Chain of a potential electric field

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Abstract
Examples are presented that geometrical images of generated electromagnetic fields are emitted by the geometrical images of the electromagnetic fields, which are the sources

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1. Boundary of a potential

A potential electric field can be obtain as a gradient of an electric potential:

$$\mathbf{E} = -\operatorname{grad}\varphi, \quad E_i = -\partial_i\varphi. \tag{1}$$

Gradient is a covector, so this electric field (1) is a covector field. The geometric image of a covector is two parallel plane elements equipped with an outer orientation (see Fig. 1[1]).

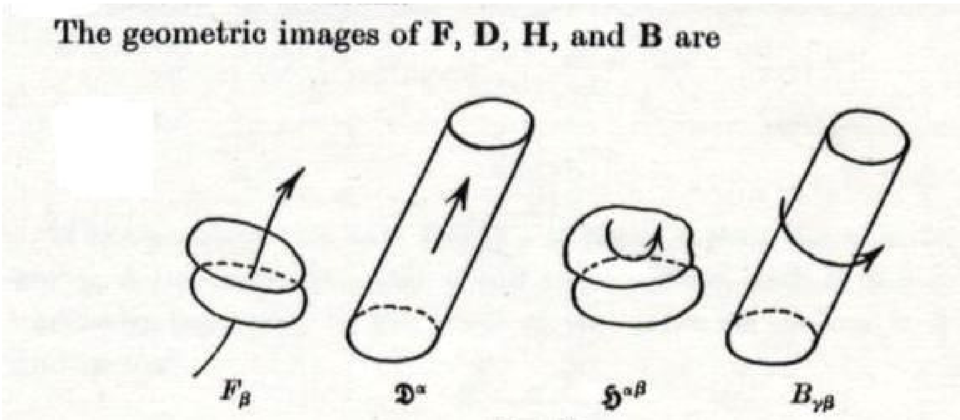

Fig. 1. Here $\mathbf{F} = \mathbf{E}$ is a covector E_i, $\mathbf{D}$ is a vector density $E^i_\wedge$, $\mathbf{H}$ is a bivector density $B^{ik}_\wedge$, $\mathbf{B}$ is a bicovector B_{ik}.

So, potential electric covector fields (1) are depicted by bisurfaces, not by field lines.

Meanwhile a scalar field, e.g. φ, may be depicted as a filling, which density is proportional to value of the scalar. Fig. 2c depicts roughly the potential of a charged sphere of radius R, $\varphi = 1/r$, $r > R$, and the corresponding covector field $\mathbf{E}$ (1). You see, the filling φ fills the closed bisurfaces of covector $\mathbf{E}$ (1), or the bisurfaces $\mathbf{E}$ bound the filling φ.

It may be said that the operation "gradient" creates a boundary of a scalar field and the field of gradient is a closed field, in correspondence with "boundary of a boundary is zero":

$$\operatorname{curl}\mathbf{E} = -\operatorname{curl}\operatorname{grad}\varphi = 0, \quad \partial_i E_k - \partial_k E_i = (-\partial_i\partial_k + \partial_k\partial_i)\varphi = 0. \tag{2}$$

So, potential electric covector field (1) is a closed field and is depicted by closed bisurfaces, Fig. 2c.

It may be said that φ fills its boundary, which is the covector $\mathbf{E}$, and even that $\mathbf{E}$ generates φ, or $\mathbf{E}$ is a source of φ (in the sense that an area is generated by its boundary).

If we interested in the electric force, which exerts on a charge q and which is a vector, we must raise the lower index of the covector E_i by the metric tensor g^{ki}:

$$\mathbf{F} = q\mathbf{E}, \quad F^k = qE^k = qE_i g^{ik}. \tag{3}$$

[1] This is figure 23 from [1].

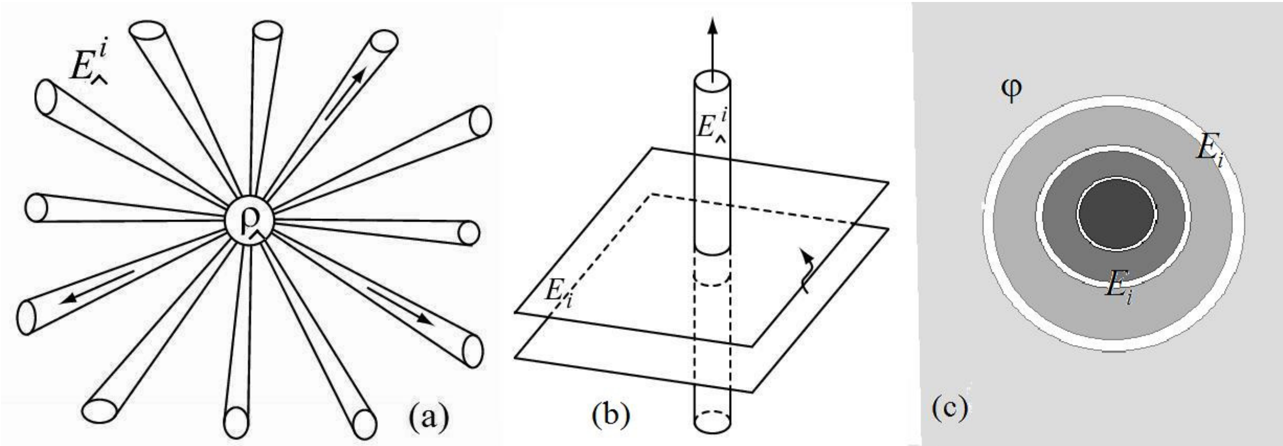

Fig. 2. Links of the potential electric field chain (6)
(a) A charge density $\rho_\wedge$ emits the tubes of electric vector density $E^i_\wedge$. **(b)** The conjugation. The vector density tube $E^i_\wedge$ changes into two parallel plane elements (bielement) of the covector E_i.
(c) The scalar field φ fills the bispheres of the covector field E_i.

2. Source of a potential electric field

The boundary of the covector potential field $\mathbf{E}$ (1) is zero, according to (2). But a potential electric field has a source. A charge density $\rho_\wedge$ is a source of the potential electric field, i.e. a charge density $\rho_\wedge$ generates the potential electric field, according to

$$\rho_\wedge = \operatorname{div}\mathbf{E}, \quad \rho_\wedge = \partial_i E^i_\wedge. \tag{4}$$

Therefore, the electric field has no boundary, but it has a source. How can this be?

Here we must recognize that the electromagnetism involves geometrical quantities of two types [1]. These are: covariant (antisymmetric) tensors, e.g. $\mathbf{E} = E_i$, $\mathbf{B} = B_{ik}$, which are named exterior differential forms or simply forms, and contravariant (antisymmetric) tensor *densities*, e.g. $\rho_\wedge$, $\mathbf{E} = E^i_\wedge$, $\mathbf{B} = B^{ik}_\wedge$ (the geometric images of E_i, $E^i_\wedge$, $B^{ik}_\wedge$, B_{ik}, see in Fig. 1). Mathematics and physicists often use Gothic fonts while writing densities. We do not use a gothic font; instead, we mark densities with the symbol "wedge" $\wedge$. For example, we name Schouten's displacement vector density $\mathfrak{D}^\alpha$ $E^i_\wedge$. This notation was used by Kunin in his Russian translation [2] of the monograph [1]. The square root of the metric tensor determinant, which is a scalar density of the weight $+1$, is denoted by $\sqrt{g}_\wedge$.

As you see, the potential electric field $E^i_\wedge$, which is generated by a charge density $\rho_\wedge$ according to (4), is a contravariant vector density. The geometric image of a vector density is a cylinder with an inner orientation. So this electric field is depicted by tubes emitted by the charge density $\rho_\wedge$ (Fig. 1a). Thus there are two different forms of the potential electric field. Covector potential electric field E_i (1) has no boundary, according to (2), but vector density potential electric field $E^i_\wedge$, according to (4), has charge density $\rho_\wedge$ as its source and its boundary.

3. The conjugation

The transition between covector E_i and vector density $E^i_\wedge$ is performed by the metric tensor density $g^{ik}_\wedge = g^{ik}\sqrt{g}_\wedge$, or $g^\wedge_{ik} = g_{ik}/\sqrt{g}_\wedge$. The transition is referred to as *the conjugation* [3,4] and is designated by the five-pointed asterisk $\star$ (in contrast to the Hodge star operation $*$), namely

$$\star E_i = g^{ik}_\wedge E_i = E^k_\wedge, \quad \star E^k_\wedge = g^\wedge_{ik} E^k_\wedge = E_i \tag{5}$$

The conjugation changes the geometric image of an electric field as it is shown in Fig. 1b.

4. Conclusion

So, we have the chain of the fields:

$$\rho_{\wedge} \quad \partial \quad E^i_{\wedge} \quad * \quad E_i \quad \partial \quad \varphi \tag{6}$$

Our symbol ∂ designates differential operations: grad, or div, or curl. These operators create boundaries. In particular, grad creates a boundary of a scalar, div creates a boundary of a tensor density, curl creates a boundary of a differential form.

In chain (6), charge density $\rho_{\wedge}$ generates the vector density $E^i_{\wedge}$. The conjugation $*$ transforms the vector density $E^i_{\wedge}$ into the closed covector E_i, which, in turn, generates potential φ.

References

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