Nonextensive Deng entropy

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Abstract

In this paper, a generalized Tsallis entropy, named as Nonextensive Deng entropy, is presented. When the basic probability assignment is degenerated as probability, Nonextensive Deng entropy is identical to Tsallis entropy.
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1. Introduction

Since firstly proposed by Clausius in 1865 for thermodynamics [1], various types of entropies are presented, such as information entropy [2] and Tsallis entropy [3]. Information entropy [2], derived from the Boltzmann-Gibbs (BG) entropy [4] in thermodynamics and statistical mechanics, has been an indicator to measures uncertainty which is associated with the probability

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density function (PDF).

Dempster-Shafer theory evidence theory[5, 6] can be seen as the generalization of probability theory. The probability measure is extended the basic probability assignment (BPA) in evidence theory.

Information entropy is efficient to handle the uncertain measure of probability distribution. However, how to measure the uncertain degree of BPA is not well addressed until the Deng entropy is presented [7]. Deng entropy is the generalization of shannon entropy since it has the same result as that of Shannon entropy when BPA is degenerated as probability. The main contribution of this paper is to generalize Tsallis entropy [3], a widely used entropy in many real applications, into a so called Nonextensive Deng entropy.

The paper is organized as follows. The preliminaries Dempster-Shafer evidence theory and some entropy functions are briefly introduced in Section 2. Section 3 presents Nonextensive Deng entropy.

2. Preliminaries

In this section, some preliminaries are briefly introduced.

2.1. Dempster-Shafer evidence theory

Dempster-Shafer theory (short for D-S theory) is presented by Dempster and Shafer [5, 6].Some basic concepts in D-S theory are introduced.

Let X be a set of mutually exclusive and collectively exhaustive events, indicated by

$$X = \{\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_{|X|}\} \tag{1}$$

where set X is called a frame of discernment. The power set of X is indicated by 2^X, namely

$$2^X = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_{|X|}\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \cdots, \theta_i\}, \cdots, X\} \tag{2}$$

For a frame of discernment $X = \{\theta_1, \theta_2, \cdots, \theta_{|X|}\}$, a mass function is a mapping m from 2^X to $[0, 1]$, formally defined by:

$$m : \quad 2^X \rightarrow [0, 1] \tag{3}$$

which satisfies the following condition:

$$m(\emptyset) = 0 \quad and \quad \sum_{A \in 2^X} m(A) = 1 \tag{4}$$

In D-S theory, a mass function is also called a basic probability assignment (BPA). Assume there are two BPAs indicated by m_1 and m_2, the Dempster's rule of combination is used to combine them as follows:

$$m(A) = \begin{cases} \frac{1}{1-K} \sum\limits_{B \cap C = A} m_1(B) m_2(C) \ , & A \neq \emptyset; \\ 0 \ , & A = \emptyset. \end{cases} \tag{5}$$

with

$$K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C) \tag{6}$$

Note that the Dempster's rule of combination is only applicable to such two BPAs which satisfy the condition $K < 1$.

D-S theory has more advantages in in handling uncertainty compared with classical probability theory.Recently, Dempster-Shafer theory evidence theory has been generalized [8, 9].

2.2. Existing entropy

Entropy is associated with uncertainty, and it has been a measure of uncertainty and disorder. The concept of entropy is derived from physics [1]. In thermodynamics and statistical mechanics, the entropy often refers to Boltzmann-Gibbs entropy [4]. According to Boltzmann's H theorem, the Boltzmann-Gibbs (BG) entropy of an isolated system S_{BG} is obtained in terms of the probabilities associated the distinct microscopic states available to the system given the macroscopic constraints, which has the following form

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i \tag{7}$$

where k is the Boltzmann constant, W is the amount of distinct microscopic states available to the isolated system, p_i is the probability of microscopic state i satisfying $\sum_{i=1}^{W} p_i = 1$. Equal probabilities, i.e. $\forall i, p_i = 1/W$, is a particular situation. In that situation, BG entropy has the following form

$$S_{BG} = k \ln W \tag{8}$$

In information theory, Shannon entropy [2] is often used to measure the information volume of a system or a process, and quantify the expected value of the information contained in a message. Information entropy, denoted as H, has a similar form with BS entropy

$$H = -\sum_{i=1}^{N} p_i \log_b p_i \tag{9}$$

where N is the amount of basic states in a state space, p_i is the probability of state i appears satisfying $\sum_{i=1}^{W} p_i = 1$, b is base of logarithm. When $b = 2$, the unit of information entropy is bit. If each state equally appears, the quantity of H has this form

$$H = \log_2 N \tag{10}$$

In information theory, quantities of H play a central role as measures of information, choice and uncertainty. For example, the Shannon entropy of the game shown in Figure **??** is $H = 0.6 \times \log_2 0.6 + 0.4 \times \log_2 0.4 = 0.9710$. But, it is worthy to notice that the uncertainty of this game shown in Figure **??** cannot be calculated by using the Shannon entropy.

According to mentioned above, no matter the BG entropy or the information entropy, the quantity of entropy is always associated with the amount of states in a system. Especially, for the case of equal probabilities, the entropy or the uncertainty of a system is a function of the quantity of states. Moreover, in that particular case, the entropy is the maximum.

2.3. Deng entropy

With the range of uncertainty mentioned above, Deng entropy can be presented as follows

$$E_d = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|} - 1} \tag{11}$$

where, F_i is a proposition in mass function m, and $|F_i|$ is the cardinality of F_i. As shown in the above definition, Deng entropy, formally, is similar with the classical Shannon entropy, but the belief for each proposition F_i is divided by a term $(2^{|F_i|}-1)$ which represents the potential number of states in F_i (of course, the empty set is not included).

Specially, Deng entropy can definitely degenerate to the Shannon entropy if the belief is only assigned to single elements. Namely,

$$E_d = -\sum_i m(\theta_i)\log\frac{m(\theta_i)}{2^{|\theta_i|}-1} = -\sum_i m(\theta_i)\log m(\theta_i)$$

2.4. Tsallis entropy

In 1988, a more general form for entropy have been proposed by Tsallis [3]. It is shown as follows:

$$S_q = -k\sum_{i=1}^{N} p_i \mathrm{ln}_q \frac{1}{p_i} \tag{12}$$

The $q - logarithmic$ function in the Eq. (12) is presented as follows [10]:

$$\mathrm{ln}_q p_i = \frac{{p_i}^{1-q}-1}{1-q}(p_i > 0; q \in \Re; ln_1 p_i = lnp_i) \tag{13}$$

Based on the Eq. (13), the Eq. (12) can be rewritten as follows:

$$S_q = -k\sum_{i=1}^{N} p_i \frac{{p_i}^{q-1}-1}{1-q} \tag{14}$$

$$S_q = -k\sum_{i=1}^{N} \frac{{p_i}^{q}-p_i}{1-q} \tag{15}$$

$$S_q = k\frac{1-\sum\limits_{i=1}^{N}{p_i}^{q}}{q-1} \tag{16}$$

Where N is the number of the subsystems.

Based on the Tsallis entropy, the nonextensive theory is established by Tsallis et.al.

3. Nonextensive Deng entropy

In this section, we derive the Nonextensive Deng entropy directly as follows

$$E_{ND} = k\frac{1 - \sum\limits_{i=1}^{N} (\frac{m(\theta_i)}{2^{|\theta_i|}-1})^q}{q-1} \tag{17}$$

It can be easily seen that, if the BPA degenerated as probability, the Nonextensive Deng entropy is degenerated as Tsallis entropy [3]. In addition, when $q = 1$, the Nonextensive Deng entropy will further degenerated as Shannon information entropy [2].

4. Conclusion

In this paper, Nonextensive Deng entropy is presented based on Deng entropy [7]. When the BPA is degenerated as probability, the Nonextensive Deng entropy is degenerated as Trallis entropy [3]. In addition, when $q = 1$, the Nonextensive Deng entropy will further degenerated as Shannon information entropy [2]. The real physical significance of Nonextensive Deng entropy is still explored.

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