Huge class of infinite series with closed-form expressions

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It is widely known that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}}{2(2k)!}$$

(where B_n denotes n-th Bernoulli number). Ramanujan gives the identity:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{4k-1}} = 2^{4k-2}\pi^{4k-1} \sum_{m=0}^{2k} \frac{(-1)^{m+1}B_{2m}B_{4k-2m}}{(2m)!(4k-2m)!}.$$

This paper continues the sequence of infinite series with closed form in terms of π, for example:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)\coth(\pi n\sqrt{i})\coth(\pi \frac{n}{\sqrt{i}})}{n^5} = \frac{127\pi^5}{37800}$$

, where $\sqrt{i} = \frac{1+i}{\sqrt{2}}$

Constructing the sequence

1.Let $\mu_1(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$. Then

$$\mu_1(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}B_{2n}}{2(2n)!}$$

(the formula also holds for 0: $\mu_1(0) = -\frac{1}{2}$).
1.1.Then for the generating function σ_1 holds:

$$\sigma_1(x) = -2\sum_{n=0}^{\infty} \mu_1(2n)x^{2n} = 1 - \sum_{n=1}^{\infty} \frac{2x^2}{n^2 - x^2}$$

1.2. But $\sigma_1(x) = \pi x \cot \pi x$
1.3. Then $\sigma_1(ix) = \pi x \coth \pi x = 1 + \sum_{n=1}^{\infty} \frac{2x^2}{n^2 + x^2}$.
2.Let $\mu_2(s) = \sum_{n=1}^{\infty} \frac{\sigma_1(in)}{n^s}$. Then

$$\mu_2(4n) = \sum_{m=0}^{2k} (-1)^{k+1} \mu_1(2k)\mu_1(4n - 2k)$$

(the formula also holds for 0: $\mu_2(0) = -\frac{1}{4}$).
2.1.Then for the generating function σ_2 holds:

$$\sigma_2(x) = -4\sum_{n=0}^{\infty} \mu_2(4n)x^{4n} = 1 - \sum_{n=1}^{\infty} \frac{4x^4\sigma_1(in)}{n^4 - x^4}$$

2.2. But

$$-4\sum_{n=0}^{\infty} \mu_2(4n)x^{4n} = (-2\sum_{n=0}^{\infty} \mu_1(2n)x^{2n})(-2\sum_{n=0}^{\infty} \mu_1(2n)(ix)^{2n}) = \sigma_1(x)\sigma_1(ix)$$

2.3. Then $\sigma_2(\sqrt{i}x) = 1 + \sum_{n=1}^{\infty} \frac{4x^4\sigma_1(in)}{n^4 + x^4}$.
3.Let $\mu_3(s) = 2\sum_{n=1}^{\infty} \frac{\sigma_1(in)\sigma_2(\sqrt{i}n)}{n^s}$. Then

$$\mu_3(8n) = \sum_{m=0}^{2k} (-1)^{k+1} \mu_2(4k)\mu_2(8n - 4k)$$

(the formula also holds for 0: $\mu_3(0) = -\frac{1}{8}$).

3.1.Then for the generating function σ_3 holds:

$$\sigma_3(x) = -8\sum_{n=0}^{\infty}\mu_3(8n)x^{8n} = 1 - \sum_{n=1}^{\infty}\frac{8x^8\sigma_1(in)\sigma_2(\sqrt{i}n)}{n^8 - x^8}$$

3.2. But

$$-8\sum_{n=0}^{\infty}\mu_3(8n)x^{8n} = (-4\sum_{n=0}^{\infty}\mu_2(4n)x^{4n})(-4\sum_{n=0}^{\infty}\mu_2(4n)(\sqrt{i}x)^{4n}) = \sigma_2(x)\sigma_2(\sqrt{i}x)$$

3.3. Then $\sigma_3((-1)^{\frac{1}{8}}x) = 1 + \sum_{n=1}^{\infty}\frac{8x^8\sigma_1(in)\sigma_2(\sqrt{i}n)}{n^8 + x^8}$.

Let's prove, that we can do it again and again. And let's prove, that $\mu_L(2^L m)$ has closed form in terms of π for every natural L and m.

Proof by induction

Let $\zeta_L = (-1)^{1/2^L}$, $\pi_L(x) = \prod_{k=1}^{L-1}\sigma_k(\zeta_k x)$, $\sigma_L(\zeta_L x) = 1 + \sum_{n=1}^{\infty}\frac{2^L x^{2^L}\pi_L(n)}{n^{2^L} + x^{2^L}}$, and there is an agreement, that the formula also holds for 0: $\mu_L(0) = -\frac{1}{2^L}$, Let's prove that $\mu_{L+1}(s) = \sum_{n=1}^{\infty}\frac{\pi_{L+1}(n)}{n^s}$ has closed form for $2^{L+1}m$ for every natural L and m.

$$\mu_{L+1}(2^{L+1}m) = \sum_{n=1}^{\infty}\frac{\pi_{L+1}(n)}{n^{2^{L+1}m}} = \sum_{n=1}^{\infty}\frac{\pi_L(n)}{n^{2^{L+1}m}}\left(1 + \sum_{N=1}^{\infty}\frac{2^L n^{2^L}\pi_L(N)}{N^{2^L} + n^{2^L}}\right) =$$

$$= \mu_L(2^{L+1}m) + 2^L\sum_{N=1}^{\infty}\pi_L(N)\sum_{n=1}^{\infty}\frac{\pi_L(n)}{n^{2^L(2m-1)}(n^{2^L} + N^{2^L})} =$$

$$= \mu_L(2^{L+1}m) + 2^L\sum_{N=1}^{\infty}\frac{\pi_L(N)}{N^{2^L}}\sum_{n=1}^{\infty}\frac{\pi_L(n)(n^{2^L} + N^{2^L} - n^{2^L})}{n^{2^L(2m-1)}(n^{2^L} + N^{2^L})} = \ldots =$$

$$= 2\mu_L(2^{L+1}m) + 2^L\sum_{k=1}^{2m-1}(-1)^{k+1}\mu_L(2^L k)\mu_L(2^L(2m-k)) - \sum_{N=1}^{\infty}\frac{\pi_{L+1}(N)}{N^{2^{L+1}m}}$$

That's why

$$\mu_{L+1}(2^{L+1}m) = \mu_L(2^{L+1}m) + 2^{L-1}\sum_{k=1}^{2m-1}(-1)^{k+1}\mu_L(2^Lk)\mu_L(2^L(2m-k))$$

Using the agreement for $\mu_L(0)$, we finally gain

$$\mu_{L+1}(2^{L+1}m) = 2^{L-1}\sum_{k=0}^{2m}(-1)^{k+1}\mu_L(2^Lk)\mu_L(2^L(2m-k))$$

So, if $\mu_L(2^Lm)$ has closed form in terms of π, $\mu_{L+1}(2^{L+1}m)$ has it too. This way we gain a new class of series with closed form in terms of π. But for large L and m the construction loses its beauty.
Also for σ_2, we can gain another nontrivial result: If we take σ_1 and change $\sigma_1(in)$ to $\sigma_1(inx)$ inside μ_2, we gain the Ramanujan's formula

$$\sum_{n=1}^{\infty}\frac{\coth(\pi nx) + x^2\coth(\frac{\pi n}{x})}{n^3} = \frac{\pi^3}{90x}(x^4 + 5x^2 + 1)$$

Its analogue for σ_2 and μ_3 is going to be

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^5}\left(\coth\frac{\pi nx}{\sqrt{i}}\coth\pi nx\sqrt{i} + x^4\coth\frac{\pi n}{x\sqrt{i}}\coth\frac{\pi n\sqrt{i}}{x}\right) =$$

$$= \frac{\pi^5}{56700x^2}(19x^8 + 343x^4 + 19)$$