}\, e^{\imath(\omega t - \vec{k}\cdot\vec{r})}. \tag{10}$$

It suffices to show that the integral diverges in some certain part the domain of integration, say, in the light cone of the future. In this domain the ordinary spherical and pseudo-spherical coordinates (6) are related as

$$ct = \zeta \cosh\eta, \quad r = \zeta \sinh\eta.$$

Note that these coordinates specify a certain frame of reference. Chose in this particular frame the wave 4-vector with $\vec{k} = 0$. Then $\omega t - \vec{k}\cdot\vec{r} = \omega\zeta\cosh\eta$ and therefore

$$g(\omega,0) = \int\limits_0^\infty \mathrm{d}\zeta\zeta^3 \int\limits_0^i nfty \mathrm{d}\eta \sinh^2\eta \int\limits_0^\pi \mathrm{d}\theta \sin\theta \int\limits_0^{2\pi} \mathrm{d}\varphi \frac{e^{\imath\omega\zeta\cosh\eta}}{\zeta^2} =$$

$$4\pi \int\limits_0^\infty \mathrm{d}\zeta\zeta \int\limits_0^\infty \mathrm{d}\eta \sinh^2\eta e^{\imath\omega\zeta\cosh\eta}.$$

The result is product of two integrals over half-line each, with integrands being oscillating functions with growing amplitudes. Each of them diverges and no way is seen how to regularize them. Lorentz invariance of the result signifies that the integral diverges for each time-like interval, thus, everywhere in the light cone of the future.

A similar divergence occurs beyond the light cones and finally it turns out that neither the integral (9) nor the integral (10) exist. So, if the Green's function for the D'Alembert equation exists, it has no Fourier transform, or, as theoretical physicists call it, representation in the momentum space. Similarly, inverse square of the wave 4-vector which they use as representation of the Feynman's propagator in the momentum space, does not correspond to any function on the space-time.

7 Change of fundamental notions and covariance breakdown

As was pointed out above, Cartesian coordinates is the only coordinate system used in quantum field theory. This restriction actually changes some fundamental notions so that in this area they have not the same meaning as in all the rest physics. First, it requires that the space-time can only be flat and hence, it can be endowed with the structure of vector space. This substitution is particularly important when considering Green's function which takes the form of single variable function $G(x-y)$ where x qand y stand for two vectors in this space. Second, the notion of coordinate transformation reduces to application of the Lorentz group. Third, the notion of covariance turns into form-invariance under Lorentz transformations. Fourth, according to the entire picture drawn by these substitutions, there is no difference between vector potential of electromagnetic field and quartet of massless scalar fields, therefore equations for the vector potential are replaced with the D'Alembert equation for each component of the vector potential. That is how one non-covariant equation penetrates the field theory.

To see what is non-covariant equation, consider the following example. Let a vector field $\vec{v}$ be constant in the space-time specified by its components $v^t = v^x = v^z = 0$, $v^y = 1$. Then, evidently, all its components satisfy the D'Alembert equation $\Box v^i = 0$. Now, pass to round cylinder coordinates $\{t, z, \rho\varphi\}$ with

$$\rho = \sqrt{x^2 + y^2}, \quad \varphi = \arctan\frac{y}{x}.$$

Components of the vector are now $v^t = v^z = 0\, v^\rho = \cos\varphi$, $v^\varphi = \frac{1}{\rho}\cos\varphi$, and none of them satisfies the equation. So, in this case the we see that the D'Alembert equation is valid in some coordinate systems and invalid in others. If it is so, the equation is physically meaningless.

Indeed, the D'Alembert equation is physically meaningful only while it is covariant. Covariance means validity of an equation disregard of choice of a coordinate system. All covariant equations can be derived without introducing any coordinate system by using exterior differential calculus and, as a result, they have coordinate-free form. The D'Alembert equation $\Box\Phi = 0$ is manifestly covariant because here the D'Alembert operator is applied to a scalar. Action of the same operator on components of a vector is a non-covariant operation, thus, physically meaningless. Nevertheless, the generally accepted form of the equation for the vector potential of electromagnetic field has the form

$$\Box A^i = 0. \tag{11}$$

This equation is non-covariant, thus, physically meaningless, but it is used mainly in Cartesian coordinates, therefore its non-covariance was not discovered.

8 Electrodynamics and tensor analysis

It was clear from the very beginning of the post-Maxwellian stage that a physical theory can neither be tied to a special class of coordinate systems, nor demand that the space-time must be flat. As well-known, any coordinate system is locally-Cartesian, therefore, a theory which is rigidly tied to Cartesian coordinates $\{t, x, y, z\}$ has strictly local meaning. A need for modification which admits usage of arbitrary coordinate systems, was evident. At first glance, such a modification seems to be just a technical problem. It was natural to expect that it will be resolved during the third stage which started circa in 1960 [3].

Before analyzing efforts to do it which were made during this stage, one more basic of electrodynamics need to be considered. It is well-known that the charge conservation law has the form the equations

$$\nabla \cdot \vec{J} = 0, \quad \partial_i J^i = 0 \tag{12}$$

in magnetostatics and electrodynamics correspondingly. In both cases the law follows from the field equations [4], particularly, from the equation (2) because

$$4\pi \partial_i J^i = \partial_i \partial_j F^{ij} \equiv 0 \tag{13}$$

due to antisymmetry of F^{ij} and commutativity of differential operators ∂_i and ∂_j. This interconnection between the field equations and the charge conservation law is a fundamental property of electrodynamics.

Besides, the fact that the law follows from the field equations signifies that these equations are meaningless and have no solutions unless unless the current density J^i in the right-hand side satisfies the equation (13). Since the charge conservation law is not a specific property of Cartesian coordinates, it is in force, like any other physical law, in any coordinate system as well as without introducing any coordinate system at all. Consequently, in a generally-covariant version of classical electrodynamics, the field equations should possess this property. Now, let us replace Cartesian coordinates and partial derivatives ∂_i with an arbitrary system and the corresponding covariant derivative D_i. It turns out that the equation(13) remains in force if and only if the space-time is flat, because only under this condition operators D_i and D_j commute. Thus, a straightforward passage from Cartesian coordinates $\{t, z, y, z\}$ and partial derivatives to arbitrary coordinates and corresponding covariant derivatives provides a generally-covariant version of classical electrodynamics which is valid under an additional condition that the space-time is flat. Evidently, this theory cannot be the ultimate one, but can be used in non-inertial frames, for example, in a uniformly accelerated one.

9 Achievements and domains of applications

After three stages of its development, classical electrodynamics took the form of a chain of three theories, each of which possesses its own domain of applications. Even pre-Maxwellian electrostatics and magnetostatics have their own domains because they are the most relevant theories in cases of stationary fields. And so is Maxwell's electrodynamics. Indeed, majority of tasks of technical electrodynamics are related to an inertial frame in which neither Lorentz invariance nor the space-time structure is significant. Evidently enough that Maxwell's electrodynamics is much more convenient for technical needs, than theories which have been created later. Consequently, no other theories are needed on this level.

The first stage of development of electrodynamics passed under strong influence of parallel development of mathematical physics, which brought numerous discoveries related to curvilinear coordinate systems and separation of scalar covariant equations. Successful development of electrostatics which is based on the Poisson equation, was a part of this process. However, all the rest equations of Maxwell's electrodynamics are non-scalar, therefore a major part of this theory lies beyond the well-developed areas of mathematical physics. Physicists of the time could guess only how to solve vector equation in Cartesian coordinates. It was expectasble that physicists will try to generalize the method of separation to non-scalar equations, but this did not happen because at the end of the stage time became the fourth dimension and construction of a new theory began in which no other coordinate systems were in use.

The only manifestly positive achievement of post-Maxwellian electrodynamics is formulation of the action principle for electromagnetic field and its source in Cartesian coordinates. All the rest achievements of this theory are results of application of the method of Green's functions mainly to the field of moving charge ("electron") and calculations of characteristics of radiation from it this way. As was pointed out above, existence of Green's functions for the equation (2) has never been proved, therefore all these achievements are, at least, doubtful. A generally-covariant theory which was built after 1960 seems to have neither positive achievements nor any domain of application.

10 Non-covariant equations in field theory

In 1974, Robert Wald citeWd proposed a model of black hole in an asymptotically uniform magnetic field. His goal was only to find out magnetic field in the space-time specified by its vacuum metric so that magnetic field does not serve as a source of gravitation. The field in question could well be obtained as the corresponding solution of the Maxwell equations at least, for the Schwartzschild space-time. However, Wald preferred to use, as he wrote, "the fact that a Killing vector in a vacuum spacetime serves as a vector potential for a Maxwell test field". In this section, we first analyze where this "fact" comes from and show that it was deduced from the equation (11) and second, show that in vicinity of a black hole, a Killing vector does not serve as a vector potential for a source-free electromagnetic field. As a result, we show that representation of magnetic field obtained this way, is wrong.

First, let us see, where this "fact" comes from. Differentiation and conversion of the Killing equation turns it into something similar to the Laplace equation. Ineed,

$$0 = g^{ij}\,\mathrm{D}(k_{j;k} + k_{k;j}) = g^{ij}\,\mathrm{D}_i\,\mathrm{D}_j\,k_k + g^{ij}\,\mathrm{D}_i\,\mathrm{D}_k\,k_j$$

where "$;i$" and D_i stand for covariant differentiation over x^i. The last term transforms as follows:

$$g^{ij}\,\mathrm{D}_i\,\mathrm{D}_k\,k_j = g^{ij}\{\mathrm{D}_k\,\mathrm{D}_i\,k_j + [\mathrm{D}_i, \mathrm{D}_k]k_j\}.$$

Now, the first term in figure brackets is zero because it contains divergence of the Killing vector. As for the second term, it disappears in vacua:

$$g^{ij}[\mathrm{D}_i, \mathrm{D}_k]k_j = g^{ij}R_{ikj}{}^l k_l = R_k{}^l k_l = 0.$$

Finally, one finds that in vacua, Killing vectors satisfy the equation

$$g^{ij}\,\mathrm{D}_i\,\mathrm{D}_k\,k_j = 0. \tag{14}$$

This equation looks similar to the D'Alembert equation for the vector potential. That is where the "fact" stated by Wald comes from.

Laplace operator is defined in the coordinate-free form (divergence of gradient) only by its action on scalar functions. In fact, Laplacian of a scalar function Φ can be written as $\triangle\Phi = g^{ij}\,\mathrm{D}_i\,\partial_j\Phi$ where the first differentiation is just taking partial derivative. Contrary to this, in the equation (14) the first differentiation is covariant, ot in other words, it contains non-differentiated component of the Killing vector multiplied by coefficients of connection. As a result, the left-hand side of this equation contains non-differentiated component of the vector multiplied twice by components of connection. Therefore, this equation differs from the commonplace Laplace equation.

The Killing vector used by Wald generates rotations and in an appropriate coordinate system has single non-zer component $k^\varphi = 1$. Evidently, this component and thereby, all components of this Killing vector satisfy the Laplace equation at least in coordinate systems with azimuthal angle φ as one of coordinates. But unlike the equation (14), this equation is for k^i's, not k_i's. The difference is, in particular, in spherical coordinates $\{t, r, \theta, \varphi\}$ that

$$k_\varphi = g_{\varphi\varphi}k^\varphi. \tag{15}$$

Thus, while k_φ satisfies one Laplace-like equation (14), k^φ does the standard one.

Now, let us see, what equation a purely azimuthal vector potential satisfies. For this end, represent it as a 1-form α

$$\alpha = A(r, \theta)\mathrm{d}\varphi \tag{16}$$

and take its exterior derivative:

$$\mathrm{d}\alpha = A_\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi - A_r \mathrm{d}\varphi \wedge \mathrm{d}r.$$

The next step is to take asterisk conjugation of the result:

$${}^*\mathrm{d}\alpha = \frac{1}{r^2 \sin\theta}\,A_\theta \mathrm{d}t \wedge \mathrm{d}r - \frac{1}{\sin\theta}\,A_r \mathrm{d}t \wedge \mathrm{d}\theta \tag{17}$$

and it remains to take exterior derivative of this 2-form:

$$\mathrm{d}\,{}^*\mathrm{d}\alpha = \left[\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial A}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 A}{\partial r^2}\right] \mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta.$$

So-called natural Laplacian of the 1-form α is thus

$${}^*\mathrm{d}\,{}^*\mathrm{d}\alpha = \left[\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial A}{\partial\theta}\right) + \frac{\partial^2 A}{\partial r^2}\right] \sin\theta\mathrm{d}\varphi.$$

According to source-free Maxwell equations, this 1-form is zero, hence, the function $A(r,\theta)$ satisfies the equation

$$\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial A}{\partial\theta}\right) + \frac{\partial^2 A}{\partial r^2} = 0$$

that apparently differs from the Laplace equation which foe a scalar $\Phi(r,\theta)$ has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0.$$

Substituting k^φ for A in the equation (16), using the equation (15), we find that this function satisfies the equation:

$$2\sin^2\theta + \sin\theta\left(\frac{1}{\sin\theta}2\sin\theta\cos\theta\right) = 0.$$

So, in a flat space-time this Killing vector serves as a source-free vector potential. Now, we repeat this calculation in the Schwartzschild space-time starting from the equation (17):

$${}^*\mathrm{d}\alpha\frac{1}{r^2\sin\theta}A_\theta\mathrm{d}t\wedge\mathrm{d}r - \frac{1}{\sin\theta}\left(1-\frac{2m}{r}\right)\mathrm{d}t\wedge\mathrm{d}\theta$$

and exterior differentiation of the result yields

$$\mathrm{d}\,{}^*\mathrm{d}\alpha = \frac{2m}{r^2\sin\theta}A_\theta\mathrm{d}t\wedge\mathrm{d}r - \frac{1}{\sin\theta}\left(1-\frac{2m}{r}\right)\mathrm{d}t\wedge\mathrm{d}\theta.$$

Finally,

$${}^*\mathrm{d}\,{}^*\mathrm{d}\alpha = \frac{2m}{r^2}\mathrm{d}\varphi$$

that is exactly the φ-component of the current density to produce this magnetic field. In other words, the Killing vector which generates rotation, does not serve as a source-free vector potential. Therefore, all results on black holes in asymptotically-uniform magnetic fields are erroneous.

11 Outlines of the new theory

The new theory already exists and is in use for long time. At the very beginning it looked like just a rewriting Maxwell equations equations in terms of exterior differential forms [7]. Discovery of isotrope complex tetrad and its property to separate Maxwell equations [8]-[10] signified that exterior calculus is actually the only possible mathematical form of electrodynamics. Change of the mathematics entailed change of the whole theory. So, the new theory is organized as follows.

Unlike post-Maxwellian electrodynamics, the new theory does not prescribe the space-time to be obligatory flat and endowed with obligatory Cartesian coordinates. So, the space-time is assumed to be an arbitrary pseudo-Riemannian (3+1)-space endowed with an arbitrary coordinate system $\{x^i\}$. Electromagnetic field is specified by a 1-form $\alpha \equiv A_i \mathrm{d}x^i$ called "vector potential" which in fact is a co-vector. The 2-form $\mathrm{d}\alpha$ stands for the strength of the field and the field equation has the form

$$^*\mathrm{d}\,^*\alpha = I \tag{18}$$

where the 3-form I in the right-hand side specifies current density. For example, in a coordinate system $\{t, x^a\}$ with t being Lorentzian time, the component $\rho \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ stands for the charge density. As was shown in our works [4, 12], equations like (18) have no Green's functions. Therefore the method of Green's functions will not be used in the new theory and the field will only be obtained by solving the Maxwell equations as they stand.

Like other field equations, this one can be solved in the most general form by the method of variables separation in some coordinate systems for the flat space-time and in Boyer-Linquist coordinates for all space-time models of the Kerr's family. This equation separates and redices to ordinary differential equations in majority of coordinate systems used in mathematical physics (see, for example, [9, 10]) and in uniformly accelerated spherical coordinates [11]. So, unlike previous versions of classical electrodynamics, the new one is much closer to classical theories in which the master equations are solved in the most general form as expansions over the complete set of particular solutions obtained by the method of variables separation.

12 Conclusion

Classical electrodynamics has been created as a generalization of electrostatics and magnetostatics to the case of time varying fields about one and half century ago and passed three 50 years long stages of its development. Every half a century a new area of theoretical investigations appeared, in which theoretical doctrines of the time did not work. Appearance of a new area always served as a good excuse to build a new theory that is much easier than to solve a problem in an existing one. Three theories have been built and reigned during the three stages passed, but gave very little real achievements.

The only practical achievement of Maxwell's electrodynamics was theoretical prediction of electromagnetic waves and all the rest achievements of this theory like discovery of Lorentz group, are rather of fundamental value. Two other theories gave actually nothing but inflation of the subject. Besides of being completely useless, they have imposed misconceptions like retarded potentials. Ideological power of useless theories has merely marginalized applied areas of electrodynamics, which had really great achievements during last century. As a result, theoretical and applied areas of classical electrodynamics became completely different areas of activity.

Mathematical background of post-Maxwellian electrodynamics is based on somewhat dogmatic foundations which include obligatory flat space-time endowed with obligatory Cartesian coordinates. These foundations replace space-time geometry with linear algebra and attach the whole of theory to plane waves. Besides, they allow one to treat vector potential of electromagnetic field as quartet of massless scalar fields and believe that each its component satisfies the D'Alembert equation. Therefore, there was no question of existence of Green's function for electromagnetic field equation.

Post-Maxwellian electrodynamics served as an underlying base for quantum electrodynamics, which actually is rather a recipe of calculations than a physical theory. Green's function for the electromagnetic field plays vital role in these calculations. Since these functions exist only thanks to identifications of electromagnetic and scalar fields that is possible only in the erroneous mathematical background of the post-Maxwellian electrodynamics, all calculations completed with use of these functions are wrong as well as quantum electrodynamics itself. Even should the identification was correct, there exists a number of problems of Green's functions for the scalar field. First, Feynman's propagator as it is used in the momentum space, is not Fourier transform of any function on the space-time. Second, There exist only one relevant Green's function for the massless scalar field, which is so-called retarded Green's function which yields retarded potentials. Hence, calculations of amplitudes completed in the momentum space have nothing to do with Green's functions as they stand in the space-time. Otherwise, an attempt to complete the same calculations in the space-time leads to new divergences, particularly, of the *ee*-scattering amplitude, which cannot be justified by renormalization of electron mass and charge.

However, let us return to classical theory. The next step was made due to development of relativistic astrophysics accompanied with discoveries in astronomy. This development required solutions of numerous problems of classical electrodynamics which cannot be solved in the framework of the post-Maxwellian electrodynamics. First of all, this new development required that this area must be free of dogmatic foundations mentioned above. As such, the field equations are to be studied in general-covariant approach, thus, field equations must be rewritten in terms of covariant derivatives. Therefore, first, all partial derivatives were replaced by covariant ones, but this was solution of the problem because this does not help to solve Maxwell equations in more general coordinate systems than Cartesian one. Substitution of covariant derivatives for partial ones brought nothing. The only result obtained this way was wrong representation of asymptotically-uniform magnetic field in vicinity of a black hole when Killing vector is used as the vector potential.

A linear theory cannot be useful unless it presents complete solutions of its own equations. A method which provides results of this kind, was worked out and published ([6]-[10]), solutions obtained could well turn theoretical electrodynamics into a useful science long ago, and this will happen only as soon as the new theory is accepted. The new theory is based on application of exterior calculus as the main mathematical tool for electrodynamics. Unlike all previous versions of electrodynamics, the new one provide complete solutions of the Maxwell equations in various coordinate systems and the most important space-time models in relativistic astrophysics.

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