The Integral Of The Logarithmic Derivative of Hardy’s Z-function

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Abstract

The integral of the logarithmic derivative of the Hardy Z function is calculated. The variational iteration method, based on the Banach fixed point theorem, which generate a rapidly convergent series expansion, is suggested as a way to calculate analytic solutions to the França-LeClair exact equation for the Riemann zeros \( \vartheta(t_n) + S(t_n) = (n - \frac{3}{2})\pi \), whose uniquely existing solution for each \( n \) is equivalent to the Riemann Hypothesis. An extension of the Berry-Keating Hamiltonian is also suggested.

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1 Introduction

The Riemann Zeta function is denoted

\[
\zeta(t) = \sum_{n=1}^{\infty} n^{-s}
\]

or any of its many alternative forms where it is extended to the entire complex plane except for the pole at \( s = 1 \) and let the Riemann-Siegel \( \vartheta \) function be defined by

\[
\vartheta(t) = -\frac{i}{2} \left( \ln \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \ln \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \right) - \frac{\ln(\pi) t}{2}
\]

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where the identity \[ \text{arg}(z) = \frac{\ln(z) - \ln(\overline{z})}{2i} \] and \( \Gamma(\overline{z}) = \Gamma(z) \) has been used. The Hardy \( Z \) function [Ivi13] can then be written as

\[
Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right) \quad (3)
\]

Furthermore, let \( S(t) \) be argument of \( \zeta \) normalized by \( \pi \) defined by

\[
S(t, \delta) = \frac{\text{arg}\left( \frac{1}{2} + \delta + it \right)}{\pi} = -\frac{i}{2} \left( \ln \zeta \left( \frac{1}{2} + \delta + it \right) - \ln\zeta \left( \frac{1}{2} + \delta - it \right) \right) \quad (4)
\]

where the latter form is preferred over the “modular” argument function for reasons of analysis and \( 0 < \delta \ll 1 \), which includes the usual definition

\[
S(t) = \lim_{\delta \to 0^+} S(t, \delta) \quad (5)
\]

Now, define the logarithmic derivative of \( Z(t) \) by

\[
Q(t) = \frac{Z(t)}{\overline{Z(t)}} = \frac{d}{dt} \ln Z(t) = -\frac{i}{4} \left( z(t) g^-(t) + z(t) g^+(t) - 2z(t) \ln(\pi) + 4 \dot{z}(t) \right) \quad (6)
\]

where

\[
\Psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)} \quad (7)
\]

is the digamma function, the logarithmic derivative of the \( \Gamma \) function and the functions \( z(t) \) and \( g^\pm(t) \) have been introduced to simplify the notation.

\[
z(t) = \zeta \left( \frac{1}{2} + it \right) \quad (8)
\]

\[
g^+(t) = \Psi \left( \frac{1}{4} + \frac{i}{2t} \right) \quad (9)
\]

\[
g^-(t) = \Psi \left( \frac{1}{4} - \frac{i}{2t} \right) \quad (10)
\]

The logarithmic derivative of \( Z(t) \) the the inverse of the Newton flow (also known as the Newton quotient) of \( Z(t) \) multiplied by negative one. \( \text{see } 2.1 \). The Newton flow of \( \zeta(t) \) is studied in [NFMS14]. The imaginary part of the integral of \( Q(t) \) is a piecewise-constant step function that jumps by 1 at each singularity of the negative of the inverse Newton flow of \( Z(t) \) which will be denoted by \( R(t) \)

\[
R(t, \delta) = i + \frac{\int Q(t, \delta) dt}{\pi} = \frac{\ln \zeta \left( \frac{1}{2} + \delta + i t \right)}{\pi} + i \left( 1 + \frac{\vartheta(t)}{\pi} \right) \quad (11)
\]

where \( N(t) = \operatorname{Re}(R(t, 0)) \) due to the Bäcklund counting formula gives the exact number of zeros on the critical strip up to level \( t \).

\[
N(t) = \frac{\vartheta(t)}{\pi} + 1 + S(t) \quad (12)
\]

which shows the inherent relationship between the functions \( N(t) \), \( S(t) \), and \( Z(t) \). A fact which is also evidenced by the formula

\[
\ln \zeta \left( \frac{1}{2} + it \right) = \ln |Z(t)| + i\pi S(t) \quad (13)
\]
[Ivi13, Corollary 1.8 p.13] Let \( L(t, \delta) \) be dual to (11) by defining

\[
L(t, \delta) = i - \frac{\int Q(t, \delta) \, dt}{\pi} = -\frac{\ln \zeta \left( \frac{1}{2} + \delta + it \right)}{\pi} + i \left( 1 + \frac{\vartheta(t)}{\pi} \right) \tag{14}
\]

then \( N(t) \) can also be expressed as

\[
N(t) = -\frac{i}{2} \left( L(t, 0) + R(t, 0) \right) \tag{15}
\]

combining this with (12) leads to the (rather redundant but also correct) expression for \( S(t) \)

\[
S(t) = N(t) - \left( \frac{\vartheta(t)}{\pi} + 1 \right) = -\frac{i}{2} \left( L(t, 0) + R(t, 0) \right) - \left( \frac{\vartheta(t)}{\pi} + 1 \right) \tag{16}
\]

also

\[
R(L(t, 0)) = \frac{L(t, 0) - R(t, 0)}{2} \tag{17}
\]

**Definition 1.** *(The Maximum Principle for Semicontinuous Functions)* The maximum principle for semicontinuous functions extends the classical result of calculus which states that at a maximum of a twice differentiable function the gradient (Jacobian) vanishes and the matrix of second derivatives (the Hessian) is nonpositive. A generalization and proof of these statements can be found in [C190].

**Theorem 2.** The real part of \( R(t, 0) \) has a local maximum at the points where \( S(t, 0) = 0 \) and \( \zeta \left( \frac{1}{2} + it \right) \neq 0 \) which is also the set of points for which \( \text{Im} \left( \zeta \left( \frac{1}{2} + it \right) \right) = 0 \) which is also the set of points for which \( \text{Re} \left( \zeta \left( \frac{1}{2} + it \right) \right) \) has a local maximum.

**Proof.** TODO: invoke the maximum principle for semicontinuous functions (Definition 1). To this end, let

\[
G(t) = \bar{R}(t, 0)^{-1} \tag{18}
\]

**Theorem 3.** The infinite set of points which satisfy the equation \( G(t_n) = 0 \) are local maxima of \( G(\cdot) \) and \( G(t) \leq 0 \forall t \in \mathbb{R} \) with equality \( G(t) = 0 \) only when \( Z(t) = 0 \)

**Definition 4.** *(Gateaux and Fréchet Differentiability)* Let \( X \) be any Banach space and \( S = \{ x \in X : \| x \| = 1 \} \) be the unit sphere. The norm \( \| x \| \) on \( S \) is said to be Gateaux differentiable if

\[
G(x, h) = \lim_{t \to 0} \frac{\| x + th \| - \| x \|}{t} \forall h \in S \tag{19}
\]

exists \( \forall x \in S \). The norm is said have the property of Fréchet differentiability if the limit in (19) holds at each \( x \in S \) uniformly in \( h \in S \). [Rao04, 5.5 Theorem 5 p.337] It is unfortunate that in the literature Gateaux differentiability is said to be 'weak' differentiability and likewise that Frechet differentiability is synonymous with 'strong' differentiability as it doesn’t seem this dividing into one as greater than the other is justified in this case.
Proof. Let $H(t)$ be the gradient (Jacobian, the first partial derivative) of $G(t)$

$$H(t) = G(t) = \frac{d}{dt} \left( \frac{d}{dt} R(t) \right)^{-1} = \frac{-4 i z(t) \pi (-z(t)^3 \tilde{g}^+(t) - z(t)^3 \tilde{g}^-(t) + 48 z(t) \tilde{z}(t) \tilde{z}(t) - 16 z(t)^2 \tilde{z}(t) - 32 z(t)^3)}{(z(t)^2 \tilde{g}^+(t) - z(t)^2 \tilde{g}^-(t) + 8 z(t) \tilde{z}(t) - 8 z(t)^2)^2}$$

where $z(t) = \zeta \left( \frac{1}{2} + it \right)$, $g^+(t) = \Psi \left( \frac{1}{4} + \frac{i}{2t} \right)$, and $g^-(t) = \Psi \left( \frac{1}{4} - \frac{i}{2t} \right)$. TODO: use Gateau-differentiability (19) and semicontinuity as in [FR75, Ch1.2 p3].

Theorem 5. (The França-LeClair Criterion for The Riemann Hypothesis) In [FL15] it is shown that the Riemann Hypothesis is true if there exists a unique solution to the infinite dimensional system of equations

$$\vartheta(t_n) + \lim_{\delta \to 0^+} S(t_n, \delta) = \left( n - \frac{3}{2} \right) \pi$$

$$\lim_{\delta \to 0^+} (\vartheta(t_n) + S(t_n, \delta)) = \left( n - \frac{3}{2} \right) \pi$$

(21)

for all integers $n \geq 1$ where $S(t, \delta)$ is the defined by (4),

Proof. Let

$$V(t, \delta) = \vartheta(t) + S(t, \delta)$$

then (21) becomes equivalent to the equation

$$V(t, 0) - \left( n - \frac{3}{2} \right) \pi = 0$$

(23)

\forall integers $n \geq 1$ where

$$V(t, 0) = \lim_{\delta \to 0^+} V(t, \delta)$$

(24) □

Figure 1. The Real and Absolute Parts of $R(t)$
Figure 2. The real part of $R(t)$ and the functions $S(t)$ and $Z(t)$ from 0.5

\[ \frac{1}{\frac{d^2}{dx^2} R(x)} \bigg|_{x=0} \]

Figure 3. The inverse of the second derivative of $R(t)$, $G(t, 0) = \tilde{R}(t)^{-1} = \frac{1}{\frac{d^2}{dt^2} R(t)}$ which has 2 branch cuts not shown in the intervals $[0.4, 0.5]$ and $[5.5, 5.6]$
Figure 4. The real and imaginary parts of $e^{R(t)}$ from 0...200

2 Appendix

Definition 6. (The Hardy-Littlewood Maximal Function [Tay97, Def. 6.4.1 p.187]) Let $\psi \in L^1(\mathbb{R})$ then the Hardy-Littlewood function $\psi^*(x)$ is defined by setting

$$\psi^*(x) = \sup_{h>0} \frac{\int_{x-h}^{x+h} |\psi(u)| du}{2h}$$

(25)

2.1 The Newton Quotient

Definition 7. The Newton quotient [NFMS14, 2.1] is defined by

$$\frac{dx}{dT} = -\frac{f(x)}{f(x)}$$

(26)
where \( f(x) = \frac{d}{dx} f(x) \). It can be understood that Newton’s iteration is equivalent to integrating this by the “forward Euler time-marching method” with unit steps in pseudotime. [Boy14, 6.7.2] Underrelaxation is equivalent to applying the Euler time-step scheme with a pseudotime less than 1.

**Theorem 8.** (Exponential Decay of the Newton Flow) [Boy14, Theorem 6.3 p.125] TODO.

### 2.2 The Calculus of Variations

#### 2.2.1 The Simplest Extremal Of The Calculus of Variations: The Euler Equation

The Euler equation is derived by minimizing the function

\[
J(x(t)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t)) \, dt \quad t \in [t_0, t_1]
\]

where \( \dot{x}(t) = \frac{d}{dt} x(t) \) is the usual standard notation for the first derivative of the function \( x(t) \). The function \( L: \mathbb{R}^3 \to \mathbb{R} \) is a real-valued function of three real-valued variables called the variational integrand. The notation might seem unnecessary at first, but the key thing is that \( x(t) \) and \( \dot{x}(t) \) in (27) will be replaced by the values of the function and its derivatives in some formulas. See [FR75, Lemma 3.3 and Theorem 3.1 p.7]

#### 2.2.2 The Variational Iteration Method

Consider the following general differential equation

\[
Lu + Nu = g(x)
\]

where \( L \) is a linear and \( N \) is a nonlinear operator respectively and \( g(x) \) is ‘source inhomogeneous term’. A so-called ‘correction functional’ for (28) can be written as

\[
u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi) (Lu_n(\xi) + Nu_n(\xi) - g(\xi)) \, d\xi
\]

where \( \lambda \) is a general Lagrange multiplier which can be optimally determined by the application of integration by parts formula and \( \tilde{u}_n \) is a ‘restricted variation’ where \( \delta \tilde{u}_n = 0 \). [BMM12] and [KDK13]

#### 2.2.3 Outline of the Algorithm

1. Minimize the Hamiltonian \( H(t, x, v, V_x) \) to derive a control law of the form

\[
u^*(t) = \psi\left(t, x, \frac{\partial}{\partial v} V(t, x)\right)
\]

2. Let \( n = 0 \) and form \( V_0(t, x) \) to compute the optimal control \( u \) and its corresponding trajectories \( x(t) \) and \( J_0(t) \)

3. Derive the Hamilton-Jacobi-Bellman equations corresponding to the optimal controls

\[
\dot{x}(t) = f(t, x(t), u(t))
\]
\[
x(t_0) = x_0
\]
\[
J(x_0, u(t)) = \phi(t_1, x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), u(t)) \, dt
\]

4. Let \( n \to n + 1 \) and solve the nonlinear equation

\[
\begin{cases}
V_i + V_x f(t, x(t), \psi) + L(t, x(t), \psi) = 0 \\
V(t_1, x_1) = \phi(t_1, x_1)
\end{cases}
\]

using the variational iteration method to obtain the expression for the value function (the function itself, not just its value at a point!) \( V(t, x) \) and substitute it into (30) to form the optimal control \( v^*(t) \)
5. Substitute the optimal control $u^*(t)$ formed in the previous step into $\dot{x}(t) = f(t, x(t), u(t))$ with initial value $x(t_0) = x_0$ to obtain the optimal trajectory $x^*(t)$ and score function $J_n(t)$.

6. If $|J_{n-1} - J_n| < \epsilon$ then $J = J_n$ and the calculation is finished.

2.3 A Possible Extension of The Berry-Keating Model

In [09] it is mentioned that Berry and Keating proposed a Hamiltonian of the form

$$H(x(t)) = \frac{(x(t)\dot{x}(t) + \dot{x}(t)x(t))}{2} = -i \left( x(t)\dot{x}(t) + \frac{1}{2} \right)$$

(35)

where $\dot{x}(t) = \frac{d}{dt}$ which seems not too far from that of

$$\mathcal{J}^2: x(t) = \left\{ (p, X) \mid x(t+z) \leq x(t) + p \cdot z + \frac{X: z \otimes z}{2} + o(|z|^2) \text{ as } z \to 0 \right\}$$

(36)

and

$$\mathcal{J}^2: x(t) = \left\{ (p, X) \mid x(t+z) \geq u(x) + p \cdot z + \frac{X: z \otimes z}{2} + o(|z|^2) \text{ as } z \to 0 \right\}$$

(37)

which are sets called the second-order superjets and subjets of $x(t)$ where $p \in \mathbb{R}^n$ is the Jacobian of $x(t \in \Omega) \in C^0(\Omega \subseteq \mathbb{R}^n)$ and $X \in \mathbb{S}^n$ is the Hessian of $x(t)$ where $\mathbb{S}^n$ is the set of $n \times n$ symmetric matrices. The main idea being that the canonical momentum $\dot{x}(t)$ is replaced with the generalized pointwise derivative which can be written as $\mathcal{J}^2 x(t)$ in case both the superjet and subjet exist and actually define the same set.

Bibliography


