Comment on "Exploring the origin of Minkowski spacetime"

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December, 2015

Abstract

In the article Chappell, Hartnett, Iannella, Iqbal, Abbott, "*Exploring the origin of Minkowski spacetime*" [2] authors almost completely revise their article [3], but there is some mathematical and other discrepancies in their articles.

Keywords: *multivector amplitude, Lorentz transformations, proper time, speed of light, geometric algebra, conserved quantities*

Introduction

In the articles ([2] and [3]) authors discussed a nice idea of a general transformation of multivectors in Cl_3 (3D real Clifford geometric algebra) that could serve as a new framework for relativity. Author of this article commented article [3] in [12] and that is known to a leading author of [3]. Nevertheless, my comments are ignored.

New theory and old assumptions

On a page 3 of [2] authors discussed the spacetime algebra (STA), Hestenes (1966), and the algebra of physical space (APS), Baylis (1996) and stated that "the authors axiomatically assume the Minkowski metric". For STA this is correct, but for the algebra of physical space (APS) is not. In fact, in APS amplitude of a paravector is defined in exactly the same way as in [2] and [3] and with the same motivation. Using form of a multivector as in [3] $M = t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb$ we define a *complex number* $z = t + jb$ (belongs to the center of the algebra) and a *complex vector* $\boldsymbol{F} = \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}}$. For a paravector $t + \vec{\boldsymbol{x}}$ we could use a form $t + \|\vec{\boldsymbol{x}}\|\hat{\boldsymbol{n}}$ and notice that we have hyperbolic numbers-like property $\hat{\boldsymbol{n}}^2 = 1$, so, natural choice is to define a paravector amplitude (square of) as in case of hypercomplex numbers: $(t + \vec{\boldsymbol{x}})(t - \vec{\boldsymbol{x}}) = t^2 - x^2$. But this is just a part of the Clifford conjugation restricted on real part of the algebra. For a whole multivector we have the same situation after noticing that $\boldsymbol{F}^2 = (\vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}})^2$ is a complex scalar (belongs to the center of the algebra) and we could write a multivector as $M = z + \boldsymbol{F} = z + z_F \hat{\boldsymbol{F}}, \quad z_F = \sqrt{\boldsymbol{F}^2}, \quad \hat{\boldsymbol{F}}^2 = 1$. So, there is an another hyperbolic number-like property and a multivector amplitude based on the Clifford conjugation is a natural choice: $M\bar{M} = (z + \boldsymbol{F})(z - \boldsymbol{F}) = z^2 - \boldsymbol{F}^2$. In [13] it is used this striking similarity with hyperbolic numbers formalism and relaying on spectral basis derived closed form for a many functions of multivector variables. In conclusion, in the paravector model of space-time the Minkowski metric is natural consequence of hyperbolic properties of vectors and is not imposed as assumption. The complex and hypercomplex properties are in the hart of Cl_3.

In [2] authors on the page 5 stated a Fundamental multivector involution theorem, but that is not clearly stated. A phrase "commutative amplitude" suggests that $MI(M)=I(M)M$ for the multivector amplitude only.A counter example is for the involution $I(M)=z^*-\boldsymbol{F}=t-\vec{\boldsymbol{x}}-j\vec{\boldsymbol{n}}-jb$, where $z^*=t-jb$, because

$$\begin{aligned}&(z+\boldsymbol{F})(z^*-\boldsymbol{F})=\\&zz^*+\boldsymbol{F}z^*-z\boldsymbol{F}-\boldsymbol{F}^2=z^*z-\boldsymbol{F}z+z^*\boldsymbol{F}-\boldsymbol{F}^2=\\&(z^*-\boldsymbol{F})(z+\boldsymbol{F}).\end{aligned} \tag{0.1}$$

So, the multivector amplitude defined in the [2] and [3] is not a unique commutative amplitude, but it is the unique commutative amplitude that in addition produces a complex scalar (so the result of multiplication is in the center of the algebra). Here are theorems from [12]:

Theorem 1. *If* $I(z)\in\mathbb{C}$, *then* $I(M)M=MI(M)$ *iff* $I(\boldsymbol{F})=\pm\boldsymbol{F}$.

The condition $I(M)M=MI(M)$ leads to

$$[z+\boldsymbol{F}][I(z)+I(\boldsymbol{F})]-[I(z)+I(\boldsymbol{F})][z+\boldsymbol{F}]=0\Rightarrow$$

$$\boldsymbol{F}I(\boldsymbol{F})-I(\boldsymbol{F})\boldsymbol{F}=0\Rightarrow I(\boldsymbol{F})=\pm\boldsymbol{F}.$$

This condition is met by the Clifford conjugation (up to a sign), but also for (see above) $z^*-\boldsymbol{F}$ etc, but $(z^*-\boldsymbol{F})(z+\boldsymbol{F})$ is not a complex scalar.

Theorem 2. *Clifford conjugation is the unique involution that meets* $M\bar{M}\in\mathbb{C}$, where $\mathbb{C}$ is the center of $Cl3$.

Proof relays on fact that $\boldsymbol{F}^2$ is a complex scalar, while $\boldsymbol{F}I(\boldsymbol{F})$ generally contains a vector $j\vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{n}}$ as a component, orthogonal to vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{n}}$, except for the Clifford conjugation $\boldsymbol{F}I(\boldsymbol{F}) = -\boldsymbol{F}^2 \in \mathbb{C}$. S straightforward proof can be easily obtained by multiplying multivectors

$$\left(t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb\right)\left(s_0 t + s_1\vec{\boldsymbol{x}} + js_2\vec{\boldsymbol{n}} + js_3 b\right), \quad s_i = \pm 1,$$

where then will appear $j\left(s_1\vec{\boldsymbol{n}}\vec{\boldsymbol{x}} + s_2\vec{\boldsymbol{x}}\vec{\boldsymbol{n}}\right) \Rightarrow s_1 = s_2$ (to have double inner product in brackets). Reasoning in [2] is incorrect (page 5, relation 6), because, for example, condition $s_2 = -s_3$ from last term is superfluous, factors b and $\boldsymbol{n}$ do commute.

Discussing a properties of a general transformation that preserve a multivector amplitude authors in [2] concluded condition $|X|^2 = |Y|^2 = 1$, but there is the possibility $|X|^2 = |Y|^2 = -1$, discussed in [12], known to the authors. In [3] this is corrected. But the phase transformation following from condition $|X|^2 = |Y|^2 = -1$ is ignored "in order to investigate the Lorentz group". Presented general multivector transformations are more complex than the "Lorentz transformations" and main results from special relativity in the framework of Lorentz transformations are not automatically valid here generally, they should be proven, if possible.

On pages 8 and 9 there is some confusion with formulas 17, 18 and 19. Starting with formula 17: $M' = e^{-j\boldsymbol{w}/2} e^{\boldsymbol{v}/2} M e^{\boldsymbol{v}/2} e^{j\boldsymbol{w}/2}$ authors claim that "using the multivectors formalism we can now write this as a single operation" $M' = e^{(\boldsymbol{v} - j\boldsymbol{w})/2} M e^{(\boldsymbol{v} + j\boldsymbol{w})/2}$, but there is no such "formalism", the formulas are just inconsistent, unless $\boldsymbol{v}$ and $\boldsymbol{w}$ are commuting. We could tray, for example, to solve the equation $e^{-j\boldsymbol{w}/2} e^{\boldsymbol{v}/2} = e^{C}$ to find $C = \left(\boldsymbol{v}' - j\boldsymbol{w}'\right)/2 \neq \left(\boldsymbol{v} - j\boldsymbol{w}\right)/2$. We could reformulate the result in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$, but that's not mentioned. Similar problem arises from the formulas 18 and 19, $\boldsymbol{v}$ and $\boldsymbol{w}$ are not the same in these formulas. If one ignores these problems, the claims in the text are the true ones.

On page 9 starts the discussion on a proper time. It is nice noticed that a real proper time is necessary to define an action and derive a conserved quantities. The problem here is that the conditions for a proper time to be real are not discussed in general, but assuming just restricted Lorentz group, meaning that "inertial" reference frame is the one with the speed of particle zero and that the speed of particle is necessary less than speed of light, leading to transformation rule $h' = \gamma^2 \left(h - \boldsymbol{v} \cdot \boldsymbol{w}\right)$ (formula 23). But on the same page the authors are discussing "the full set of possible transformations" and conclude that from $h' = 0$ follows $h = \boldsymbol{v} \cdot \boldsymbol{w}$, but this is generally true if the formula 23 is correct, which means in the frame of restricted Lorentz transformations and certainly not in the frame of "the full set of possible transformations". The authors also conclude that the relativistic factor γ is real, but their conclusion is valid if the speed of particle v is less then the speed of light – fact that's not proven in "the full set of possible transformations". Nevertheless, later in the text the authors are discussed possibility of a superluminal effects, but this is not a consequence of their analysis, rather it is a noticed possibility in final obtained formulas.

Let us rethink 3D Clifford algebra based on the real vectors. There is several subspaces in it, for example those based on grades: real scalars, vectors, bivectors and pseudoscalars. A one dimensional motion of a particle is just the one possibility widely explored in restricted Lorentz group (regarded frequently as the only "physical" in the special relativity). But there are the other subspaces of the algebra and why not to regard their symmetries as the equally possible sources of dynamics? For example, an electron is rather strange object, not the classical one for sure, it possesses a spin and it is not like a fast Einstein train or spaceship. Regarding the restricted Lorentz transformations in an electron theory seems to me as a strong restriction. What that it means an “inertial reference frame” when we talk about an electron. If we accept a mechanical point of view regarding “inertial reference frame” then we are taking just one possible symmetry of the 3D space as important, namely a 1D translational symmetry. From this follows conclusion about the maximal speed of particles. But here we are talking about a general transformations, wider then the restricted Lorentz group, and shouldn’t we take all possibilities that such a theory is offering us? In [2] the authors nicely conclude about a conserved quantities, but why the momentum is preferred one (as is in the special relativity)?

In [2] the author of this article was discussed other logical possibilities from demand that proper time was real. Just briefly, from general expression for multivector amplitude, comparing real and imaginary part we have

$$t^2 - x^2 + n^2 - b^2 = t'^2 - x'^2 + n'^2 - b'^2 \tag{0.2}$$

$$tb - \vec{x} \cdot \vec{n} = t'b' - \vec{x}' \cdot \vec{n}'. \tag{0.3}$$

Defining differential of the multivector $dX = dt + d\vec{x} + jd\vec{n} + jdb$, we have the multivector amplitude of the differential

$$|dX| = dt^2 - dx^2 + dn^2 - db^2 + 2j\left(dbdt - d\vec{x} \cdot d\vec{n}\right),$$

so we can ask the question which conditions must to be met to be defined real proper time τ.

There is a possibility already discussed here to define proper time as $d\tau = |dX'|_{v'=0} \in \mathbb{R}$ (“rest frame”), but then generally remains dependence of the ratio dt'/dt on a quantities from different referent frames. One can easily obtain a proper time assuming that all quantities in $|dX'|$, except dt', are equal to zero. Assuming that this is not the case (for example, an electron has a spin in every reference frame) and still regarding proper time to be real, the imaginary part of the multivector amplitude must be zero for every referent frame:

$$dbdt - d\vec{x}\cdot d\vec{n} = dt^2\left(d\dot{b} - d\dot{\vec{x}}\cdot d\dot{\vec{n}}\right) = dt^2\left(h - d\dot{\vec{x}}\cdot d\dot{\vec{n}}\right) = 0 \Rightarrow h = d\dot{\vec{x}}\cdot d\dot{\vec{n}},$$

where we defined $d\dot{b} = h$ and $h' = d\dot{\vec{x}}'\cdot d\dot{\vec{n}}'$. Defining $\dot{\vec{n}} = \vec{w}$ we have $h = \vec{w}\cdot\vec{v}$. A bivector part of a multivector is not transforming like an area [3], so is reasonable to assume vector $\vec{w}$ to be proportional to some *angular momentum-like quantity* (AMLQ). Now $\vec{w}\cdot\vec{v}$ may be associated with flow of AMLQ. It turns out that this quantity could be associated with a new law of conservation.

One could regard conditions for a real proper time $d\tau$ to be:

i) $d\tau \in \mathbb{R}$

ii) $dt'/dt \equiv \gamma\left(M, M'\right) = \gamma\left(M\right) = dt'/d\tau.$

The condition ii) is natural, relativistic factor γ now depends on quantities from single reference frame only. From i) and ii) follows

$$\left|dX\right| = \left|dX'\right| = d\tau^2 = dt^2 - dx^2 + dn^2 - db^2,$$

$$1 = \frac{dt^2}{d\tau^2}\left(1 - \frac{dx^2}{dt^2} + \frac{dn^2}{dt^2} - \frac{db^2}{dt^2}\right) = \gamma^2\left(1 - v^2 + w^2 - h^2\right),$$

$$\gamma = 1/\sqrt{1 - v^2 + w^2 - \left(\vec{w}\cdot\vec{v}\right)^2} = 1/\sqrt{1 - v^2 + w^2 - w^2v^2\cos^2\alpha}. \tag{0.4}$$

Recalling that the factor γ is real (ratio of two reals) we have the condition

$$1 - v^2 + w^2 - w^2v^2\cos^2\alpha > 0 \Rightarrow v_{\max} < \sqrt{\frac{1 + w^2}{1 + w^2\cos^2\alpha}}. \tag{0.5}$$

For $\cos\alpha = \pm 1$ is $v_{\max}$ = 1, but $v_{\max}$> 1 otherwise. So, for a vector $\vec{w}$ given a physical meaning, it follows that the maximum speed varies. A natural assumption is that we do not require $w' = 0$ generally because it could be an internal characteristic of a system (like spin) and could not be reduced to zero by the selection of a suitable reference frame, i.e., there is no a reference frame for an electron ceased to be a fermion.

We have real $\left|dX'\right| = dt'^2\left(1 - v'^2 + w'^2 - w'^2 v'^2 \cos^2\alpha'\right)$, so it would be easiest to conclude that the $w' = 0$, $v' = 0$, as discussed. Regarding $v' = 0$ a relativistic factor γ becomes dependent on w', so, remains possibility $-v'^2 + w'^2 - w'^2 v'^2 \cos^2\alpha' = 0$, which means

$$v'_\tau = \frac{w'}{\sqrt{1 + w'^2 \cos^2\left(\alpha'\right)}}, \tag{0.6}$$

and one has a real proper time in a referent frame of moving particle. What could be a physical meaning of that? In relativistic physics one usually relays on a real scalars and real vectors and defines a proper time regarding $\vec{\boldsymbol{p}} = 0$. But under bilinear transformations that preserve a multivector amplitude one could regard $-v'^2 + w'^2 - w'^2 v'^2 \cos^2\alpha' = 0$, which is equivalent to $\gamma = 1$. This could be possible to justify physically, because after extending the Lorentz transformations and including all the other motions and their symmetries there is no preferable momentum-zero condition, but rather „center of energy-momentum-AMLQ-flow-zero“ condition, whatever that means. A conclusion on limiting speed 1 is based on preferring the momentum as the main form of motion in space-time. Also, the important motivation for the use of geometry contained in Cl_3 is just equal treatment of all kind of movements (for the author surely). It is interesting that the speed v'_τ generally could be greater than 1, having upper limit $1/\cos\alpha'$ (but there is a question of limiting AMLQ somehow). Instead of the "inertial reference frame" of the special relativity all we demand for a proper time to be real is the condition $\gamma = 1$.

Having a (really) real proper time we could define derivative of a multivector by the proper time

$$V=\frac{dX}{d\tau}=\frac{dt}{d\tau}+\frac{d\vec{x}}{dt}\frac{dt}{d\tau}+j\frac{d\vec{n}}{dt}\frac{dt}{d\tau}+j\frac{db}{dt}\frac{dt}{d\tau}=\gamma\left(1+\vec{v}+j\vec{w}+jh\right), \tag{0.7}$$

$$|V|=1\Rightarrow\frac{d|V|^2}{d\tau}=\frac{dV}{d\tau}\bar{V}+V\frac{d\bar{V}}{d\tau}=0, \tag{0.8}$$

which we could understand as a kind of orthogonality of multivectors (velocity and acceleration). Defining $A=dV/d\tau$ we have $A\bar{V}+V\bar{A}=A\bar{V}+\overline{A\bar{V}}=0$ which means that multivector $A\bar{V}$ (or $V\bar{A}$) is a complex vector. In [2] the condition (0.8) is stated as $dV^2/d\tau=0$, suggesting that $V^2=const$, but it is not generally.

It is interesting that giving the energy to a particle we have

$$\gamma=E/m=1/\sqrt{1-v^2+w^2-\left(\vec{w}\cdot\vec{v}\right)^2}\Rightarrow$$

$$v=\sqrt{\frac{1+w^2-m^2/E^2}{1+w^2\cos^2\alpha}}=\sqrt{\frac{1+\left(l/E\right)^2-m^2/E^2}{1+\left(l/E\right)^2\cos^2\alpha}}\geq\sqrt{1-m^2/E^2},$$

So, one could expect that after acceleration a particle should be faster than a particle without spin (but possessing equal mass). An effect for an electron is rather small and could be a challenge for experimental physicists.

Conclusion

Starting from the articles [2, 3] it is shown a few consequences of the introduction of a bilinear transformations of the multivectors that preserve the multivector amplitude and commented some statements from [2]. There is some interesting possibilities not discussed in [2], but discussed here and in [12]. A particles with a spin, like an electron, should possess the properties not contained in the Einstein special relativity. For them, speed of light is not a limiting speed.

Acknowledgments

I would like to thank Slavica Vučko for her advises and support.

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