The Integral Of The Logarithmic Derivative of Hardy's Z-function

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[1]

Abstract

The integral of the logarithmic derivative of the Hardy Z function is calculated and used as a basis for the construction of an entire function $\chi(t) = -\chi(1-t)$ which shares roots at exactly the same points of the Riemann zeta function, but with additional (complex) roots which interlace the roots of ζ and also having roots at the positive odd integers, 0, and the negative even integers.

Table of contents

1 Introduction

1.1 The Riemann Zeta And Hardy Z Function

The Riemann Zeta function is denoted

$$\zeta(t) = \sum_{n=1}^{\infty} n^{-s} \tag{1}$$

or any of its many alternative forms where it is extended to the entire complex plane except for the pole at $s=1$ and let the Riemann-Siegel ϑ function be defined by

$$\vartheta(t) = -\frac{i\left(\ln\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) - \ln\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\right)}{2} - \frac{\ln(\pi)\,t}{2} \tag{2}$$

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where the identity $\arg(z) = \frac{\ln(z) - \ln(\bar{z})}{2i}$ and $\overline{\Gamma(z)} = \Gamma(\bar{z})$ has been used. The Hardy Z function[Ivi13] can then be written as

$$Z(t) = e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2} + i t\right) \tag{3}$$

which can be again rotated back to the ζ function again

$$\zeta(t) = \frac{Z\left(\frac{i}{2} - i t\right)}{e^{i\vartheta\left(\frac{i}{2} - i t\right)}} \tag{4}$$

due to the identity

$$t = \frac{1}{2} + i\left(\frac{i}{2} - i t\right) \tag{5}$$

Furthermore, let $S(t)$ be argument of ζ normalized by π defined by

$$\begin{aligned} S(t) &= \frac{\arg\left(\zeta\left(\frac{1}{2} + i t\right)\right)}{\pi} \\ &= -\frac{\frac{i}{2}\left(\ln\zeta\left(\frac{1}{2} + i t\right) - \ln\zeta\left(\frac{1}{2} - i t\right)\right)}{\pi} \end{aligned} \tag{6}$$

Now, define the logarithmic derivative of $Z(t)$ by

$$Q(t) = \frac{\dot{\mathrm{Z}}(t)}{Z(t)} = \frac{\frac{\mathrm{d}}{\mathrm{dt}} Z(t)}{Z(t)} = \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{lnZ}(t) = = \frac{\frac{i}{4}(z(t) g^{-}(t) + z(t) g^{+}(t) - 2\, z(t) \ln(\pi) + 4\, \dot{z}(t))}{z(t)} \tag{7}$$

where

$$\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln(\Gamma(x)) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\Gamma(x)}{\Gamma(x)} \tag{8}$$

is the digamma function, the logarithmic derivative of the Γ function and let the functions $z(t)$ and $g^{\pm}(t)$ have been introduced to simplify the notation.

$$z(t) = \zeta\left(\frac{1}{2} + i t\right) \tag{9}$$

$$g^{+}(t) = \Psi\left(\frac{1}{4} + \frac{i}{2t}\right) \tag{10}$$

$$g^{-}(t) = \Psi\left(\frac{1}{4} - \frac{i}{2t}\right) \tag{11}$$

The function $Q(t)$ has singularities at $\pm\frac{i}{2}(4n - 3)$ with residues

$$\operatorname*{Res}_{t = \frac{i}{2}(4n-3)}(Q(t)) = \frac{-8\zeta(2 - 2n) n^2 + 4\,\zeta(2 - 2n) n}{16\,\zeta(2 - 2n) n^2 - 8\zeta(2 - 2n) n} = -\frac{1}{2} \tag{12}$$

and

$$\operatorname*{Res}_{t = -\frac{i}{2}(4n-3)}(Q(t)) = \frac{-2\zeta(2n - 1) + 4\,\zeta(2n - 1) n}{8\,\zeta(2n - 1) n - 4\zeta(2n - 1)} = \frac{1}{2} \tag{13}$$

The logarithmic derivative of $Z(t)$ the the inverse of the Newton flow(also known as the Newton quotient) of $Z(t)$ multiplied by negative one. The imaginary part of the integral of $Q(t)$ is a piecewise-constant step function that jumps by 1 at each singularity of the negative of the inverse Newton flow of $Z(t)$ which will be denoted by $R(t)$

$$R(t) = i + \frac{\int Q(t)\mathrm{d}t}{\pi} = \frac{\ln\zeta\left(\frac{1}{2}+it\right)}{\pi} + i\left(1 + \frac{\vartheta(t)}{\pi}\right) \tag{14}$$

where $N(t) = \mathcal{I}(R(t,0))$ due to the Bäcklund counting formula gives the exact number of zeros on the critical strip up to level t.

$$N(t) = \frac{\vartheta(t)}{\pi} + 1 + S(t) \tag{15}$$

which shows the inherent relationship between the functions $N(t)$, $S(t)$, and $Z(t)$. A fact which is also evidenced by the formula

$$\ln\zeta\left(\frac{1}{2} + it\right) = \ln|Z(t)| + i\pi S(t) \tag{16}$$

[Ivi13, Corrollary 1.8 p.13]

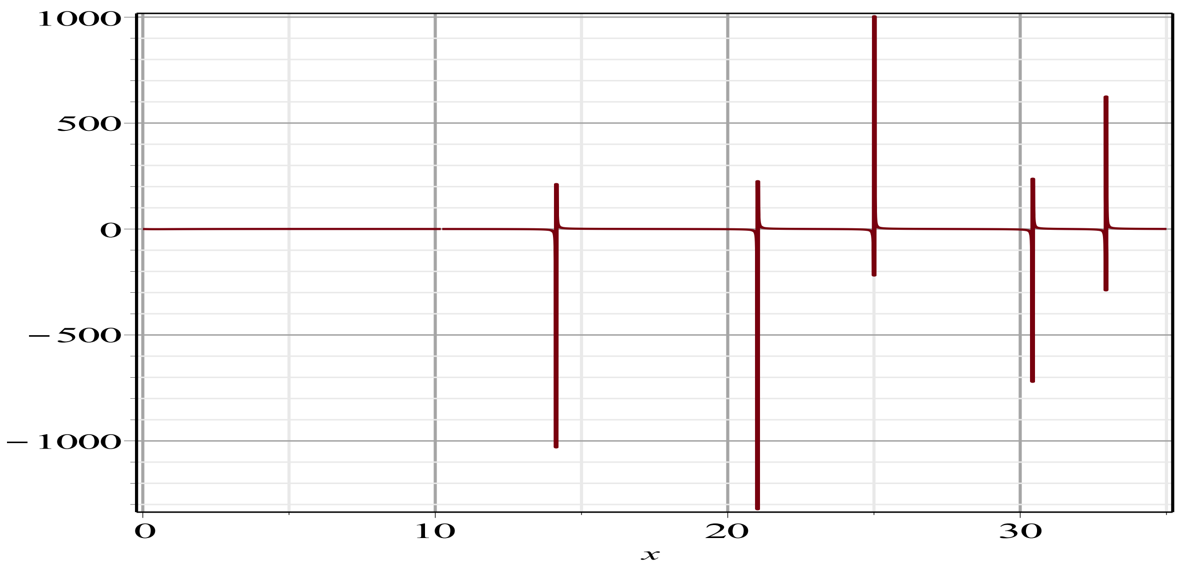


Figure 1. $Q(t)$ over the range $t = 0...35$

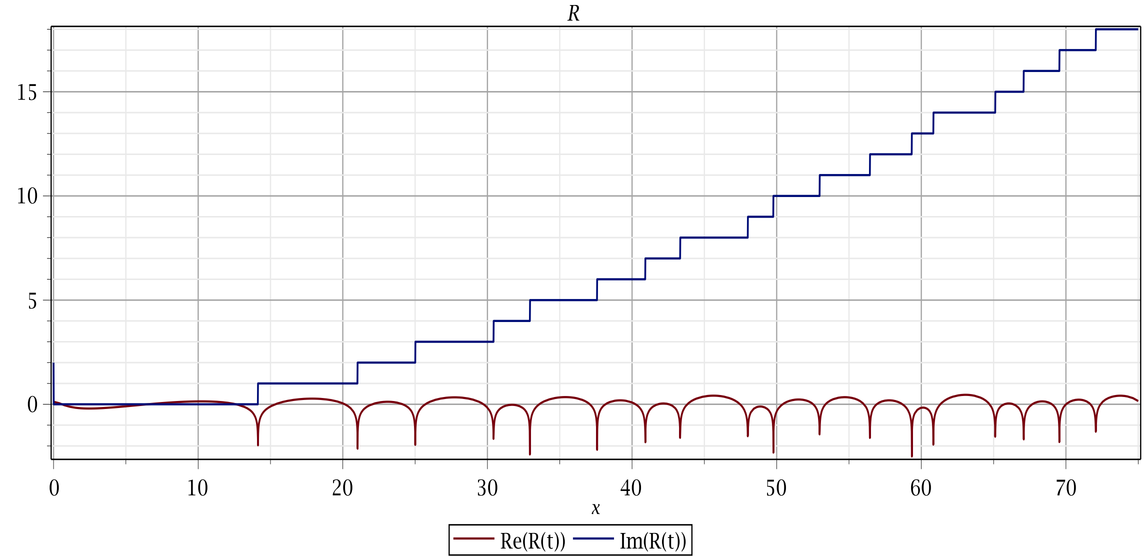


Figure 2. The Real and Imaginary Parts of $R(t)$

Let $G(t)$ be the multiplicative inverse of the second derivative of $R(t)$

$$\begin{aligned} G(t) &= \ddot{R}(t)^{-1} = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}t^2}R(t)} \\ &= \frac{8\,z(t)^2\pi}{\dot{g}^-(t)z(t)^2 - \dot{g}^+(t)z(t)^2 - 8\ddot{z}(t)z(t) + 8\,\dot{z}(t)^2} \end{aligned} \tag{17}$$

and $H(t)$ be the derivative of $G(t)$

$$H(t) = \dot{G}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t^2}R(t)^{-1}\right) = \frac{-4\,i\,z(t)\pi(-z(t)^3\ddot{g}^+(t) - z(t)^3\ddot{g}^-(t) + 48z(t)\dot{z}(t)\ddot{z}(t) - 16\,z(t)^2\,\dddot{z}(t) - 32\,\dot{z}(t)^3)}{(z(t)^2\,\dot{g}^+(t) - z(t)^2\,\dot{g}^-(t) + 8\,z(t)\ddot{z}(t) - 8\dot{z}\,(t)^2)^2} \tag{18}$$

where $z(t)$, $g^+(t)$, and $g^-(t)$ are defined in (9)-(11).

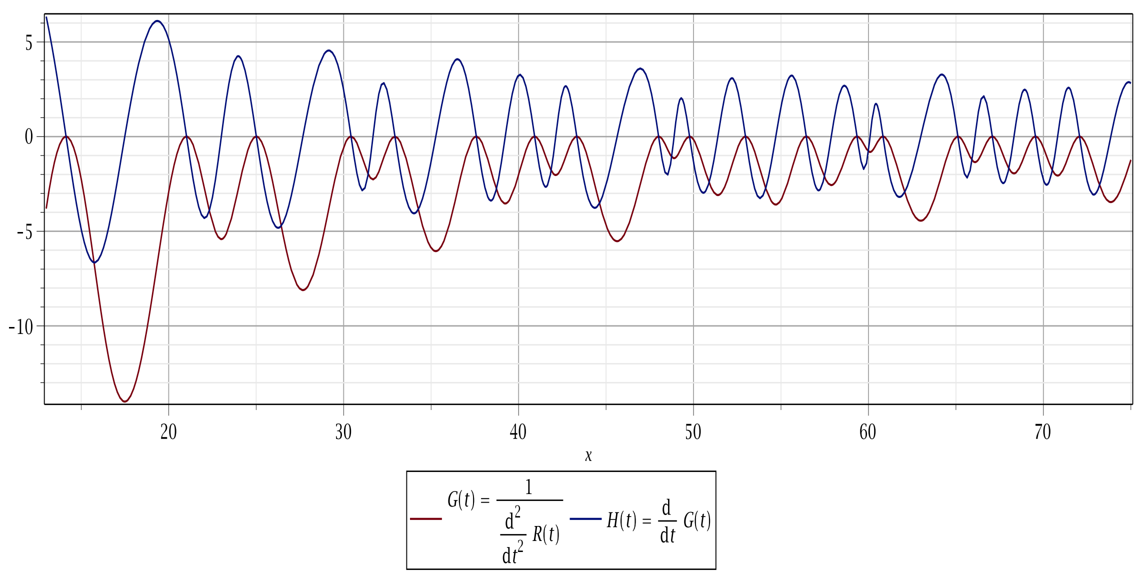


Figure 3. $G(t) = \ddot{R}(t)^{-1} = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}t^2}R(t)}$ is the the multiplicative inverse of the second derivative of $R(t)$, which has 2 branch cuts not graphed in this figure at approximately 0.44917867.. and 5.5611757.. and its derivaitve, $H(5)$

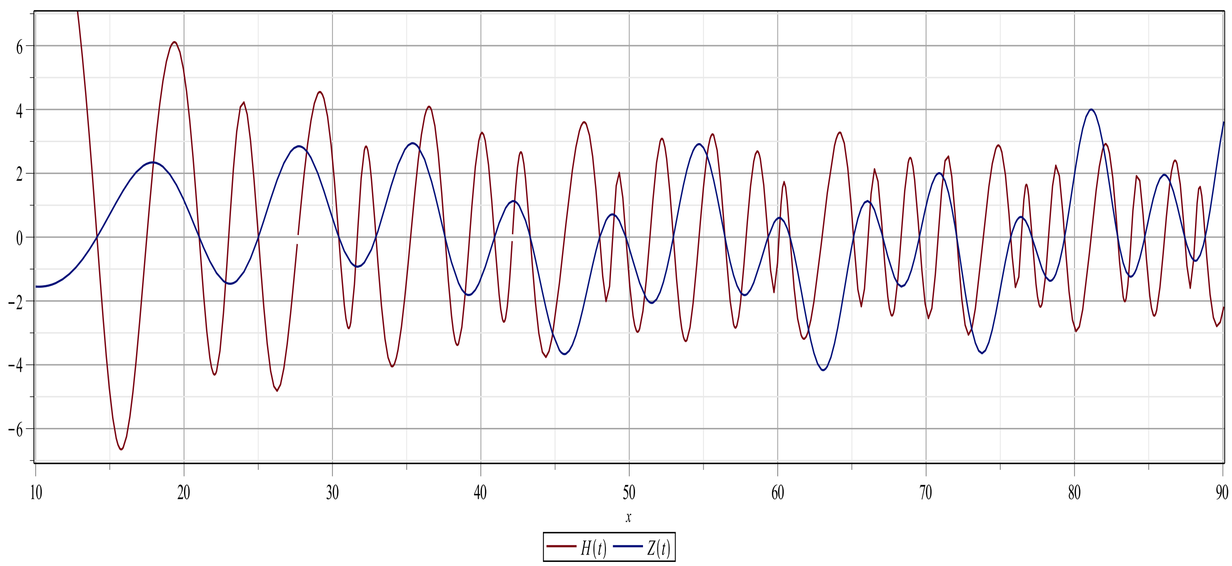


Figure 4. $H(t)$ compared to the Hardy Z function $Z(t)$ over the range $t=10...100$

1.1.1 An Entire Function Having a Subset of Roots Coinciding with Those of ζ

Now, using the same identites implicit in (4) to transform $Z(t)$ back into $\zeta(t)$, define the entire function

$$\begin{aligned}\chi(t) &= H\left(\frac{i}{2} - i t\right)\\ &= \frac{4 i \pi \zeta(t)\left(\left(\ddot{\Psi}\left(\frac{t}{2}\right) - \ddot{\Psi}\left(\frac{1}{2} - \frac{t}{2}\right)\right)\zeta(t)^3 - 48\,\zeta(t)\dot{\zeta}(t)\ddot{\zeta}(t) + 16\,\zeta(t)^2\dddot{\zeta}(t) + 32\,\dot{\zeta}(t)^3\right)}{\left(\left(\dot{\Psi}\left(\frac{t}{2}\right) - \dot{\Psi}\left(\frac{1}{2} - \frac{t}{2}\right)\right)\zeta(t)^2 + 8\,\zeta(t)\ddot{\zeta}(t) - 8\dot{\zeta}(t)^2\right)^2}\end{aligned} \tag{19}$$

which has zeros on the real line at the positive odd integers, the negative even integers, and zero.

$$\lim_{t \to 2n-1} \chi(t) = 0 \forall n > 0 \tag{20}$$

$$\lim_{t \to 0} \chi(t) = 0 \tag{21}$$

$$\lim_{t \to 2n} \chi(t) = 0 \forall n < 0 \tag{22}$$

and also satisfies the functional equation

$$\chi(t) = -\chi(1-t) \tag{23}$$

which implies that $\chi\left(t + \frac{1}{2}\right)$ is an odd entire function. The function $\chi(t)$ has as a subset of its roots, the roots of the Rieman zeta function $\zeta(t)$, the converse is not true, since $\chi(t)$ is a function of $\zeta(t)$ and its first, second, and third derivatives.

$$\{\chi(t) = 0 \colon \zeta(t) = 0\} \tag{24}$$

Apparently, $\chi(t)$ is purely imaginary real values of t. The inner-most range of the χ function is where it takes its values with the greatest magnitude. That is,

$$\chi(2) = -\chi(-1) = -i9447.759120... \tag{25}$$

for all other even (and negative odd) integers, the absolute value is less than 1 and appears to always decrease.

$$\chi(4) = -\chi(-3) = -i0.2816402... \tag{26}$$

$$\chi(6) = -\chi(-5) = -i0.0589526... \tag{27}$$

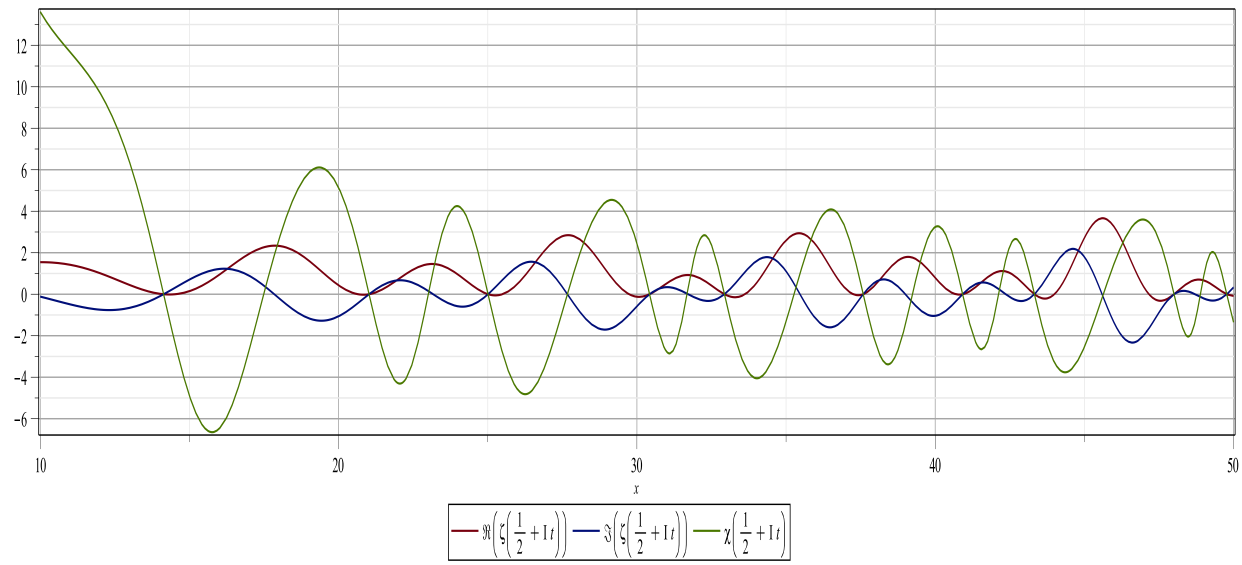


Figure 5.

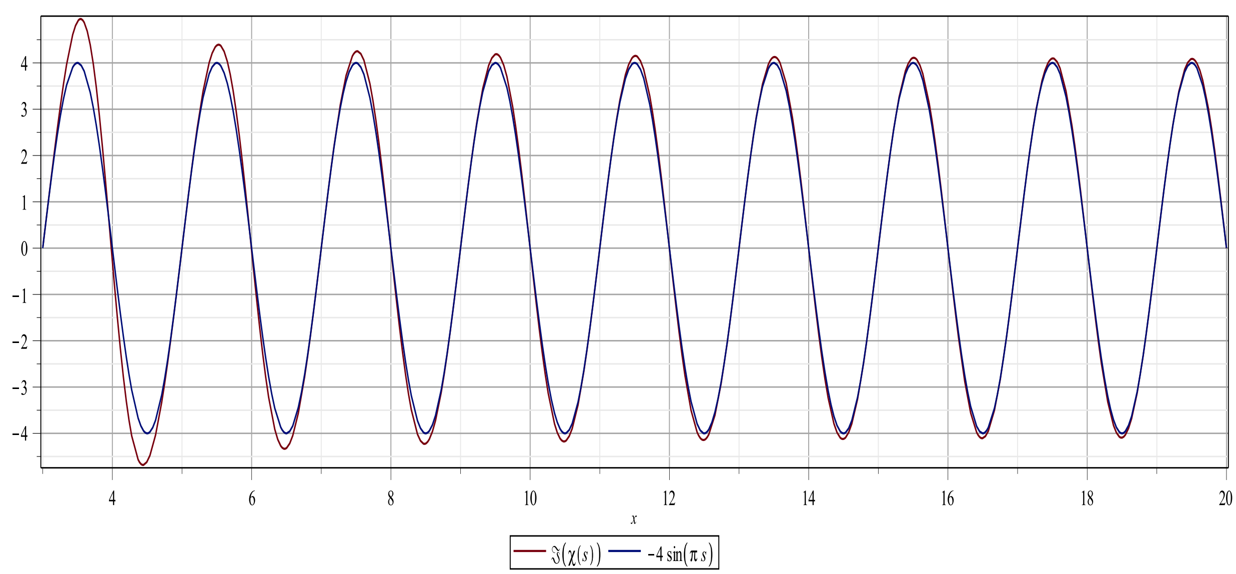
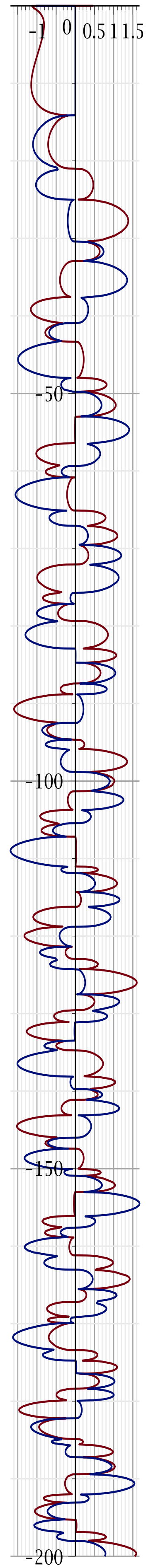


Figure 6. The imaginary part of $\chi(s)$ on the real line compared to the sine function, the real part is identically zero for real values of s,


-1
0
0.5
1
1.5
-50
-100
-150
-200

1.1.2 The Kontorovich-Lebedev Transformation

The Kontorovich-Lebedev transformation

$$\mathcal{M}_{i\tau}[f] = \int_0^\infty K_{i\tau}(x)\, f(x) \mathrm{d}x \forall t \geqslant 0 \tag{28}$$

has the Mellin transform

$$\int_0^\infty \mathcal{M}_{i\tau}(x) x^{s-1} \mathrm{d}x = 2^{s-2}\,\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)\zeta(s) \forall \mathrm{Re}(s) > 1 \tag{29}$$

which has singularities at $\{1, \infty, 2-2n \pm i\tau\}$. The residue around the singular point at $s=1$ is

$$\operatorname*{Res}_{s=1}\left(\int_0^\infty \mathcal{M}_{i\tau}(x) x^{s-1} \mathrm{d}x\right) = \frac{\pi}{4} \tag{30}$$

and the limit as $s \to \infty$ is

$$\lim_{s\to\infty} \int_0^\infty \mathcal{M}_{i\tau}(x) x^{s-1} \mathrm{d}x = \infty \tag{31}$$

[12][Yak96, 1.2]

1.2 Brownian Excursions

Let b_t be Brownian bridge, which is continuous-time stochastic process whose probability distribution function is the conditional probability distribution function of a Wiener process W_t given the boundary-condition that $\{W_0 = W_1 = 0\}$, that is, the process both starts and ends with a value of 0. The expected value of the s-th power of the normalized range of the the brownian-bridge over the interval 0 to 1 can then be written

$$E[Y^s] = 2\,\xi(s) \tag{32}$$

where

$$Y = \sqrt{\frac{2}{\pi}}\left(\max_u b_u - \min_u b_u\right) \forall u \in [0, 1] \tag{33}$$

and $\xi(s)$ is the Riemann eta function.[BPY01]

Bibliography

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[BPY01] Philippe Biane, Jim Pitman, and Marc Yor. Probability laws related to the jacobi theta and riemann zeta functions, and brownian excursions. *Bulletin of the American Mathematical Society*, 38(4):435–465, 2001.

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[Yak96] S.B. Yakubovich. *Index Transforms*. World Scientific, 1996.