GR as a Nonsingular Classical Field Theory*

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Abstract

Thirring and Feynman showed that the Einstein equation is simply a partial differential classical field equation, akin to Maxwell's equation, but its solutions are required to conform to the GR principles of general covariance and equivalence. It is noted, with many examples, that solutions of such equations can contravene required physical principles when they exhibit unphysical boundary conditions. From the equivalence principle and the necessity of the tensor contraction theorem to the general covariance of the Einstein equation, it is shown that metric tensors are physical only where all their components, and also those of their inverse matrix, are smoothly well-defined, and their signature is that of the Minkowski metric. Thus the "horizons" of the empty-space Schwarzschild solution metrics are clearly unphysical, which is traced to their violation of the boundary condition pertaining to the minimum energetically-allowed radius of a specified positive effective mass. It is also noted that the abstract "time" of "comoving" ostensible "coordinate systems" can't be registered by the clock of any GR observer, and that the metric feature which follows from that "time" clashes with well-known GR metric properties. In examples of the transformation to "standard" coordinates of "comoving coordinate" results which have unphysical metric singularities, unphysical boundary conditions in "time" are observed to be excised, and the metric singularities are seen to be excised in tandem.

Introduction

The Thirring-Feynman systematic physical development of the Einstein equation within a purely Minkowskian framework [1, 2] is ample reason to regard that equation *as a straightforward partial differential classical field equation very akin to the Lorentz-covariant Maxwell electromagnetic field equation*, but one whose *solutions* are physical *only* when they are *consistent* with the *postulated* gravitational physical principles of equivalence and general covariance—the Einstein equation *itself* of course manifests *formal* general covariance.

Though this fact isn't commonly *explicitly* pointed out, *solutions* of the partial differential equation of a classical field theory *sometimes violate* the physical *postulates* of that theory; such a seeming paradox *readily arises* when a solution of the partial differential classical field equation *exhibits boundary conditions which happen to be inconsistent with those physical postulates.*

For example, the four Maxwell equations of source-free electromagnetism,

$$\nabla\cdot\mathbf{E}=0,\quad \nabla\times\mathbf{E}=-(1/c)\dot{\mathbf{B}},\quad \nabla\cdot\mathbf{B}=0\quad\text{and}\quad \nabla\times\mathbf{B}=(1/c)\dot{\mathbf{E}},$$

are clearly satisfied by *all* static uniform $\mathbf{E}$ and $\mathbf{B}$ fields. However *unless* those particular solutions *completely vanish*, their corresponding *electromagnetic field energy*, namely $(1/2)\int d^3\mathbf{r}\,(|\mathbf{E}|^2+|\mathbf{B}|^2)$, *diverges*, revealing their *unphysical nature*; indeed nonzero static uniform electromagnetic fields, along with *all* field solutions of the above source-free Maxwell equations *which fail to be square-integrable*, are necessarily *discarded* in source-free electromagnetic theory in order not to violate the physical postulate of finite energy.

The *divergent energies* of these unphysical electromagnetic field solutions are strikingly reminiscent of the *divergent wave-function normalizations* which occur for a class of *unphysical wave-function solutions of Schrödinger equations.* (Note that it is straightforward to recast any complex-valued Schrödinger equation of quantum mechanics into the *form* of a real-valued linear classical-field equation system by separating the real and imaginary parts of both its Hermitian Hamiltonian operator and of its wave function.)

The stationary-state Schrödinger equation for the simple harmonic oscillator,

$$(1/2)[-(\hbar^2/m)(d^2/dx^2)+m\omega^2x^2]\psi_{E_{\rm osc}}(x)=E_{\rm osc}\psi_{E_{\rm osc}}(x),$$

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has for each nonnegative value of $E_{\rm osc}$ two linearly-independent solutions (parabolic cylinder functions). When $x \to +\infty$ or $x \to -\infty$, all linear combinations of those two solutions are either strongly unbounded or else strongly approach zero. But it is *only* when $E_{\rm osc}$ takes on one of the *discrete* values $[n + (1/2)]\hbar\omega$, $n = 0, 1, 2, \ldots$, that there *exists* a linear combination of the two solutions which *isn't* strongly unbounded *under at least one of the two circumstances* $x \to +\infty$ and $x \to -\infty$.

All the remaining nonnegative values of $E_{\rm osc}$ are therefore associated to solutions of the stationary-state harmonic oscillator Schrödinger equation *that are not normalizable and hence are unphysical.* Such non-normalizable, unphysical Schrödinger-equation solutions *are all discarded.*

The *discrete* negative energy spectrum of the hydrogen atom is *likewise* associated with the *discarding* of the non-normalizable, unphysical Schrödinger-equation solutions.

Non-normalizability isn't the *only* unphysical wave-function attribute associated with a wave function's *exhibiting unphysical boundary conditions.* The stationary-state Schrödinger equation for the simple rotator of moment of inertia I is,

$$-(1/2)(\hbar^2/I)(d^2/d\theta^2)\psi_{E_{\rm rot}}(\theta) = E_{\rm rot}\psi_{E_{\rm rot}}(\theta),$$

which for each nonnegative value of $E_{\rm rot}$ has the two linearly-independent wave-function solutions,

$$\psi^{\pm}_{E_{\rm rot}}(\theta) = C^{\pm}_{E_{\rm rot}} \exp\left[\pm i(E_{\rm rot}(2I/\hbar^2))^{\frac{1}{2}}\theta\right].$$

These solutions *breach* the physically-required boundary condition *of being periodic in* θ *with period* 2π unless $E_{\rm rot}$ assumes one of the discrete values $(n\hbar)^2/(2I)$, $n = 0, 1, 2, \ldots$. The rotator wave-function solutions for the *remaining* nonnegative values of $E_{\rm rot}$ are *unphysical*; they are therefore *all discarded.*

Turning now to a requirement that arises from the *structure* of the Einstein tensor and equation in regard to that equation's compliance with the principle of general covariance, we note that because the Einstein tensor involves *contractions* of the Riemann tensor, the validity of the *tensor contraction theorem* is indispensable to the Einstein equation's general covariance.

Space-time transformation constraints for valid tensor contraction

Demonstration of the tensor contraction theorem for the space-time transformation $\bar{x}^{\alpha}(x^{\mu})$ at the space-time point x^{μ} requires that at x^{μ} the following relation must hold [3],

$$(\partial\bar{x}^{\alpha}/\partial x^{\mu})(\partial x^{\nu}/\partial\bar{x}^{\alpha}) = \delta^{\nu}_{\mu}. \tag{1}$$

Eq. (1) of course follows from the chain rule of the calculus when every component of the Jacobian matrix $(\partial\bar{x}^{\alpha}/\partial x^{\mu})$ is a well-defined finite real number and the same is true of every component of its inverse matrix. But if any component of that Jacobian matrix or of its inverse matrix is ill-defined as a finite real number, that will also be true of the left-hand side of Eq. (1), while its right-hand side remains well-defined as a finite real number, i.e., under those circumstances Eq. (1) *is self-inconsistent.*

Therefore, in the context *of respecting the general covariance of the Einstein equation*, for which the tensor-contraction theorem is *indispensable*, a space-time transformation is physical *only* at space-time points where *every component* of its Jacobian matrix and also of the *inverse* of that matrix is well-defined as a finite real number.

Since the principle of equivalence requires every metric tensor to locally be the *congruence* of the Minkowski metric tensor with a space-time transformation [4], the foregoing characterization of the *physical* space-time points of those transformations in the context of GR necessarily *also* impacts the characterization of the *physical* space-time points of metric tensors in the context of GR.

Metric constraints due to the constraints on physical transformations

The principle of equivalence as it affects metric tensors [4], taken together with the results of the previous section regarding the *physical* points of a space-time transformation in the context of GR, implies that a metric tensor can only be *physical* in the context of GR at space-time points where every component of both it and its inverse is a well-defined finite real number and where its signature is equal to the $(+, -, -, -)$ signature of the Minkowski metric tensor.

Therefore a metric tensor *solution* of the Einstein equation is *unphysical* in the context of GR at any space-time point where it or its inverse *has singular components.* As has been pointed out in the Introduction, such solution singularities would likely reflect *unphysical boundary conditions* exhibited by those solutions. In any case *it certainly isn't expected* that a properly formulated *classical physics* theory manifests *mathematical singularities that can legitimately be regarded as physical.*

We now focus on the unphysical singular "horizon" points of spherically-symmetric static empty-space Schwarzschild metric tensor solutions of the Einstein equation to attempt to identify the unphysical boundary condition which these empty-space singular metric solutions exhibit. The singular "horizon" points in empty space are obtained under the assumption that a static source of fixed *positive* effective mass $M > 0$ *can have arbitrarily small size*, just as is the case in *nonrelativistic* Newtonian gravity theory. But attempting to *assemble* an arbitrarily *small* source of fixed *positive* effective mass $M > 0$ unleashes a potentially *unlimited* source of *negative gravitational attractive energy* which enters into *the relativistic effective mass*, blocking the attempt.

Is the Schwarzschild "horizon" really located in empty space?

In the static picture, let's try to *assemble* a positive effective mass of arbitrarily small size by progressively *reducing the separation* d between *two* idealized point masses which each have positive mass $M_>/2$. The effective mass M of this system is given by,

$$Mc^2 = M_>c^2 - G(M_>/2)^2/d.$$

When $d \to \infty$, $M \to M_>$. But when $d \to 0$ for *fixed* $M_>$, $M \to -\infty$!

We can *evade* this negative effective-mass catastrophe by *optimally choosing* $M_>$ *at each value of* d *so as to maximize* M. That *maximum* of M at d is attained when $M_>(d) = 2(c^2/G)d$, and it has the value,

$$M_{\max}(d) = (c^2/G)d. \tag{2}$$

The $M_{\max}(d)$ result of Eq. (2) shows conclusively that as $d \to 0$ a point object of *positive* effective mass *absolutely cannot ensue.*

In *addition*, Eq. (2) draws our attention to an inherent self-gravitational *limit* on a system's effective mass that is proportional to *its largest linear dimension*, with a proportionality constant of order (c^2/G). Therefore a system of effective mass M *must have its largest linear dimension be of order* $(G/c^2)M$ *or greater.*

This is the *same* order as that of the radius of the unphysical Schwarzschild "horizon" resulting from effective mass M, which makes it plausible that that "horizon" *might always lie inside its source*, where the *empty-space* Schwarzschild solution of course *doesn't even apply.*

That this is *indeed* the case is *strongly supported* by the fact that in spherically-symmetric "standard" coordinates the self-gravitationally shrinking dust ball of effective mass M treated by Oppenheimer and Snyder *never* (quite) shrinks to the radius $2(G/c^2)M$ [5], which is *precisely* the radius of the Schwarzschild-solution "horizon" that *also* has a source of effective mass M and is expressed in those *same* spherically-symmetric "standard" coordinates.

In summary, the unphysical singular "horizon" of the *empty-space* Schwarzschild solution is *boundary-condition disallowed* because the potentially unlimited negative energy of gravitational attraction makes it *energetically impossible* for a spherically-symmetric source of *relativistic* positive effective mass $M > 0$ *to be as small as its corresponding Schwarzschild radius*, let alone *arbitrarily* small. A positive mass of arbitrarily small size *is* permissible in *nonrelativistic* Newtonian gravity theory, where negative gravitational energy *cannot alter* net gravitating mass (namely effective mass).

Unphysical boundary conditions in time from "comoving coordinates"

An artfully subtle insinuation into GR of *unphysical boundary conditions in time* (and with those, of *apparent* metric singularities that in fact *cannot transpire at physically realizable times*) occurs via purported "coordinate systems" whose definition of "time" *cannot in fact be registered by a clock possessed by any GR observer whatsoever.* Indeed the "time" in "comoving" ostensible "coordinates" is defined by the clocks *of an infinite number of observers* [6]. The *purpose* of that *observationally unrealizable abstraction* is to *compel* $g_{00}(x^\mu)$ to be equal to unity *whether any gravitational field is present or not* [7], but that goal is itself *at loggerheads* with the GR theoretical deductions that in the weak-field static limit $(g_{00} - 1)/2$ becomes the Newtonian gravitational potential ϕ [8], and that in the static limit $(g_{00})^{-\frac{1}{2}}$ is the gravitational time dilation factor [9].

It therefore isn't surprising that unphysical metric singularities which transpire at a definite "time" in unphysical "comoving coordinates" [10] *turn out to be absent at any finite time* in "standard" coordinates, a fact first encountered by Oppenheimer and Snyder [5]. Just as Oppenheimer and Snyder found that in "standard" coordinates a self-gravitationally contracting ball of uniform dust never at any finite time becomes as small as its Schwarzschild radius [5], it *likewise* is the case that in "standard" coordinates an expanding ball of uniform dust never at any finite time in the past was as small as its Schwarzschild radius [11]. These entirely *singularity-free* results are of course the unavoidable consequences of relativistic energy conservation in the face of potentially unlimited negative attractive gravitational energy, but they stand in the starkest imaginable *contrast* to the *ostensible* finite-"time" metric and energy-density *singularities* [10] manifested for these dust balls *in grossly unphysical* "comoving coordinates". Unfortunately the expanding dust ball's *nonexistent* early-"time" singularity which "occurred" in *unphysical* "comoving coordinates" has given rise to *completely untenable* references to the finite "age" of the expanding dust ball *since the "occurrence" of that nonexistent singularity* [12].

From the "standard" coordinate system point of view, however, there *is* a finite time when the expanding dust ball's *rate of expansion* reaches its *peak*, because at sufficiently early times gravitational time dilation drastically slows down the dust ball's expansion, while at sufficiently late times the dust ball's rate of expansion gravitationally decelerates in the familiar nonrelativistic way. Therefore the dust ball's finite "age since its inflationary expansion-rate peak" *does* make physical sense [11].

References

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[6] S. Weinberg, op. cit., Section 11.8, Second paragraph, p. 338.

[7] S. Weinberg, op. cit., Eq. (11.8.1), p. 339.

[8] S. Weinberg, op. cit., Eq. (3.4.5), p. 78.

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[12] S. Weinberg, op. cit., Sentence below Eq. (15.1.24), p. 473.