Generating Chaos from Love - Mathematically

Sai Venkatesh Balasubramanian

Sree Sai Vidhya Mandhir, Mallasandra, Bengaluru-560109, Karnataka, India.
saivenkateshbalasubramanian@gmail.com

Abstract

The present paper pertains to the formulation and generation of a signal based frequency controlled chaos based on the Heart Curves, an assortment of parametric functions so named due to their ability to generate cardoid (heart-shaped) curves. Specifically, the variable in two parametric functions is taken as an additively coupled sum of sinusoids with competing frequencies. By adapting the x and y components of these functions into signals, the time derivatives are computed and used to form the iterative maps and phase portraits. It is seen that the four phase portraits display to varying degrees, ornamental patterns, characteristic of quasiperiodicity and chaos. Using these, bifurcation diagrams are plotted in order to investigate the chaotic behavior. It is seen that the nature of chaos in the generated signals depend on the frequency ratio of the driving signals, thus pertaining to a case of signal based chaos, which has the key advantage of easy tunability, forming the novelty of the present work.

Keywords: Nonlinear Dynamics, Heart Curves, Frequency Controlled Chaos, Bifurcation Analysis

1. Introduction

The flagship child of Nonlinear Dynamics, termed Chaos Theory, with its characteristic 'sensitive dependence on initial conditions', has emerged as one of the defining highlights of twentieth century science, with applications in diverse fields including biology, astrophysics, engineering and mechanics [1]-[16]. The development of various techniques to characterize and study nonlinear dynamics and chaotic behavior such as Bifurcation Plots and Iterative Maps, stemming from the development of computation and visualization technologies have enabled observations of complex and intricate patterns pertaining to long term evolution [1, 2].

In the electrical engineering domain, development has occurred in leaps and bounds following the ability to generate chaotic signals using op-amp based realizations of nonlinear differential equations, examples if which are the Chua Circuits, and this is eventually translated into real-time applications such as secure communications and cryptography [17]-[21]. However, such implementations use system-based parameters such as resistors and capacitors as the initial conditions, with a clear disadvantage of difficulty in tuning when implemented at high frequencies as Integrated Circuits (IC) [22, 23].

In the present work, this issue is addressed in a radical and innovative way using the concept of signal based chaos, where the initial conditions are not physical parameters, but rather the signal based properties (amplitude, frequency and phase) of the inputs in a driven chaotic system. With this motivation, the present work focuses on the romantically popular 'Heart Curves', an assortment of parametric equations involving trigonometric, exponential and logarithmic functions, known so because of their ability to generate cardoid curves, shapes that look like the symbol of love.[24]. Specifically, two such parametric equations are considered, each containing an x and a y function, and for each of the four forms denoted by a generalized $L(q)$, the variable q is set to an additively coupled sinusoidal signal with competing frequencies, becoming the 'driving signal' of the system, where the frequency ratio between the input signals acting as the control parameter r. The phase portrait is plotted for an r value of π, wherein the presence of ornamental patterns indicate quasi-periodicity and chaos in the system. Following this, the iterative map is formed by computing the derivative of $E(q)$ and expressing it as a difference equation. Using the iterative map, the Bifurcation plots are then plotted, which indicate the regions of order and chaos in these signal forms.

The results discussed in the present work indicate that for specific values of r in signals derived from the heart curves, chaotic behavior is observed, thus pertaining to the case of signal based chaos, which is physically realized using Field Programmable Gate Arrays (FPGA), with much simpler circuitry and easier tunability than conventional chaos generator circuits, and this forms the novelty of the present work.

2. Phase Portrait Analysis of The Heart Curves

The present work considers two parametric equations of the heart curves, L_1 and L_2, each containing the x and y components, denoted as $L_1x(q)$, $L_1y(q)$, $L_2x(q)$ and $L_2y(q)$, denoted for generality as $L(q)$ [24]. Based on this notation, the following procedure is used for the investigation of nonlinear dynamics in the heart curves:

1. The variable q is denoted as an additively coupled signal of two sinusoids of frequencies f_1 and $f_2 = rf_1$, as

   $$q = sin(2\pi f_1 t) + sin(2\pi r f_1 t) \tag{1}$$

   with r denoting the ratio between the frequencies, and acting as the key control parameter.
2. Using this substitution, $L(q)$ is rewritten as a time-varying signal $L(t)$, and its time derivative $L'(t)$ is computed.
3. The dynamics of $L(x)$ are studied using the Phase Portrait, which is a plot of $L'(t)$ in terms of $L(t)$ for a given r, illustrating the phase space dynamics and qualitatively serving as a tool to assess sensitivity and ergodicity. Since r denotes the frequency ratio of the driving signals, an irrational number such as π is set as the value of r, in order to maximize the frequency and phase mismatches between the driving signals. The detection of ornamental and rich patterns in a phase portrait is a clear indicator of the presence of chaos.
4. In order to form the iterative map, $L(t)$ and $L'(t)$ are discretized into $L(i)$ and $L'(i)$ respectively and a difference equation of the form $L'(i) = L(i+1) - L(i)$ is formed. This difference equation is rearranged to give the expression of 'next' sample $L(i+1)$ in terms of 'current' samples $L(i)$, as $L(i+1) = L(i) + L'(i)$, this equation termed the 'Iterative Map' due to its recurrence nature. For systems depicting phase portraits indicative of chaos, the bifurcation diagram, plotting the output L as a function of r is obtained. This diagram clearly illustrates for what values of r, the system exhibits chaotic and non-chaotic behavior.

The four Heart Curve Function components are given as follows, with the heart curves generated by these functions illustrated in Fig. (1):

$$L_1x(q) = sin(q)cos(q)ln(|q|) \tag{2}$$

$$L_1y(q) = |q|^{0.3}(cos(q))^{0.5} \tag{3}$$

$$L_2x(q) = 16sin^3(q) \tag{4}$$

$$L_2y(q) = 13cos(q) - 5cos(2q) - 2cos(3q) - cos(4q) \tag{5}$$

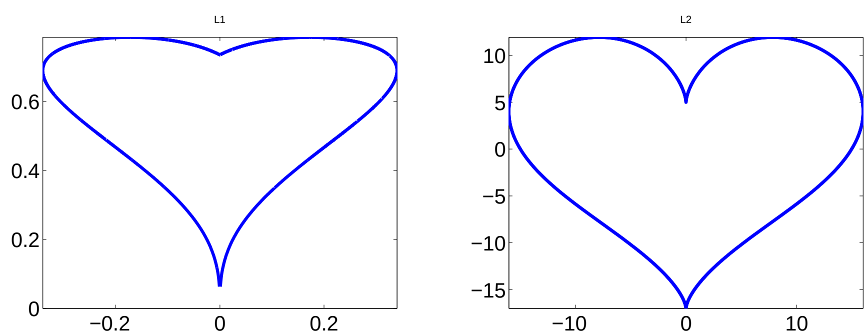

Figure 1: Heart Curves Generated by $L_1(q)$ and $L_2(q)$

By substituting x with the additive sum of sinusoids as mentioned above, we obtain the 'Heart Curve Signals' $L(t)$. Using this, the time derivative $L'(t)$ is computed, and is plotted for an r value of π in Fig. (2) and Fig. (3).

As seen from the phase portraits, it can be observed that all the four forms show, to varying degrees, ornamental patterns characteristic of either quasi-periodic or chaotic behavior, where to distinguish between these two, further bifurcation analysis is required. It is noteworthy that the phase portrait of $L_2y(t)$ shows the most characteristic signatures of quasiperiodicity among the four, whereas phase portraits of $L_1x(t)$ and $L_1y(t)$ are more indicative of chaos.

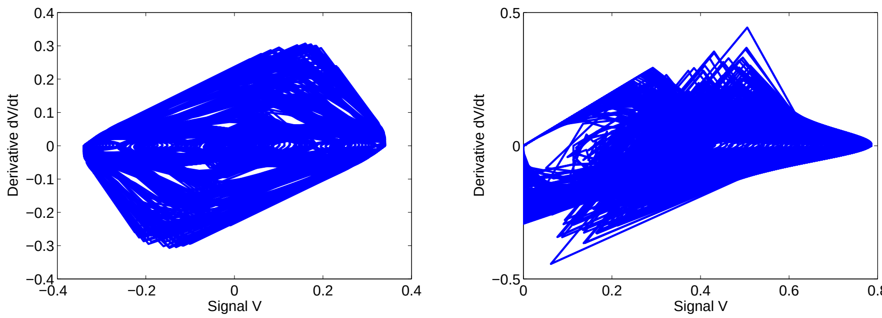

Figure 2: Phase Portraits of $L_1x(t)$ (left) and $L_1y(t)$ (bottom right)

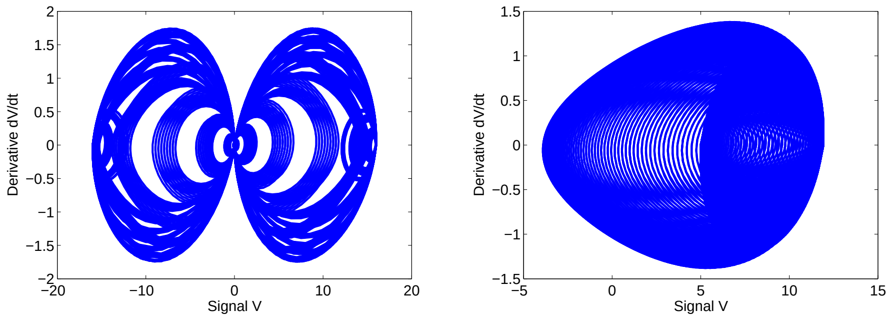

Figure 3: Phase Portraits of $L_2x(t)$ (left) and $L_2y(t)$ (bottom right)

3. Bifurcation Analysis

In this section, the bifurcation analysis for $L(t)$ is explored. It must be noted that, on account of the coupled frequency formation of x, these forms do not give rise to bifurcations in the traditional sense; rather they exhibit a quasi-periodic route to chaos, as typically seen in other coupled phase based chaotic systems such as the standard circle map [25, 26, 27, 28, 29].

The time derivatives of the four heart curves are given as follows:

$$z = sin(2\pi f_1 t) + sin(2\pi r f_1 t) \tag{6}$$

$$z' = 2\pi f_1 cos(2\pi f_1 t) + 2\pi r f_1 cos(2\pi r f_1 t) \tag{7}$$

$$L_1x'(t) = \frac{zz' sin(z) cos(z)}{|z|^2} + z' cos^2(z) log(|z|) - z' sin^2(z) log(|z|) \tag{8}$$

$$L_1y'(t) = \frac{0.3zz' cos^{0.5}(z)}{|z|^{1.7}} - \frac{z'|z|^{0.3} sin(z)}{2cos^{0.5}(z)} \tag{9}$$

$$L_2x'(t) = 48z' sin^2(z) cos(z) \tag{10}$$

$$L_2y'(t) = -13z' sin(z) + 10z' sin(2z) + 6z' sin(3z) + 4z' sin(4z) \tag{11}$$

Thus, we obtain the following difference equation by discretizing the above equations and setting $L'(i) = L(i+1) - L(i)$.

$$L(i+1) = L(i) + L'(i) \tag{12}$$

The above equation is termed the 'iterative map', and the corresponding bifurcation diagrams are plotted for L as a function of r in Fig. (4) and Fig. (5).

From the bifurcation plots, it is seen that while the four forms display largely non-chaotic quasiperiodic behavior as well as chaotic behavioral trends with dense 'grassy' patches characteristic of chaotic behavior.

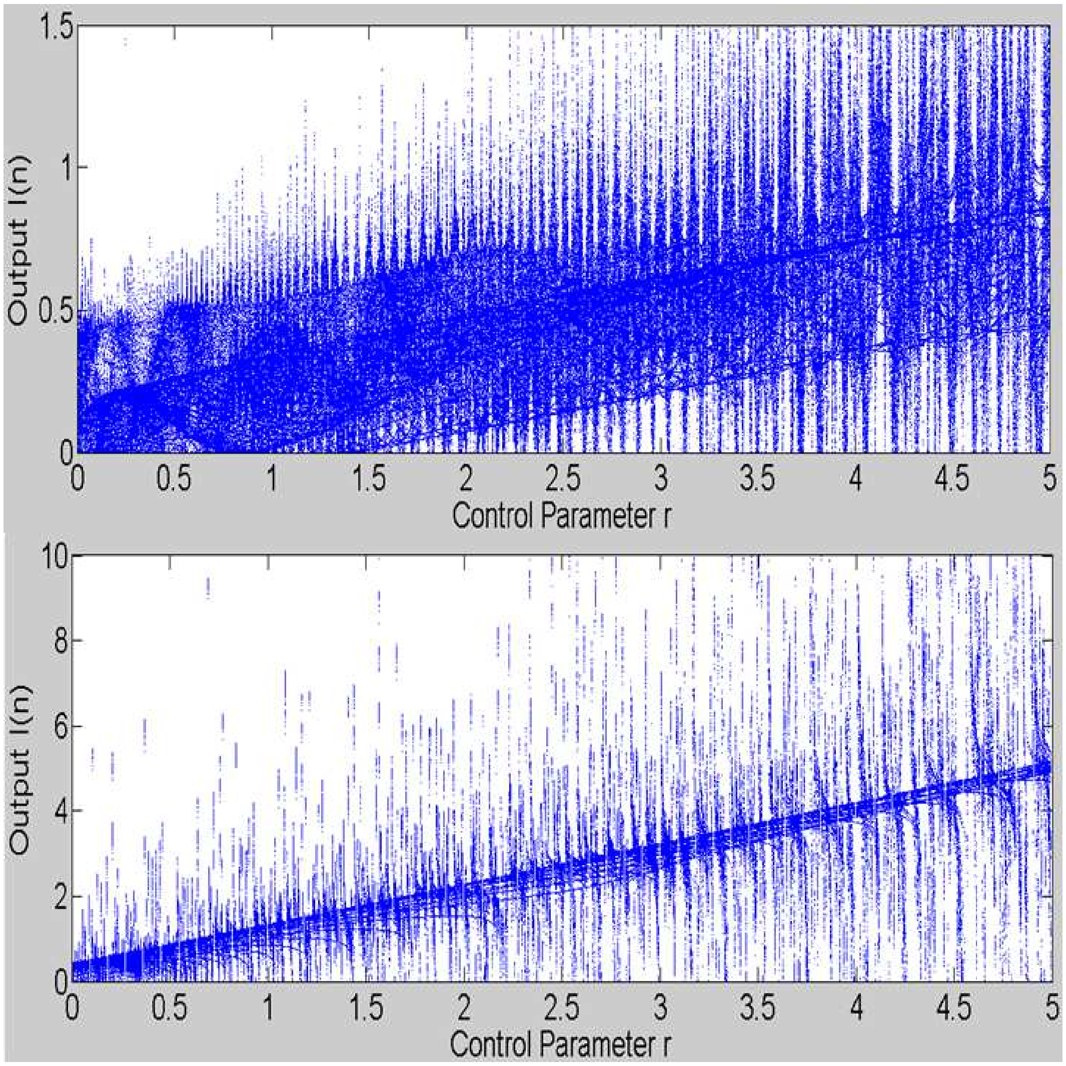

Figure 4: Bifurcation Plots of L_1 forms for x (top) and y (bottom) components

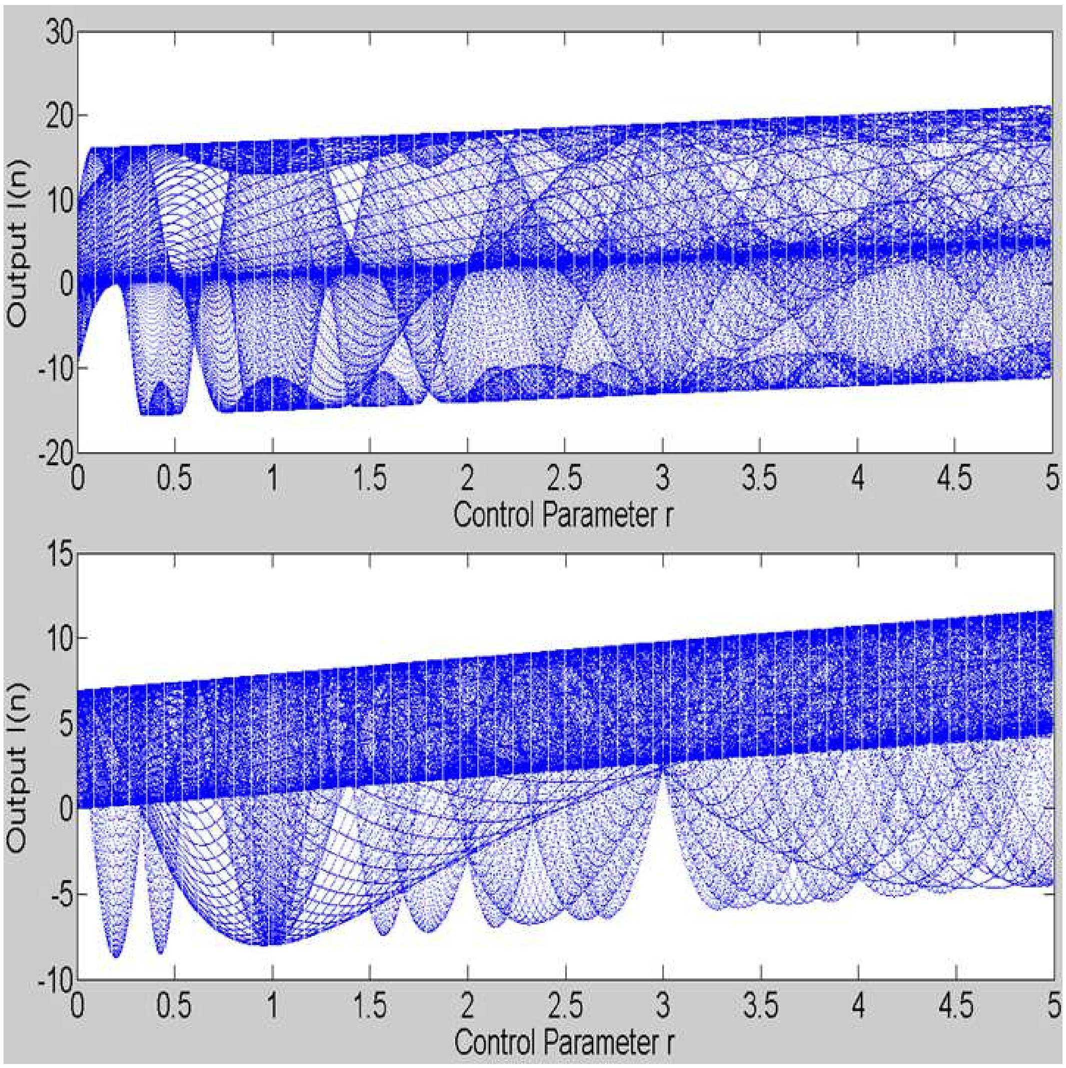

Figure 5: Bifurcation Plots of L_2 forms for x (top) and y (bottom) components

4. Conclusion

Motivated by issues of tunability in system based chaos generation circuits, the present work proposes a radical and innovative solution using signal based chaos, and to achieve this, four forms of two parametric equations of heart curves are considered. These forms are adapted into signals by substituting the variable x as an additively coupled sinusoidal signal, viewing the output as a representative of a driven nonlinear coupled system. Following this, the derivative of the output is computed and used to form a difference equation, which yields the iterative map of the proposed system. This iterative map is studied using phase portraits exhibiting varying degrees of ornamental patterns characteristic of quasiperiodic or chaotic behavior. Hence, the bifurcation analysis of these forms are presented describing the nature of quasiperiodic and chaotic behavior in the four forms.

Finally, it is noteworthy that since the behavior of the output signal depends on frequency ratio r, this ratio serves as a secure 'key', enabling the use of the Heart Curve based 'Frequency Controlled Chaos' in secure communication and encryption systems. The signal oriented approach to generating chaos from mathematical functions, coupled with the easy tunability hence obtained forms the novelty of the present work.

References

[1] M. Ausloos, M. Dirickx, *The Logistic Map and the Route to Chaos: From the Beginnings to Modern Applications*, (Springer, US, 2006).

[2] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* ,(Westview Press, Cambridge, 2008).

[3] F. Cramer, *Chaos and Order the Complex Structure of Living Systems.* (Springer, 1993).

[4] D. S. Coffey, *Self-organization, complexity and chaos: the new biology for medicine.* Nature medicine **4**, 882-885 (1998).

[5] G. Contopoulos, *Order and chaos in dynamical astronomy*, (Springer Science and Business Media, 2002).

[6] J. Laskar, *Large-scale chaos in the solar system*, Astronomy and Astrophysics **287**, L9-L12 (1994).

[7] A. B. Cambel, *Applied chaos theory-A paradigm for complexity.* (Academic Press, Inc., 1993).

[8] K. Aihara, *Chaos engineering and its application to parallel distributed processing with chaotic neural networks.* Proceedings of the IEEE, **90** 919-930 (2002).

[9] G. Chen, *Controlling chaos and bifurcations in engineering systems.* (CRC press, 1999).

[10] R. B. Stull, *An introduction to boundary layer meteorology.* (Springer Science and Business Media, 1988).

[11] M. F. Barnsley, A. D. Sloan, *Chaotic Compression*, Computer Graphics World, **3** (1987).

[12] K. E. Barner G. R. Arce, *Nonlinear Signal and Image Processing: Theory, Methods, and Applications*, (CRC Press, U.S, 2003).

[13] S. Saini, J. S. Saini. *Secure communication using memristor based chaotic circuit.* Parallel, Distributed and Grid Computing (PDGC), 2014 International Conference on. IEEE, (2014).

[14] S. Shaerbaf, S. A. Seyedin. *Nonlinear Multiuser Receiver for Optimized Chaos-Based DS-CDMA Systems.*, Iranian Journal of Electrical andElectronic Engineering **7**, 149 (2011): 149.

[15] L. Kocarev, *From chaotic maps to encryption schemes.* Circuits and Systems, (1998).

[16] G. Jakimoski, L. Kocarev. *Chaos and cryptography: block encryption ciphers based on chaotic maps.* IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications **48** 163-169 (2001).

[17] E.Bilotta and P.Pantano, *A gallery of Chua attractors*, (World Scientific, Singapore, 2008).

[18] L. Chua, *A universal circuit for studying and generating chaos. I. Routes to chaos.* Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on **40**732-744 (1993).

[19] G. Kolumban, M. P. Kennedy, L. O. Chua. *The role of synchronization in digital communications using chaos. II. Chaotic modulation and chaotic synchronization*, Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on **45**, 1129-1140 (1998).

[20] L. Chua, *Chaos synchronization in Chua's circuit*, Journal of Circuits, Systems, and Computers **3**, 93-108 (1993).

[21] L. Chua, *Experimental chaos synchronization in Chua's circuit.* International Journal of Bifurcation and Chaos **2**, 705-708 (1992).

[22] B. Razavi, *RF Microelectronics*, (Prentice Hall, US, 2011).

[23] M.Chan, K.Hui, C.Hu, P.K.Ko, *A robust and physical BSIM3 non quasi static transient and AC small signal model for circuit simulation*, IEEE Transactions on Electron Devices. **45**, 834 (1998).

[24] E. W. Weisstein, *Heart Curve*, MathWorld - A Wolfram web resource, (1972).

[25] J. Briggs, *Fractals: The patterns of chaos: A new aesthetic of art, science, and nature.* (Simon and Schuster, 1992).

[26] M. Lakshmanan, S. Rajaseekar, *Nonlinear dynamics: integrability, chaos and patterns.* (Springer Science and Business Media, 2012).

[27] I R. Epstein, K. Showalter. *Nonlinear chemical dynamics: oscillations, patterns, and chaos.* The Journal of Physical Chemistry *100* 13132-13147 (1996).

[28] R. Gilmore, M. Lefranc, *The Topology of Chaos*, (Wiley,US, [2002]).

[29] J. M. T. Thompson, H. B. Stewart, *Nonlinear Dynamics and Chaos* (Wiley,UK, [2002]).

[30] R. G. James, K. Burke, J. P. Crutchfield, *Chaos forgets and remembers: Measuring information creation, destruction, and storage*, Int. J Bifurcation Chaos. **378**, 2124 (2014).

[31] M. T. Rosenstein, J. J. Collins, C. J. De Luca, *A practical method for calculating largest Lyapunov exponents from small data sets*, Physica D, **65**, 117, (1993).