

$E = mc^2$ is a classical physics result and the relativistic formula is

$$E = E_0 / \sqrt{1 - v^2 / c^2}$$

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Abstract

The mass-energy formula $E = mc^2$ is thought to be derived by Einstein from special relativity. The present study shows that since the formula has also been derived from classical physics by Einstein, it has a deep-rooted connection with classical physics. The formula is implied by Maxwell's electromagnetic momentum $P = E / c$ and the Newtonian definition of momentum $P = mv$. It can be derived from classical physics with c as the constant velocity of light in its medium ether. The present study also shows that within the framework of classical physics, this classical physics based formula is correct in other inertial frames that move relative to the ether frame as well. In contrast, Einstein's derivation in 1905 seems logically flawed as a relativistic proof, because it relies on the classical kinetic energy definition, approximates at low velocity and fails to show mass-energy equivalence in the same reference frame. Therefore, treating $E = mc^2$ as a quantity like the momentum $P = mv$, which applies to both classical physics and special relativity if relativistic mass is used in the equation, appears to be more consistent with the logic of special relativity. Then the truly relativistic formula should be $E = E_0 / \sqrt{1 - v^2 / c^2} = m_0 c^2 / \sqrt{1 - v^2 / c^2}$ derived by Laue and Klein, which corresponds to the formula of relativistic mass.

Key words: Lorentz transformation; Mass-energy equation; Special relativity; conservation of momentum; conservation of energy; reference frame.

1. Introduction

The mass-energy formula $E = mc^2$ has a prominent role in both physics research and public perception of science. The formula explains the power of nuclear bombs as well as the energy source of stars (Einstein 1946a; Rhodes 1986; Bahcall et al. 2001), and stimulates the imagination of the general public. It also underlies key components of the Dirac equation, which has accounted for the fine details of the hydrogen spectrum and implied the existence of antimatter (Dirac 1928). Although Einstein (1905) derived mass-energy equivalence initially as an approximation, the accuracy of the formula has been confirmed by experiments to a high level of precision (Rainville et al. 2005).

The explicit expression of $E = mc^2$ was first proposed by Planck (Planck 1907, 1908; Stark 1907), but it is generally believed that Einstein (1905) derived the mass-energy formula $E = mc^2$ from special relativity. Fernflores (2012) asserts in *Stanford Encyclopedia of Philosophy*: “Einstein correctly described the equivalence of mass and energy as “the most important upshot of the special theory of relativity” (Einstein 1919), for this result lies at the core of modern physics”. Although there are still some disputes on Einstein’s discovery of the mass-energy equation and some researchers have argued that Einstein’s derivation might be logically flawed (Ives 1952; Jammer 1961; Ohanian 2009; Hecht 2011; Ma 2014), nobody seems to question whether the mass-energy equation is really an exclusively relativistic result.

It has been long known that the mass-energy equation appears to be implied in Maxwell’s electromagnetic theory (Maxwell 1865; Poynting 1884; Poincaré 1900), and Lewis (1908) has provided a derivation within the framework of classical physics. Since the mass-energy equation might be derived within the framework of classical physics, it could be a result from classical physics rather than special relativity. The aim of this study is to show that $E = mc^2$ is actually a formula common to both classical physics and special relativity, and the relevant relativistic formula is $E = E_0 / \sqrt{1 - v^2 / c^2}$. This study will prove this by examining Einstein’s first derivation of mass-energy relation in 1905 and his last derivation in 1946 and providing logically more consistent corresponding derivations.

It must be emphasized here that, this study does not question the validity of the mass-energy equation, nor does it question the validity of special relativity. The main fact this study intends to establish is that, the mass-energy equation has a status similar to that of the conservation of momentum rather than that of time dilation or length contraction. The mass-energy equation and the conservation of momentum are valid in both classical physics and special relativity; therefore, they are not relativistic conclusions. Time dilation and length contraction are not compatible with classical physics, hence they are relativistic.

2. Criteria for being relativistic

To determine whether the mass-energy formula is common to both classical physics and special relativity, we need to establish the criteria for being relativistic. What qualifies a formula as a relativistic result? The following criterion could be used:

Proposition 1. A formula is relativistic if and only if the formula in its general form or specific forms can be derived only when assumptions or results unique to special relativity have been applied.

With this criterion, we can readily tell whether a formula or physical law is relativistic or not. If a formula can be derived without using any relativistic assumptions or results, no matter whether it is valid in special relativity, it is not a relativistic formula. Many laws in physics are valid in both classical physics and special relativity, but we cannot say those laws are consequences of special relativity simply because they are valid in special relativity. For example, the Newton's third law and the conservation of momentum are still valid in special relativity, but they are not relativistic results or conclusions. Some conclusions in physics are not valid in classical physics or compatible with it, such as time dilation and length contraction, so that they are relativistic results. Although the concept of relativistic mass has been dismissed by many physicists (Okun 2009), it is obviously not a concept in classical physics.

Proposition 1 treats the necessity of using uniquely relativistic assumptions or results to derive a formula as a basic criterion for it to be relativistic. If the mass-energy equation can be derived without using relativistic assumptions or results, then it is not a

relativistic formula. When some relativistic conditions have been assumed, but the derivation of a formula must use a uniquely classical assumption or result, can it be relativistic formula? The following criterion could be used as an answer for this question.

Proposition 2. If the derivation of a formula must use a result or assumption unique to classical physics, the formula cannot be viewed as relativistic.

Proposition 2 puts a more restrictive constraint on what can be considered being relativistic. Some researchers may argue that special relativity contains classical physics, so using classical physics to derive a formula does not affect its relativistic nature. However, if a formula can only be derived under some conditions unique to classical physics (although they are low speed approximations of relativistic conditions), it cannot be extended to higher speed scenarios, so that it is not relativistic. For the mass-energy equation, if a unique classical condition or relation has to be used in its derivation, then it is not a relativistic formula.

The mass-energy equation is about the equivalence between mass and energy, but to energy measured in which reference frame is a mass measured in one reference frame equivalent? Is an object's mass measured in reference frame A equivalent to its energy measured in the same reference frame (i.e. frame A), or its energy measured in another reference frame? To my knowledge, this question has not been raised or discussed so far. The following restriction might be imposed with respect to this question:

Proposition 3. In the mass-energy equation $E = mc^2$, energy E and mass m are measured in the same reference frame rather than different reference frames.

Proposition 3 requires us to keep track of the reference frames involved in measuring mass and energy during a derivation. Obviously, an object's mass m measured in one reference frame (e.g. frame A) cannot have the same mass-energy relationship $E = mc^2$ with values of its energy E measured in all reference frames, i.e. $E_{\text{any reference frame}} = m_A c^2$ is incorrect, since the values of E measured in other reference frames depend on their velocities relative to frame A.

In classical physics, the issue of different reference frames is less noticeable, because at low velocity the variations of an object's total energy in different reference frames due to kinetic energy differences between different reference frames are negligible compared with the energy implied by its rest mass. In special relativity, an object's kinetic energy in some reference frames can be much larger than the energy implied by its rest mass, so identifying the reference frames where mass and energy are measured is essential for valid derivation of mass-energy relationships. If the derivation gives the equivalence between mass in frame A and energy in frame B in the form of $E = mc^2$ while the two frames move relative to each other, we know it is unlikely to be a correct derivation.

3. Einstein's non-relativistic derivation of mass-energy formula in 1946

Einstein (1946b) gave his last derivation of the mass-energy equivalence in 1946, which is based on conservation of momentum and Maxwell's classical theory of electromagnetism. Since the derivation is quite short, its key part is quoted here (Fig.1).

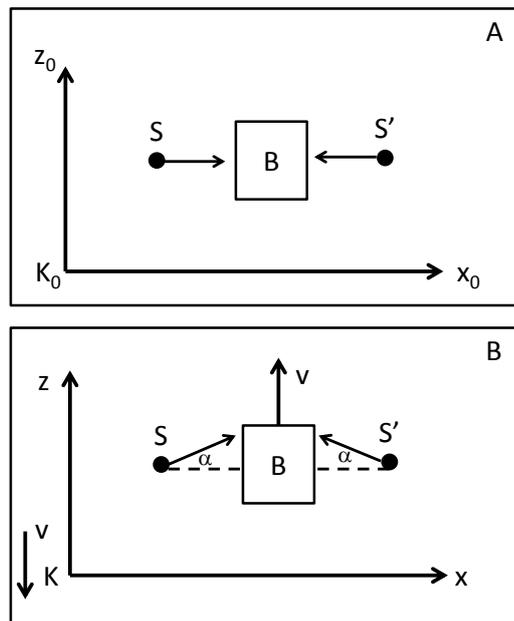


Fig.1. An object B absorbing two wave complexes (S and S') from opposite directions with energy $E/2$ each. A. Object B is at rest in frame K_0 . B. In frame K which moves along z -axis negative direction of frame K_0 with

velocity v , object B is moving in the z -axis positive direction with velocity v , and the two wave complexes have an angle α with the x -axis, $\sin \alpha = v/c$.

“We now consider the following system. Let the body B rest freely in space with respect to the system K_0 . Two complexes of radiation S , S' each of energy $E/2$ move in the positive and negative x_0 direction respectively and are eventually absorbed by B . With this absorption the energy of B increases by E . The body B stays at rest with respect to K_0 by reasons of symmetry. Now we consider this same process with respect to the system K , which moves with respect to K_0 with the constant velocity v in the negative Z_0 direction. With respect to K the description of the process is as follows:

The body B moves in positive Z direction with velocity v . The two complexes of radiation now have directions with respect to K which make an angle α with the x axis. The law of aberration states that in the first approximation $\alpha = \frac{v}{c}$, where c is the velocity of light. From the consideration with respect to K_0 we know that the velocity v of B remains unchanged by the absorption of S and S' .

Now we apply the law of conservation of momentum with respect to the z direction to our system in the coordinate-frame K .

I. Before the absorption let m be the mass of B ; mv is then the expression of the momentum B (according to classical mechanics). Each of the complexes has the energy $E/2$ and hence, by a well-known conclusion of Maxwell's theory, it has the momentum $\frac{E}{2c}$. Rigorously speaking this is the momentum of S with respect to K_0 . However, when v is small with respect to c , the momentum with respect to K is the same except for a quantity of second order of magnitude ($\frac{v^2}{c^2}$ compared to 1). The z -component of this momentum is $\frac{E}{2c} \sin \alpha$ or with sufficient accuracy (except for quantities of higher order of magnitude) $\frac{E}{2c} \alpha$ or $\frac{E}{2} \cdot \frac{v}{c^2}$. S and S' together therefore

have a momentum $E \frac{v}{c^2}$ in the z direction. The total momentum of the system before absorption is therefore

$$mv + \frac{E}{c^2} \cdot v. \quad [(1)]$$

II. After the absorption let m' be the mass of B . We anticipate here the possibility that the mass increased with the absorption of the energy E (this is necessary so that the final result of our consideration be consistent). The momentum of the system after absorption is then

$$m'v$$

We now assume the law of the conservation of momentum and apply it with respect to the z direction. This gives the equation

$$mv + \frac{E}{c^2} \cdot v = m'v. \quad [(2a)]$$

or

$$m' - m = \frac{E}{c^2}. \quad [(2b)]$$

This equation expresses the law of the equivalence of energy and mass. The energy increase E is connected with the mass increase $\frac{E}{c^2}$. Since energy according to the usual definition leaves an additive constant free, we may choose the latter that

$$E = mc^2. \quad (3)$$

There is no special relativity involved in Einstein's derivation in 1946, which is a demonstration that derivation of $E = mc^2$ does not require special relativity. Using Maxwell's theory of electromagnetism and conservation of momentum, Lewis (1908) also derived $E = mc^2$. Poincaré (1900) implicitly derived the mass-energy relation from

classical physics. Since neither Einstein's derivation in 1946 nor Lewis' derivation in 1908 requires assumptions unique to special relativity, according to our Proposition 1, the mass-energy formula $E = mc^2$ is not a result of special relativity.

4. $P = E/c$ is not a relativistic formula

Some commentators claim that Einstein's derivation in 1946 is also relativistic, because $P = E/c$ of the classical electromagnetic theory, which contains the invariant c , is a relativistic formula. According to them, while $P = mv$ is also covariant under Galilean transformations, $P = E/c$ is covariant exclusively under Lorentz transformations thus cannot be considered to be a classical formula.

It seems laughable to claim that a formula that appeared dozens of years before Lorentz ether theory and special relativity is a relativistic formula. Can $P = E/c$ be a classical formula? Of course, it was first derived as a classical formula from Maxwell's classical theory of electromagnetism. In classical physics, light is a type of electromagnetic wave propagating in its medium ether, and c is the constant velocity of light in its medium (frame). Special relativity postulates the constancy of the speed of light, which is actually an expansion of classical physics' constant velocity of light in its medium ether frame into all inertial frames. $P = E/c$ is exactly a description of the momentum of wave packets in the ether frame in Maxwell's classical electromagnetic theory.

Being not covariant under Galilean transformations does not disqualify $P = E/c$ from being a classical formula. Mechanics features of sound are not covariant under Galilean transformations, but physics of sound is still a part of classical physics. In classical physics, light was treated as a mechanical wave in its medium ether, which is why $P = E/c$ is not covariant under Galilean transformations. Unlike motion of material objects, mechanical waves have their privileged frame even in special relativity. A wave's medium is a privileged frame for that wave, which cannot be covariant under Galilean transformations or Lorentz transformations. The motion of sound is not covariant under either Galilean transformations or Lorentz transformations.

Whether a formula is classical physics formula or relativistic formula depends on which it is derived from and which system it is consistent with. $P = E/c$ is derived from classical electromagnetic theory and is consistent with ether-based classical theory, therefore, it is primarily a classical formula. After the advent of special relativity, physicists reinterpreted the meaning of c , making it the speed of light in any inertial frames rather than only in ether frame. This reinterpretation of c does not change the fact that $P = E/c$ in ether frame is a classical physics formula.

Since $P = E/c$ in ether frame is a classical physics formula and in Newtonian mechanics $P = mv$, we can obtain $E = mc^2$ for light wave packets in their ether frame. The velocity of light wave packets is c , so $m = P/v = P/c$ in ether frame and Maxwell's electromagnetic momentum $P = E/c$ implies

$$m = \frac{P}{v} = \frac{E/c}{c} = \frac{E}{c^2},$$

$$E = mc^2 \tag{4}$$

This explains why Preston (1875), Poincaré (1900), De Pretto (1903) and Hasenöhrl (1904) had proposed or derived similar mass-energy relations well before Einstein postulated the constancy of the speed of light. Becquerel used conversion of mass into energy to explain the radioactive energy of radium in 1900, and the conversion ratio that he used is in the same order of magnitude as the mass-energy equation (Nature 2000). Rutherford (1904) and Soddy (1904) also proposed conversion of mass into energy as a source of radioactive energy before special relativity.

As $E = mc^2$ for light wave packets in their ether frame is implied in classical physics, we can ask what the relationships between mass and energy in other reference frames should be. Following the design of the Michelson-Morley experiment, we can consider first the scenario where the direction of light rays is perpendicular to the direction of the velocity of the reference frame in question. Here the velocity of the reference frame is that relative to the ether frame, as in the Michelson-Morley experiment. Since in classical physics the velocity of light follows the Huygens principle, we have the two-way

velocity of light when the direction of light rays is perpendicular to the direction of the velocity of the reference frame c_N ,

$$c_N = \frac{d}{t} = \sqrt{c^2 - v^2} \quad (5)$$

In equation (5), d is the length of two-way light path, t the time interval needed for the light ray to cover the length d , and v the velocity of the reference frame relative to the ether frame. The two-way velocity of light has to be used because one-way velocity of light cannot be measured.

Using the classical momentum formula $P = mv$, we obtain the momentum of light wave packets when the direction of light rays is perpendicular to the direction of the velocity of the reference frame,

$$P_N = mc_N = m\sqrt{c^2 - v^2} = mc\sqrt{1 - \frac{v^2}{c^2}} = P\sqrt{1 - v^2/c^2} \quad (6)$$

We may draw an analogy from the influence of frame velocity on kinetic energy from the classical kinetic energy formula $K = \frac{1}{2}mv^2$. As energy is proportion to the square of velocity while momentum is proportional to the velocity, for a velocity change from c to $\sqrt{c^2 - v^2}$ we have the energy of light wave packets when the direction of light rays is perpendicular to the direction of the velocity of the reference frame,

$$E_N = E\left(\frac{c^2 - v^2}{c^2}\right) \quad (7)$$

In equations (6) and (7), m and E are the mass and the energy implied by the momentum of light wave packets in the ether frame respectively, and P_N and E_N are the momentum and energy of the light wave packets in the frame moving relative to the ether frame at v respectively. If we use the values of momentum, energy and velocity of light measured in this frame, and $P_N = E_N/c_N$, we obtain the relationship between mass and energy

$$\begin{aligned} mc_N &= E_N/c_N \\ E_N &= mc_N^2 \end{aligned} \quad (8)$$

If we substitute equations (5), (6) and (7) into equation (8), we still have the same $E = mc^2$,

$$E \left(\frac{c^2 - v^2}{c^2} \right) = m(\sqrt{c^2 - v^2})^2 = m(c^2 - v^2)$$

$$E = mc^2$$

So $E = mc^2$ is true in this moving frame as well. This result shows that we can also derive equation (8) from $E = mc^2$ in the ether frame.

When the direction of light rays is parallel to the direction of the velocity of the reference frame, the two-way velocity of light measured by the moving frame is

$$c_P = \frac{d}{t} = \frac{c^2 - v^2}{c} \quad (9)$$

We have

$$P_P = m \frac{c^2 - v^2}{c} = mc \left(1 - \frac{v^2}{c^2} \right) = P \left(1 - \frac{v^2}{c^2} \right) \quad (10)$$

For a velocity change from c to $(c^2 - v^2)/c$, we have

$$E_P = E \frac{(c^2 - v^2)^2}{c^4} \quad (11)$$

If we use the values of momentum, energy and velocity of light measured in this frame, and $P_P = E_P/c_P$, we obtain the relationship between mass and energy

$$mc_P = E_P/c_P$$

$$E_P = mc_P^2 \quad (12)$$

In equations (10) - (12), P_P and E_P are the momentum and energy of the light wave packets in the frame moving relative to the ether frame at v respectively, when the direction of light rays is parallel to the direction of the velocity of the reference frame.

If we substitute equations (9) - (11) into equation (12), we obtain

$$E \frac{(c^2-v^2)^2}{c^4} = m \left(\frac{c^2-v^2}{c} \right)^2 = m \frac{(c^2-v^2)^2}{c^2}$$

$$E = mc^2$$

Therefore, in classical physics, $E = mc^2$ is true in all inertial reference frames. Again, we can also derive equation (12) from $E = mc^2$ in the ether frame.

In the above analysis I have shown that in classical physics $E = mc^2$ or $E_A = mc_A^2$, where A stands for any inertial reference frame, is true in all inertial reference frames, if energy, momentum and the velocity of light are all measured in the same reference frame. Being consistent with the invariant mass across different frames in classical physics, the mass implied by $P = mv$ and $P = E/c$ is also invariant across different frames, while P and the velocity of light change across different frames.

$E_A = mc_A^2$ tells us the relationship between mass and energy in the same reference frame, and we may ask a further question on the relationship between energy values measured in two different reference frames, for example, between those measured in the ether frame and in a moving frame. We could obtain the relationship between energy values measured in the ether frame and in a frame moving relative to it at v by using the velocity of light in the ether frame c for both frames.

In the scenario where the direction of light rays is perpendicular to the direction of the velocity of the reference frame, instead of $P_N = E_N/c_N$ if we use

$$P_N = E_N/c \tag{13}$$

and substitute equations (6) and (7) into it, we have

$$mc \sqrt{1 - \frac{v^2}{c^2}} = E(1 - v^2/c^2)/c$$

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{E_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{14}$$

In equation (14), we let $E_0 = mc^2$ be the energy measured in the ether frame. Equation (14) shows the relationship between energy values measured in the ether frame and the frame moving relative to the ether frame at v , when the direction of light rays is perpendicular to the direction of the velocity of the reference frame. This equation has the same function form as that discovered by Laue (1911) and Klein (1918) for special relativity.

Again, in the scenario where the direction of light rays is parallel to the direction of the velocity of the reference frame, instead of $P_p = E_p/c_p$ if we use

$$P_p = E_p/c \quad (15)$$

by substitute equations (9), (10) and (11) into equation (15), we obtain

$$mc \left(1 - \frac{v^2}{c^2}\right) = E \frac{(c^2 - v^2)^2}{c^5}$$

$$E = \frac{mc^4}{c^2 - v^2} = \frac{mc^2}{1 - v^2/c^2} = \frac{E_0}{1 - v^2/c^2} \quad (16)$$

The equation (16) shows that when the direction of light rays is parallel to the direction of the velocity of the reference frame, the relationship between energy values measured in the ether frame and the moving frame is different from that when the direction of light rays is perpendicular to the direction of the velocity of the reference frame.

The difference between equations (14) and (16) reflects the fact that electromagnetism is not covariant under Galilean transformations. The design of the Michelson-Morley experiment was to measure the velocity of the earth relative to the ether frame by exploiting the difference between $\sqrt{1 - v^2/c^2}$ and $1 - v^2/c^2$, but they failed to find a significant one. Fitzgerald (1889) and Lorentz (1902, 1904) hypothesized length contraction to explain the null result in the attempts to measure the difference between $\sqrt{1 - v^2/c^2}$ and $1 - v^2/c^2$. With Fitzgerald-Lorentz length contraction, equation (16) will have the same function form as that of equation (14).

From analysis in this section, we see that classical physics could not only logically have $P = E/c$ and $E = mc^2$, but also logically arrive almost at the relativistic formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{E_0}{\sqrt{1-v^2/c^2}}$. Therefore, it is not logically justified to assert that $P = E/c$ is a relativistic formula and $E = mc^2$ is an exclusively relativistic equation. The contribution of Lorentz ether theory in this is to ensure light rays in any directions have the same relationship with the ether frame via the Fitzgerald-Lorentz length contraction, and special relativity replaces the ether frame with any inertial frame.

5. Einstein's derivation in 1905 and its flaws as a relativistic proof

It is Einstein's first derivation (Einstein 1905) that links the mass-energy equation with special relativity. The derivation is based on a thought experiment that is unlikely to be achievable in laboratory (Ohanian 2009; Hecht 2011). Its key part is quoted here.

“Let a system of plane waves of light, referred to the system of co-ordinates (x, y, z) , possess the energy L ; let the direction of the ray (the wave-normal) make an angle ϕ with the axis of x of the system. If we introduce a new system of co-ordinates (ξ, η, ζ) moving in uniform parallel translation with respect to the system (x, y, z) , and having its origin of co-ordinates in motion along the axis of x with the velocity v , then this quantity of light—measured in the system (ξ, η, ζ) —possesses the energy

$$L^* = L \frac{1 - \frac{v}{c} \cos \phi}{\sqrt{1 - v^2 / c^2}} \quad [(17)]$$

where c denotes the velocity of light. We shall make use of this result in what follows.

Let there be a stationary body in the system (x, y, z) , and let its energy—referred to the system (x, y, z) be E_0 . Let the energy of the body relative to the system (ξ, η, ζ) moving as above with the velocity v , be H_0 .

Let this body send out, in a direction making an angle ϕ with the axis of x , plane waves of light, of energy $\frac{1}{2}L$ measured relatively to (x, y, z) , and simultaneously an equal quantity of light in the opposite direction. Meanwhile the body remains at rest with respect to the system (x, y, z) . The principle of energy must apply to this process, and in fact (by the principle of relativity) with respect to both systems of co-ordinates. If we call the energy of the body after the emission of light E_1 or H_1 respectively, measured relatively to the system (x, y, z) or (ξ, η, ζ) respectively, then by employing the relation given above we obtain

$$E_0 = E_1 + \frac{1}{2}L + \frac{1}{2}L \quad [(18)]$$

$$H_0 = H_1 + \frac{1}{2}L \frac{1 - \frac{v}{c} \cos \phi}{\sqrt{1 - v^2/c^2}} + \frac{1}{2}L \frac{1 + \frac{v}{c} \cos \phi}{\sqrt{1 - v^2/c^2}} = H_1 + \frac{L}{\sqrt{1 - v^2/c^2}} \quad [(19)]$$

By subtraction we obtain from these equations

$$H_0 - E_0 - (H_1 - E_1) = L \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right). \quad [(20)]$$

The two differences of the form $H - E$ occurring in this expression have simple physical significations. H and E are energy values of the same body referred to two systems of co-ordinates which are in motion relatively to each other, the body being at rest in one of the two systems (system (x, y, z)). Thus it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other system (ξ, η, ζ) , only by an additive constant C , which depends on the choice of the arbitrary additive constants of the energies H and E . Thus we may place

$$H_0 - E_0 = K_0 + C \quad [(21)]$$

$$H_1 - E_1 = K_1 + C \quad [(22)]$$

since C does not change during the emission of light.” (Einstein 1905)

Equations (21) and (22) are the key in Einstein’s derivation, which is equivalent to a statement that (the change in) non-kinetic energy has the same value in all reference frames, i.e. the difference in energy values of an object measured in two reference frames is only the difference in its values of kinetic energy. This assertion by Einstein has been a major source of controversy regarding the validity of Einstein’s derivation in 1905. Ives (1952), Jammer (1961) and Arzeliés (1966) think that the mass-energy equation is implied by equations (21) and (22); without justifying them, Einstein’s derivation is invalid. However, the current definition of kinetic energy in relativistic mechanics has implied equations (21) and (22), which weakens the objection of Ives, Jammer and Arzeliés. From equations (21) and (22), Einstein derived an approximate mass-energy equivalence.

“So we have

$$K_0 - K_1 = L \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) \quad [(23)]$$

The kinetic energy of the body with respect to (ξ, η, ζ) diminishes as a result of the emission of light, and the amount of diminution is independent of the properties of the body. Moreover, the difference $K_0 - K_1$, like the kinetic energy of the electron (§ 10), depends on the velocity.

Neglecting magnitudes of fourth and higher orders we may place

$$K_0 - K_1 = \frac{1}{2} \frac{L}{c^2} v^2.” (Einstein 1905) \quad (24)$$

Equation (23) is a logical consequence of (21) and (22), which states the difference in the values of an object’s kinetic energy measured in one reference frame at two time points (i.e. $K_0 - K_1$) equals the difference between the changes of total energy measured in that frame (i.e. $H_0 - H_1$) and the frame where the object is stationary (i.e. $E_0 - E_1$) at these

two time points. The right hand side of equation (24) is an approximate of the right hand side of equation (23), which gives an appearance of the classical expression of kinetic energy. From this approximate, Einstein draws the conclusion that “if a body gives off the energy L in the form of radiation, its mass diminishes by L/c^2 ”.

The transition from equation (23) to (24) does show Einstein’s ingenuity in dealing with difficult problems in physics, but as a relativistic proof of the mass-energy equation, it lacks sufficient logical rigour.

Firstly, K_0 and K_1 are obviously relativistic kinetic energy, which would not be equal to $\frac{1}{2}mv^2$ because $K = \frac{1}{2}mv^2$ is a classical formula. If relativistic kinetic energy $K_{rel} \neq \frac{1}{2}mv^2$, we cannot say that $K_{rel} = \frac{1}{2}\frac{L}{c^2}v^2$ implies $L = mc^2$ or $E = mc^2$. At least, we cannot say that $K_{rel} = \frac{1}{2}\frac{L}{c^2}v^2$ implies a precise relationship $L = mc^2$ or $E = mc^2$.

Secondly, the mass-energy relationship from Einstein’s derivation seems to be velocity dependent. When v is larger, such as $v = 0.8c$, magnitudes of fourth and higher orders cannot be neglected. So $E = mc^2$ derived implicitly by Einstein in 1905 is only an approximate when v is relatively small, it is not a universal relation applicable to objects at all velocities. Einstein (1946a) acknowledged the imprecision of his mass-energy equation by noting that “It is customary to express the equivalence of mass and energy (though somewhat inexactly) by the formula $E = mc^2$ ”.

Thirdly and more importantly, according to our Proposition 3, mass and energy should be measured in the same reference frame, but in equation (24) $K_0 - K_1$ and L (hence L/c^2) are not measured in the same reference frame. L is the radiation energy measured in the frame where the emitting body is stationary, while K_0 and K_1 are kinetic energy measured in the frame where the emitting body is moving with velocity v . As mass-energy equivalence should not be one in frame (x, y, z) and one in frame (ξ, η, ζ) , Einstein’s

“relativistic” derivation fails to show equivalence between mass and energy measured in the same reference frame.

6. Derivation of mass-energy equation from conservation of momentum

Einstein’s equations (21) and (22) are among the main controversial points regarding the validity of Einstein’s derivation (Ives 1952; Jammer 1961; Arzeliés 1966). The two equations are consistent with classical physics where the difference between the values of an object’s energy measured by two reference frames in relative motion is only kinetic energy. We know from the preceding section that Einstein failed to provide a valid derivation even with equations (21) and (22). Einstein (1935) tried to prove rest energy $E_0 = m$ by asserting without proof that total energy $E = E_0 + m \left(\frac{1}{\sqrt{1 - v^2 / c^2}} - 1 \right)$ and kinetic energy is $m \left(\frac{1}{\sqrt{1 - v^2 / c^2}} - 1 \right)$. However, he did not give a derivation of $E_0 = \Delta m_0 c^2$. So far, all proofs of the mass-energy formula based on special relativity fail to show a mass-energy equivalence in the same reference frame.

Without equations (21) and (22), Einstein could have started with the conservation of momentum to derive the mass-energy relation. Then in the frame (x, y, z) where the radiating body is at rest, we have

$$P_{S0} = P_{S1} + \frac{E_S}{2c} - \frac{E_S}{2c} = P_{S1} = 0 \quad (25)$$

In equation (25), P stands for momentum, the subscript S indicates the frame where the radiating body is stationary, and $\frac{E}{2c}$ is the momentum of light wave packet in one direction (as in Maxwell’s classical electromagnetic theory, here Einstein’s L is replaced with the more conventional E for energy).

In the frame (ξ, η, ζ) where the radiating body is moving at the velocity v ,

$$P_{V0} = P_{V1} + \frac{E_S}{2c} \frac{1 + \frac{v}{c} \cos \phi}{\sqrt{1 - v^2/c^2}} - \frac{E_S}{2c} \frac{1 - \frac{v}{c} \cos \phi}{\sqrt{1 - v^2/c^2}} = P_{V1} + E_S \frac{\frac{v}{c^2} \cos \phi}{\sqrt{1 - v^2/c^2}} \quad (26)$$

In equation (26), the subscript V indicates the moving frame. When $\phi = 0$,

$$\Delta P_V = P_{V0} - P_{V1} = \frac{\frac{v}{c^2} E_S}{\sqrt{1 - v^2/c^2}} \quad (27)$$

Since $P_V = m_V v = m_S v / \sqrt{1 - v^2/c^2}$ (here relativistic mass m_V is used for illustration purpose),

$$\Delta m_S = E_S / c^2. \quad (28)$$

In the frame where the radiating body is stationary, when energy E is emitted, there is a loss of mass $\Delta m = E/c^2$. This mass-energy equivalence in the same reference frame is exact rather than approximate, which has been confirmed by experiments. From a measurement's point of view, equation (28) is equivalent to equations (8) and (12); that is $E = mc^2$ when the three quantities are measured in the same reference frame.

From equation (27) and $P_V = m_V v$, we can also obtain

$$\Delta m_V c^2 = E_S / \sqrt{1 - v^2/c^2}$$

Since $\Delta m_S c^2 = E_S$, let $\Delta m_V c^2 = E_V$, which is the energy (value) measured in the frame moving relative to the radiating body, we obtain

$$E_V = E_S / \sqrt{1 - v^2/c^2} = \Delta m_S c^2 / \sqrt{1 - v^2/c^2}. \quad (29)$$

Equation (29) is the relativistic formula describing the relationship between values of the same energy measured in two reference frames, which depends on their relative velocity v .

If we use subscript 0 to indicate measurements obtained in the frame where the radiating body is stationary, our new derivation reveals what Einstein should have proved is the equation (14) derived by Laue (1911) and Klein (1918)

$$E = E_0 / \sqrt{1 - v^2 / c^2} .$$

Equation (14) corresponds to the relativistic mass equation (Lorentz 1904)

$$m = m_0 / \sqrt{1 - v^2 / c^2} .$$

The essence of Einstein's derivation in 1905 is actually an approximation of equation (14),

$$E - E_0 = \frac{E_0}{\sqrt{1 - v^2 / c^2}} - E_0 = E_0 \left(\frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \dots \right) \approx \frac{1}{2} \frac{E_0}{c^2} v^2 . \quad (30)$$

Expanding the relativistic mass equation and using classical kinetic energy expression

$K = \frac{1}{2} m v^2$ can get the same relationship when v is small,

$$\Delta m = m - m_0 = \frac{m_0}{\sqrt{1 - v^2 / c^2}} - m_0 = m_0 \left(\frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \dots \right) \approx \frac{1}{2} \frac{m_0 v^2}{c^2} = \frac{E_0}{c^2} \quad (31)$$

However, both equation (31) and Einstein's derivation in 1905 describe relationships between variables measured in different frames, which violate Proposition 3, and need classical kinetic energy formula, which violates Proposition 2.

Therefore, the relativistic result should be $E = E_0 / \sqrt{1 - v^2 / c^2}$, which is just a different expression of the relativistic mass equation $m = m_0 / \sqrt{1 - v^2 / c^2}$. This relationship between energy values measured in two reference frames has been shown by Laue (1911), using conservation of energy-momentum tensor and assuming that there is no energy flow in the rest frame. Klein (1918) extended Laue's results to closed systems with or without flow of energy.

7. Logical shortcomings in Einstein's derivation in 1946 and correct derivation using Einstein's premise

Einstein's derivation in 1946 has the shortcoming of not distinguishing different values measured in the two reference frames. A wave complex has different energy values in two frames K_0 and K with relative motion. In equations (1) and (2), the energy values of the wave complexes are those measured in frame K_0 , while the momentums are measured in frame K . The derivation is logically inconsistent, because mass-energy equivalence should be the equivalence when both mass and energy are measured in the same reference frame.

To derive a more precise mass-energy equation, we need to know in which reference frame the variables are measured. We can firstly add subscripts to the variables so that we can keep track of the reference frames in which they are measured. We re-write equations (2a) and (2b) as

$$m_{K1}v + v \cdot E_K / c^2 = m_{K2}v. \quad (32a)$$

$$\Delta m_K = m_{K2} - m_{K1} = E_K / c^2. \quad (32b)$$

In equation (32), m_{K1} is the mass before the absorption in the moving frame, m_{K2} the mass after the absorption in the moving frame, and E_K the energy measured in the moving frame. From equation (32b), we obtain the mass-energy equation in the moving frame

$$E_K = \Delta m_K c^2. \quad (32c)$$

So far, the derivation is in classical physics with electromagnetic waves having momentum. What is the relationship between mass and energy in the stationary frame?

Lorentz relativistic mass formula has given us the relationship between values of a mass in different reference frames. Using Lorentz relativistic mass formula, we obtain from equation (32b)

$$\frac{\Delta m_0}{\sqrt{1-v^2/c^2}} = \frac{m_{02}}{\sqrt{1-v^2/c^2}} - \frac{m_{01}}{\sqrt{1-v^2/c^2}} = \frac{E_K}{c^2},$$

which gives

$$E_K = \Delta m_0 c^2 / \sqrt{1-v^2/c^2}. \quad (33)$$

When $v=0$, we have the mass-energy equation in the stationary frame

$$E_0 = \Delta m_0 c^2. \quad (34)$$

Therefore, the relativistic energy formula is still equation (14), i.e., what Laue (1911) and Klein (1918) have found

$$E = E_0 / \sqrt{1-v^2/c^2}.$$

Equation (14) is the correct formula for relationship of relativistic energy values between two reference frames with relative motion. The result reveals the symmetry between changes in relativistic mass and in relativistic energy in the moving frame. The equation $E = mc^2$ can be obtained approximately from the correct relativistic equation only when classical kinetic definition $K = \frac{1}{2}mv^2$ is used and the requirement of measuring mass and energy in the same reference frame (Proposition 3) is dropped.

8. Discussions

Strictly speaking, the two derivations presented in this paper and many other derivations so far are only illustrations of the mass-energy equivalence contained in Newtonian mechanics and Maxwell's electromagnetic theory with special scenarios. If in Newtonian mechanics there were another type of momentum which had no corresponding mass or inertia, $P_{WithoutMass} \neq mv$, equations (2), (28) and (32) and all other similar equations would not be valid. If $m \equiv P/v$, we can obtain the mass-energy equation directly from

$P \equiv mv$ and electromagnetic momentum $P = E/c$. When a material object with mass m is converted completely into electromagnetic waves, the total energy released is $E = mc^2$.

The mass-energy equation can be derived from classical physics without involving special relativity, while in Einstein's "relativistic" derivation $E = mc^2$ cannot be obtained without resorting to classical kinetic energy definition and approximation at small velocity. Even with classical kinetic energy definition and approximation at small velocity, Einstein still failed to prove $E = mc^2$ for mass and energy measured in the same reference frame.

Although $P = mv$ and $P = E/c$ in classical physics and $E = E_0/\sqrt{1-v^2/c^2}$ in special relativity have been known to physicists for a long time, most physicists and the general public still strongly believe that $E = mc^2$ is an exclusively relativistic result, overlooking the deep-rooted connection of the mass-energy equation with classical physics. Therefore, establishing the true identity of $E = mc^2$ is not only important in physics, but also philosophically and historically significant. Attributing a formula contained in classical physics exclusively to special relativity distorts the historical process of human understanding of the nature.

From the present study, we may draw the following conclusions:

Firstly, the mass-energy equation $E = mc^2$ is contained in Maxwell's classical electromagnetic theory and the momentum definition of Newtonian mechanics. With the momentum definition in Newtonian mechanics $P \equiv mv$ and Maxwell's electromagnetic momentum $P = E/c$, the mass-energy equation $E = mc^2$ should be a logical consequence.

Secondly, all logically valid derivations of $E = mc^2$, where both mass m and energy E are measured in the same reference frame, rely on the two classical equations $P \equiv mv$ and $P = E/c$. No matter whether a derivation is under classical or relativistic conditions, the two equations must be held true. If the two equations are denied in any of those derivations, it is not possible to arrive at $E = mc^2$ logically. If these two equations are held

true, the mass energy equation $E = mc^2$ can be obtained directly without the special scenarios assumed for those derivations.

Thirdly, since $E = mc^2$ can be derived without resorting to any relativistic result, it is a formula from classical physics, applicable to both classical physics and special relativity when relativistic mass is used in the equation.

Fourthly, the relativistic transformation of energy (values) between different reference frames is $E = E_0 / \sqrt{1 - v^2 / c^2}$.

Fifthly, Einstein's "relativistic" derivation in 1905 relies on the classical kinetic energy definition, describes implicitly a relationship between mass and energy measured in different reference frames and leads only to an approximation at low velocity for a velocity dependent equation; hence it is not logically valid as a relativistic proof of the mass-energy equation $E = mc^2$.

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