Toward an Understanding of Electromagnetic Phenomena

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On the basis of ordinary mathematical methods we discuss new classes of solutions of the Maxwell s equations discovered in papers by D. Ahluwalia, M. Evans and H. Múnera et al.
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Recently several authors have found additional solutions to the relativistic wave equations. These solutions are listed here:

1. The $E = 0$ solution of the Maxwell s $j = 1$ equations [1] which was found by considering the characteristic equation (in the momentum representation).
2. The $\mathbf{B}^{(3)}$ Evans-Vigier field [2], which was obtained as a cross-product of the transverse modes of electromagnetism: $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = iB^{(0)}\mathbf{B}^{(3)*}$ and cyclics.
3. Non-plane-wave solutions of the Klein-Gordon equation [3a,b] due to Múnera *et al.*, which were obtained by using unconventional basis functions and coupling *anzatz* , see [3a, Eqs. (11,12)].
4. The Múnera and Guzmán generalized solution of Maxwell s equations in terms of potentials [3c,d].
5. The Chubykalo and Smirnov-Rueda method of separated potentials , ref. [4], which enables us to regard a function with implicit time-dependence as a full-value solution of Maxwell s (and/or the D Alembert) equations.

Why did so many new and unexpected solutions appear at once? Let us look at this issue using the ordinary methods of solving the system of partial differential equations [5,6].

It is well known that the set of dynamical Maxwell s equations are equivalent to the following, *e.g.*, [7, Eqs.(4.21,4.22)]:*

$$\nabla \times [\mathbf{E} + i\mathbf{B}] - i\frac{\partial}{\partial t}[\mathbf{E} + i\mathbf{B}] = 0\,, \tag{1a}$$

$$\nabla \times [\mathbf{E} - i\mathbf{B}] + i\frac{\partial}{\partial t}[\mathbf{E} - i\mathbf{B}] = 0\,. \tag{1b}$$

This is a system of partial differential equations. It is easy to see that the second equation is just the parity conjugate $(\mathbf{x} \to -\mathbf{x})$ of the first one if one uses the usual interpretation of $\mathbf{E}$, a *vector*, and $\mathbf{B}$, an *axial vector*.

In the framework of this paper we shall look for solutions of (1a) in the generalized form

$$\mathbf{A} \equiv \mathbf{E} + i\mathbf{B} \sim \mathbf{a}\exp(\lambda t + \kappa \cdot \mathbf{r})\,,$$

where λ and κ are some unknown parameters, which provide characteristic polynomials, and $\mathbf{a} = \mathrm{column}(a_1\ a_2\ a_3)$ is some *constant* vector, which is defined by the boundary and/or normalization conditions. Thus, at this time we will not restrict our discussion to plane waves. By using the method of characteristic polynomials for the differential equation

$$\left[\frac{\partial}{\partial t} + \mathbf{J}\cdot\nabla\right]^{ij} A^j = 0\,, \tag{2}$$

with $\left(\mathbf{J}^i\right)^{jk} = -i\epsilon^{ijk}$, we obtain the algebraic equation for the parameters λ and κ :

$$Det\left[\lambda + (\mathbf{J}\cdot\kappa)\right]^{ij} = 0\,. \tag{3}$$

This has solutions $\lambda = 0$ and $\lambda = \pm|\kappa|$. In fact, we have repeated the procedure in ref. [1], but since this point of view is the most general, we do not as yet know how λ and κ are connected with energy and momentum. Thus, the general solution of the first Maxwell equation (1a) may be given, for instance, in the form:

$$\begin{aligned}\mathbf{E} + i\mathbf{B} = \mathbf{A}_1 \exp\left[\alpha_1\left(|\kappa|t + \kappa\cdot\mathbf{r}\right)\right] + \\ + \mathbf{A}_2 \exp\left[\alpha_2\left(-|\kappa|t + \kappa\cdot\mathbf{r}\right)\right] + \,, \\ + \mathbf{A}_3 \exp\left[\alpha_3(\kappa\cdot\mathbf{r})\right]\end{aligned} \tag{4}$$

with the complex vectors $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ and the constants α_i to be defined from normalization and boundary conditions. The following remarks are warranted:

a) Plane waves are obtained only if we associate $\lambda = \pm iE$ and $\kappa = \pm i\mathbf{k}$, which is not obligatory. It becomes

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* Issues relating to the source equations will be discussed in detail elsewhere.

A more rigorous treatment will be given in the extended version.

clear that the Maxwell equations may describe physical states which are different from plane waves, so that the hypothesis on the quanta of light waves may be regarded as a particular case only, *cf.* [3a,4];

b) The solution with $\lambda = 0$ enters into the general solution of the system of differential equations. It may be removed only by means of a special choice of boundary conditions;

c) In general, κ can be replaced by $-\kappa$ (an analog of the space inversion transformation in the momentum representation), *i.e.* the solution can be written in several forms, which should be equivalent in physical content;

d) In the same way, one can find the general solution of the second equation (1b).

While these issues can certainly be analyzed further (and more rigorously) we will refrain from further analysis here. An extended version will be published elsewhere. However, below we shall show that *non-plane-wave* solutions of the Maxwell s equations also arise from the different viewpoint [2], that these solutions are *not* zero and that the field connected with these unusual modes may be *irrotational* under certain conditions. Firstly, we write *particular plane-wave* solutions of the Maxwell s equations in the form

$$\mathbf{A}(\mathbf{r}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}, \quad \mathbf{B}(\mathbf{r}) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}, \tag{5}$$

with the objects $\mathbf{a} = \mathrm{column}(a_1\ a_2\ a_3)$ and $\mathbf{b} = \mathrm{column}(b_1\ b_2\ b_3)$ at the exponents being constant *vectors* with respect to the space inversion operation. In order to form an *axial vector*, the space-inverted vectors must be added to the defined vectors.§ Thus, we obtain

$$\mathbf{C}(\mathbf{r}) = \frac{1}{2}\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i(\omega t + \mathbf{k}\cdot\mathbf{r})} \right) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \sin(\mathbf{k}\cdot\mathbf{r}) e^{i\left(\omega t - \frac{\pi}{2}\right)}, \tag{6a}$$

$$\mathbf{D}(\mathbf{r}) = \frac{1}{2}\left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} e^{-i(\omega t + \mathbf{k}\cdot\mathbf{r})} \right) = \begin{pmatrix} b_1 \\ b_2 \\ b_2 \end{pmatrix} \sin(\mathbf{k}\cdot\mathbf{r}) e^{-i\left(\omega t - \frac{\pi}{2}\right)}. \tag{6b}$$

We shall further prove the following theorems:

Theorem 1. *The quantity* $\mathbf{F} = \mathbf{C} \times \mathbf{D}$ *is conserved in time:*

$$\frac{\partial}{\partial t}\mathbf{F} = 0. \tag{7}$$

Proof. By straightforward calculation one can find the explicit form of the *axial vector* $\mathbf{F}$. It is as follows:

$$\mathbf{F} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \sin^2(\mathbf{k}\cdot\mathbf{r}). \tag{8}$$

By definition the $\mathbf{a}$ and $\mathbf{b}$ are the constant *vectors.* Thus, Eq. (8) contains no dependence on the time t, so $\partial \mathbf{F}/\partial t = 0$. The theorem is proved.

Theorem 2. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *chosen in the form (5) satisfy Maxwell s equations (1a,1b) respectively (or* vice versa*), the quantity* $\mathbf{F} = \mathbf{C} \times \mathbf{D}$ *a) is irrotational; b) satisfies both equations (1a) and (1b); c) is zero in all space if and only if* $\mathbf{A}$ *or* $\mathbf{B}$ *is zero.*

Proof. In order to prove a) and b) it is sufficient to prove that $(\mathbf{J}\cdot\nabla)^{ij}\mathbf{F}^j = 0$, because of the operator identity $\nabla\times \equiv \mathbf{curl}$, the definition of the $j = 1$ matrices and the proof of **Theorem 1.** By direct calculation we arrive at

$$\begin{aligned}(\mathbf{J}\cdot\nabla)^{ij}\mathbf{F}^j &= i\nabla\times\mathbf{F} = i\sin 2(\mathbf{k}\cdot\mathbf{r})\{\mathbf{k}\times[\mathbf{a}\times\mathbf{b}]\} \equiv \\ &\equiv i\sin 2(\mathbf{k}\cdot\mathbf{r})\{\mathbf{a}(\mathbf{k}\cdot\mathbf{b}) - \mathbf{b}(\mathbf{k}\cdot\mathbf{a})\} \equiv \\ &\equiv i\sin 2(\mathbf{k}\cdot\mathbf{r})\{\mathbf{a}\times[\mathbf{k}\times\mathbf{b}] - \mathbf{b}\times[\mathbf{k}\times\mathbf{a}]\}.\end{aligned} \tag{9}$$

After using the Maxwell s equations (1a,1b) one finds $\mathbf{k}\times\mathbf{a} = -i\omega\,\mathbf{a}$ and $\mathbf{k}\times\mathbf{b} = +i\omega\,\mathbf{b}$.** Substituting these relations in (9) we demonstrate that $\mathbf{F}$ is irrotational and, thus, combining this statement with the previous one (conservation of $\mathbf{F}$ in time) we prove that the quantity $\mathbf{F}$ satisfies both Maxwell s equations (1a) and (1b). Following the usual terminology, it may be termed longitudinal .

Let us now assume that $\mathbf{F} = \mathbf{0}$ in all space. If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ this can occur only if $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ for the *propagating wave* states. By definition, they are complex *vectors.* So, if we write $\mathbf{c} = \Re e\mathbf{a}$, $\mathbf{d} = \Im m\mathbf{a}$, $\mathbf{e} = \Re e\mathbf{b}$ and $\mathbf{f} = \Im m\mathbf{b}$, we can deduce that in order for the cross product we are seeking to be equal to zero, we must have

$$\mathbf{c}\times\mathbf{e} = +\mathbf{d}\times\mathbf{f}, \quad \mathbf{d}\times\mathbf{e} = -\mathbf{c}\times\mathbf{f}, \tag{10}$$

First, we consider the case where $\mathbf{c}$ and $\mathbf{e}$ are not collinear, $\mathbf{d}$ and $\mathbf{f}$ are not collinear, *i.e.* the first relation is not equal to zero. This condition is fulfilled if and only if the *real* vectors $\mathbf{c}, \mathbf{d}, \mathbf{e}$ and $\mathbf{f}$ are all coplanar. Thus, let us choose two vectors $\mathbf{c}$ and $\mathbf{d}$, which are assumed to be linearly independent; the other two can then be expanded as follows

$$\mathbf{e} = a_{11}\mathbf{c} + a_{12}\mathbf{d}, \ \mathbf{f} = a_{21}\mathbf{c} + a_{22}\mathbf{d}$$

with real coefficients a_{ij}. Considering $\mathbf{c}\times\mathbf{e}$ and $\mathbf{d}\times\mathbf{f}$, we have demonstrated that the quantity $a_{12} = -a_{21}$.

Here and below the notation may have nothing to do with the accustomed notation for the vectors of electric and magnetic fields.

§ We are still working in the coordinate representation and want to form an axial vector with respect to $\mathbf{r} \to -\mathbf{r}$. We are not concerned with the properties of the vectors with respect to $\mathbf{k} \to -\mathbf{k}$.

** If we assumed that **A** is a particular solution of (1b) and **B** is a particular solution of (1a), we would have opposite signs in the written relations.

Considering the second equation in (10), we are convinced that $a_{11} = a_{22}$. Thus, $\mathbf{b} = \mathbf{e} + i\mathbf{f} = (a_{11} - ia_{12})(\mathbf{c} + i\mathbf{d})$ and, hence, $\mathbf{b} \sim c_1 e^{i\beta}\mathbf{a}$. We now have a contradiction with the statement that $\mathbf{A}$ and $\mathbf{B}$, which are *not* phase free, satisfy different Maxwell s equations (1a) and (1b). Next, if $\mathbf{d} = \lambda\mathbf{c}$, we deduce from the set (10) that this can occur if and only if $\lambda^2 = -1$, which is again in contradiction with the fact that $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$ are *real* vectors. Finally, if $\mathbf{c} = \lambda_1\mathbf{e}$ and, then, $\mathbf{d} = \lambda_2\mathbf{f}$ we deduce that:

$$\mathbf{d} \times \mathbf{e} = \lambda_2\mathbf{f} \times \mathbf{e} = -\lambda_1\mathbf{e} \times \mathbf{f}$$

and, therefore, $\lambda_1 = \lambda_2 = \lambda$. Again, $\mathbf{b} \sim (1/\lambda)\mathbf{a}$ and we have a contradiction with the conditions of the theorem. So, using the from the inverse statement method we can say that $\mathbf{a} \times \mathbf{b}$ cannot be equal to zero and, hence, $\mathbf{F} \neq 0$. End of proof.

Theorem 3. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are solutions of the same equations (1a) or (1b) and* $\omega = \pm|\mathbf{k}|$, *one can deduce the following* relation *for the axial vector* $\mathbf{F}$ *and the corresponding polar* **curl** $\mathbf{F}$ *:*

$$\mathbf{curl}(\mathbf{curl}\mathbf{F}) + 4\nabla^2\mathbf{F} = 0. \qquad (11)$$

Proof. The theorem is proved by direct calculation. We have

$$\nabla \times \mathbf{F} = \mp 4i\omega \cot(\mathbf{k}\cdot\mathbf{r})\mathbf{F}, \qquad (12)$$

The signs depend on whether $\mathbf{A}$ and $\mathbf{B}$ simultaneously satisfy the first equation (1a), the sign is , or the second one, the sign is $+$. Next,

$$\nabla^2\mathbf{F} = 2\mathbf{k}^2\cos 2(\mathbf{k}\cdot\mathbf{r})[\mathbf{a}\times\mathbf{b}] = \\ = 2\mathbf{k}^2\frac{\cos 2(\mathbf{k}\cdot\mathbf{r})}{\sin^2(\mathbf{k}\cdot\mathbf{r})}\mathbf{F}, \qquad (13)$$

and, if we take into account (8,12),

$$\nabla \times (\nabla \times \mathbf{F}) = -8\omega^2\frac{\cos 2(\mathbf{k}\cdot\mathbf{r})}{\sin^2(\mathbf{k}\cdot\mathbf{r})}\mathbf{F}. \qquad (14)$$

By substituting these equations in (11), we have demonstrated the validity of the theorem. It is necessary to stress that Eq. (11) is a *relation*, which was obtained after taking into account certain constraints between $\mathbf{k}$, $\mathbf{a}$, $\mathbf{b}$ and ω. It cannot be regarded as a dynamical *equation*. This is due to the operator identity $\mathbf{curlcurl} \equiv \mathbf{graddiv} - \nabla^2$. If we rewrite (11) taking this identity into account, we demonstrate that the corresponding equation does *not* have solutions unless $\mathbf{F} = \text{constant}$, and/or $\mathbf{k}\cdot\mathbf{r} = \pm\pi/4, \pm 3\pi/4\ldots$, or $\mathbf{k} \equiv 0$.

We may thus conclude that Maxwell s electromagnetic theory seen through the eyes of a mathematician/theoretical physicist has a richer structure compared with the viewpoint that has been current since it was proposed that light had a quantum field nature. In a recent series of the papers (see [8] for references), we have analyzed the shortcomings of this viewpoint and the advantages compared to the more general Weinberg formalism[7]. The issue of whether the former is equivalent to the latter still requires further rigorous examination.

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