Matrix theorems and interchange for lattice group-valued series in the filter convergence setting

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Abstract

We investigate some properties of unconditional convergence of series taking values in lattice groups. We give some matrix and Schur-type theorems in the filter convergence context for lattice group-valued measures, and deduce an interchange theorem for series.

In the literature there have been several studies about Schur-type and different kinds of matrix theorems and related topics, in connection with limit theorems for measures with values in various types of structures, like for example normed spaces, topological and lattice groups. Some basic matrix theorems were proved in the Riesz space context, with respect to the so-called $(*)$-convergence, whose nature is topological. However, there are some Riesz spaces, in which order and (D)-convergence are not generated by any topology, for example the space $L^0(X, \mathcal{M}, \mu)$, where μ is a σ-additive and σ-finite non-negative extended real-valued measure. In this space, order and (D)-convergence coincide with almost everywhere convergence and $(*)$-convergence is equal to convergence in measure.

There have been some recent investigations and developments about matrix and Schur-type theorems for series taking values in abstract spaces, in connection with some interchange theorems, by requiring convergence of sequences of series on each element of a family of subsets of $\mathbb{N}$, satisfying suitable properties.

In this paper we use filter (D)-convergence, since this tool is easier to handle than filter order convergence, for example in replacing a "series" of regulators with a single regulator by using the well-known Fremlin theorem. Moreover, in connection with (D)-convergence, we deal with unconditional convergence of series, giving a characterization and relating it with σ-additivity of suitable lattice group-valued measures. We extend some earlier results and, under suitable conditions on the involved filters of $\mathbb{N}$, we prove some matrix and Schur-type theorems, by requiring only filter (D)-convergence of suitable sequences of series rather than (D)-convergence in the usual sense. Furthermore, we deduce an interchange theorem of series. We use a sliding hump argument, the Fremlin lemma and the Maeda-Ogasawara-Vulikh representation theorem, relating some properties of convergence of lattice group-valued series with the corresponding ones of real-valued series.

Let $\mathcal{F}$ be a free filter of a countable set Q. A subset of Q is *$\mathcal{F}$-stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of Q.

A free filter $\mathcal{F}$ of Q is said to be *diagonal* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$.

Given an infinite set $I \subset Q$, a *blocking* of I is a countable partition $\{D_k : k \in \mathbb{N}\}$ of I into nonempty finite subsets.

A free filter $\mathcal{F}$ of Q is said to be *block-respecting* iff for every $I \in \mathcal{F}^*$ and for each blocking $\{D_k : k \in \mathbb{N}\}$ of I there is a set $J \in \mathcal{F}^*$, $J \subset I$ with $\sharp(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where $\sharp$ denotes the number of elements of the set into brackets.

A family $\mathcal{W}$ of subsets of $\mathbb{N}$ is said to *satisfy property* (M) iff for each sequence $(F_k)_k$ in $\mathcal{I}_{\text{fin}}$, such that $\max F_k < \min F_{k+1}$ for each $k \in \mathbb{N}$, there exist a set $B \in \mathcal{W}$ and a finite set $D \subset \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}\setminus D} F_k \subset B \subset \bigcup_{k\in\mathbb{N}} F_k$.

Theorem 0.1. *Let R be a super Dedekind complete and weakly σ-distributive lattice group, $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $\mathcal{W}$ satisfy property (M) and $(a_{i,j})_{i,j}$ be a double sequence in R, such that:*

0.1.1) $(D)\lim_i a_{i,j} = 0$ for every $j \in \mathbb{N}$;

0.1.2) the series $(D\mathcal{F})\sum_{j\in B} a_{i,j}$ $(D\mathcal{F})$-converges (that is, the sequence $\Big(\sum_{j\in B\cap[1,n]} a_{i,j}\Big)_n$ $(D\mathcal{F})$-converges) for each $B \in \mathcal{W}$ and $i \in \mathbb{N}$ with respect to a single regulator, independent of B and i;

0.1.3) the family $\Big(\sum_{j\in B} a_{i,j}\Big)_{i\in\mathbb{N},\, B\in\mathcal{I}_{\text{fin}}}$ is equibounded;

0.1.4) for every infinite subset $B \in \mathcal{W}$ the sequence $\Big((D\mathcal{F})\sum_{j\in B} a_{i,j}\Big)_i$ $(D\mathcal{F})$-converges to 0 with respect to a regulator $(z_{t,l})_{t,l}$ independent of B.

Then we get:

0.1.5) the series $\sum_{j=1}^{\infty} a_{i,j}$ is unconditionally convergent for any $i \in \mathbb{N}$;

0.1.6) there is a (D)-sequence $(d_{t,l})_{t,l}$ such that for any subset $A \subset \mathbb{N}$ the sequence $\Big(\sum_{j\in A} a_{i,j}\Big)_i$ $(D\mathcal{F})$-converges to 0 with respect to $(d_{t,l})_{t,l}$;

0.1.7) if $\mathcal{F}$ is also diagonal, then 0.1.5) and 0.1.6) follow directly from 0.1.j) , j=2,3,4.

A consequence of Theorem 0.1 is the following Schur-type theorem.

Theorem 0.2. *Let R and $\mathcal{W}$ be as in Theorem 0.1, $\mathcal{F}$ be a diagonal and block-respecting filter of $\mathbb{N}$ and $(a_{i,j})_{i,j}$ satisfy conditions 0.1.j), j=2,3,4. Then we get:*

0.2.1) $(D\mathcal{F})\lim\limits_i \Big(\sum_{j=1}^{\infty} |a_{i,j}|\Big) = 0;$

0.2.2) $(D\mathcal{F})\lim\limits_i \Big(\bigvee_{A\subset\mathbb{N}} \Big(\sum_{j\in A} a_{i,j}\Big)\Big) = 0;$

0.2.3) if $m_i : \mathcal{P}(\mathbb{N}) \to R$, $i \in \mathbb{N}$, are defined by

$$m_i(A) := \sum_{j\in A} a_{i,j}, \quad A \in \mathcal{P}(\mathbb{N}), \tag{1}$$

then for every $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ there is an $\mathcal{F}$-stationary set $J \subset I$ with

$$(D)\lim_n \Big(\bigvee_{i\in J} v(m_i)[n, +\infty[\Big) = 0.$$

Corollary 0.3. *Let R, $\mathcal{F}$ and $\mathcal{W}$ be as in Theorem 0.2, and $(x_{i,j})_{i,j}$ be a double sequence in R, such that:*

0.3.1) the limit $x_{0,j} := (D\mathcal{F})\lim\limits_i x_{i,j}$ exists in R for every $j \in \mathbb{N}$.

0.3.2) the series $(D\mathcal{F})\sum_{j\in B} x_{i,j}$ $(D\mathcal{F})$-converges for each $B \in \mathcal{W}$ and $i \geq 0$ with respect to a single regulator, independent of B and i;

0.3.3) the family $\Big(\sum_{j\in B} x_{i,j}\Big)_{i\geq 0, B\in\mathcal{I}_{\text{fin}}}$ is equibounded.

Furthermore, suppose that

0.3.4) for every infinite subset $B \in \mathcal{W}$ the sequence $\Big((D\mathcal{F})\sum_{j\in B}(x_{i,j} - x_{0,j})\Big)_i$ $(D\mathcal{F})$-converges to 0 with respect to a regulator $(z_{t,l})_{t,l}$ independent of B.

Then we get

0.3.5) $(D\mathcal{F})\lim\limits_i \sum_{j=1}^{\infty} x_{i,j} = \sum_{j=1}^{\infty} x_j$ *and*

0.3.6) $(D\mathcal{F})\lim\limits_i \sum_{j\in A} x_{i,j} = \sum_{j\in A} x_j$ *uniformly with respect to $A \subset \mathbb{N}$.*

As a consequence of Corollary 0.3, we prove an interchange theorem for lattice group-valued series in the setting of filter convergence.

Theorem 0.4. *Let R, $\mathcal{F}$ and $\mathcal{W}$ be as in Theorem 0.2, $(x_j)_j$ be a sequence in R and $(x_{i,j})_{i,j}$ be a double sequence in R, such that:*

0.4.1) the series $(D\mathcal{F})\sum_{i=1}^{\infty} x_{i,j}$ $(D\mathcal{F})$-converges in R to x_j for every $j \in \mathbb{N}$;

0.4.2) the series $(D\mathcal{F})\sum_{j\in B} x_{i,j}$ *and* $(D\mathcal{F})\sum_{j\in B} x_j$ $(D\mathcal{F})$*-converge for each* $B \in \mathcal{W}$ *and* $i \in \mathbb{N}$ *with respect to a single regulator, independent of* B *and* i*;*

0.4.3) the families $\Bigl(\sum_{i=1}^{q}\Bigl(\sum_{j\in B} x_{q,j}\Bigr)\Bigr)_{q\in\mathbb{N}, B\in\mathcal{I}_{\rm fin}}$ *and* $\Bigl(\sum_{j\in B} x_j\Bigr)_{B\in\mathcal{I}_{\rm fin}}$ *are equibounded;*

0.4.4) $(D\mathcal{F})\Bigl(\sum_{i=1}^{\infty}\Bigl(\sum_{j\in B} x_{i,j}\Bigr)\Bigr) = \sum_{j\in B} x_j$ *for every infinite subset* $B \in \mathcal{W}$*, where the convergence of the series is intended with respect to a single regulator independent of* B*.*

Then we have

0.4.5) $(D\mathcal{F})\sum_{i=1}^{\infty}\Bigl(\sum_{j=1}^{\infty} x_{i,j}\Bigr) = \sum_{j=1}^{\infty}\Bigl((D\mathcal{F})\sum_{i=1}^{\infty} x_{i,j}\Bigr)$*;*

0.4.6) $(D\mathcal{F})\sum_{i=1}^{\infty}\Bigl(\sum_{j\in A} x_{i,j}\Bigr) = \sum_{j\in A}\Bigl((D\mathcal{F})\sum_{i=1}^{\infty} x_{i,j}\Bigr)$ *uniformly with respect to* $A \subset \mathbb{N}$*.*