A conjecture for Ulam Sequences

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Abstract: A conjecture on the quasi-periodic behaviour of Ulam sequences.

An Ulam sequence is an increasing sequence $a_n \ n \geq 1$ of positive integers such that each element after the second is the smallest positive integer greater than its predecessor which is the sum of two previous distinct elements of the sequence in exactly one way [1]. Such a sequence is determined by its first two elements. For example if $a_1 = 1$ and $a_2 = 2$ then the sequence begins:

$$1,2,3,4,6,8,11,13,16,18,26,\ldots$$

Numerical studies of Ulam sequences suggests that surprisingly they have a positive density rather than becoming rarer for increasing n. Although the numbers are often chaotically distributed they have been found by Steinerberger to have a natural wavelength $\lambda = \frac{2\pi}{\alpha}$ where in the case above $\alpha = 2.5714474995\ldots$ and $\cos(\alpha a_n) < 0$ for the first 10 million numbers on the sequence except 2,3,47 and 69 [2].

The main point of this paper is to make a more specific conjecture as follows

For any Ulam sequence a_n there is a natural wavelength $\lambda \geq 2 \in \mathbb{R}$ such that if r_n is the residual of a_n mod λ in the interval $[0,\lambda)$ then for any $\varepsilon > 0$ there are only a finite number of elements in the Ulam sequence such that $r_n < \frac{\lambda}{3} - \varepsilon$ or $r_n > \frac{2\lambda}{3} + \varepsilon$.

This means that the residuals almost always lie in the middle third of their possible range. When a number does not fall in this range we call it an **outlier**. For the above sequence the outliers are 2, 3, 8, 13, 36, 47, 53, 57, 69, 97, …

Steinerberger's observation that $\cos(\alpha a_n) < 0$ in all but a few cases follows from the conjecture with $\varepsilon = \frac{\lambda}{12}$ since this requires that the Ulam numbers fall in the middle half of the range where the cosine is negative in all but a finite number of exceptions.

It is easy to see that the sum of two numbers which are not outliers must itself be an outlier. This means that most numbers in the Ulam sequence include at least one outlier in their sum despite and because of the rarity of outliers. If the conjecture could be proven it would therefore go some way towards explaining the behaviour of the sequences.

In the case of Ulam sequences where $a_1 = 2$ and $a_2 \geq 5$ is odd it is known that the sequence has only a finite number of even elements and is eventually periodic [3,4]. This confirms the conjecture for these cases with $\lambda = 2$

References

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[4] J. Schmerl and E. Spiegel, The regularity of some 1-additive sequences. J. Combin. Theory Ser. A 66 (1994), no. 1, 172-175.