# On generalized harmonic numbers, Tornheim double series and linear Euler sums *

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**Abstract**

In this paper, direct links between generalized harmonic numbers, linear Euler sums and Tornheim double series are established in a more perspicuous manner than is found in existing literature. The high point of the paper is the discovery of certain combinations of Euler sums that are reducible to Riemann zeta values.

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# 1 Introduction

## 1.1 Generalized harmonic numbers and linear Euler sums

Generalized harmonic numbers have a long history, having been studied since the time of Euler. The $r^{th}$ generalized harmonic number of order $n$, denoted by $H_{r,n}$ in this paper, is defined by

$$H_{r,n} = \sum_{s=1}^{r} \frac{1}{s^n} \,,$$

where $H_{r,1} = H_r$ is the $r^{th}$ harmonic number and $H_{0,n} = 0$. The generalized harmonic number converges to the Riemann zeta function, $\zeta(n)$:

$$\lim_{r\to\infty} H_{r,n} = \zeta(n), \quad \mathcal{R}[n] > 1 \,, \tag{1.1}$$

since $\zeta(n) = \sum_{s=1}^{\infty} s^{-n}$ .

Of particular interest in the study of harmonic numbers is the evaluation of infinite series involving the generalized harmonic numbers, especially linear Euler sums of the type

$$E(m, n) = \sum_{\nu=1}^{\infty} \frac{H_{\nu,m}}{\nu^n} \,.$$

The linear sums can be evaluated in terms of zeta values in the following cases: $m = 1$, $m = n$, $m + n$ odd and $m + n = 6$ (with $n > 1$), (see [1]).

Evaluation of Euler sums, E(m,n) of odd weight, $m+n$ in terms of $\zeta$ values can be accomplished through Theorem 3.1 of [1].

As for the case $m = 1$, we have:

**THEOREM 1.1** (Euler). *For $n-1 \in \mathbb{Z}^+$ holds*

$$2E(1,n) = 2\sum_{\nu=1}^{\infty} \frac{H_\nu}{\nu^n} = (n+2)\zeta(n+1) - \sum_{j=1}^{n-2} \zeta(j+1)\zeta(n-j)\,.$$

## 1.2 Tornheim double series and relation to linear Euler sums

Tornheim double series, T(r,s,t), is defined by

$$T(r,s,t) = \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{1}{\mu^r \nu^s (\mu+\nu)^t}$$

and named after Leonard Tornheim who made a systematic and extended study of the series in a 1950 paper, [2]. $T(r,s,t)$ has the following basic properties [5]:

$$T(r,s,t) = T(s,r,t)\,, \tag{1.2a}$$

$$T(r,s,t) \text{ is finite if and only if } r+t>1, s+t>1 \text{ and } r+s+t>2\,, \tag{1.2b}$$

$$T(r,s,0) = \zeta(r)\zeta(s)\,, \tag{1.2c}$$

$$T(r,0,t) + T(t,0,r) = \zeta(r)\zeta(t) - \zeta(r+t)\,, \quad r \geq 2 \tag{1.2d}$$

and

$$T(r,s-1,t+1) + T(r-1,s,t+1) = T(r,s,t), \quad r \geq 1, s \geq 1. \tag{1.2e}$$

In light of (1.1), the useful identity

$$\sum_{\nu=1}^{N} \frac{1}{(\mu+\nu)^t} = H_{N+\mu,t} - H_{\mu,t}\,, \tag{1.3}$$

leads to

$$\sum_{\nu=1}^{\infty} \frac{1}{(\mu+\nu)^t} = \zeta(t) - H_{\mu,t}\,, \tag{1.4}$$

which establishes the link between the Hurwitz zeta function, $\zeta(t,\mu)$, the Riemann zeta function and the generalized harmonic numbers (see also equation (1.19) of [9]) as

$$\zeta(t,\mu) = \zeta(t) - H_{\mu-1,t}\,, \tag{1.5}$$

since

$$\zeta(t,\mu) = \sum_{\nu=0}^{\infty} \frac{1}{(\mu+\nu)^t} \, .$$

The identity (1.4) also brings out the direct connection between the linear Euler sums and the Tornheim double series, namely,

$$E(n,m) = \zeta(n)\zeta(m) - T(m,0,n), \quad n > 1, m > 1 \, . \tag{1.6}$$

Differentiating the identity

$$\frac{1}{\nu} - \frac{1}{\mu+\nu} = \frac{\mu}{\nu(\mu+\nu)} \, , \tag{1.7}$$

$n-1$ times with respect to $\nu$ gives

$$\frac{1}{\nu^n} - \frac{1}{(\mu+\nu)^n} = \sum_{p=0}^{n-1} \frac{\mu}{\nu^{p+1}(\mu+\nu)^{n-p}} \, , \quad n \in \mathbb{N}_0 \, ,$$

from which, by summing over $\nu$, employing (1.3), we obtain

$$H_{N,n} - H_{N+\mu,n} + H_{\mu,n} = \sum_{p=0}^{n-1} \sum_{\nu=1}^{N} \frac{\mu}{\nu^{p+1}(\mu+\nu)^{n-p}} \, ,$$

and hence, in the limit $N \to \infty$ we have (see also [7] and [8] for alternative derivations of the particular case of $n = 1$)

$$H_{\mu,n} = \sum_{p=0}^{n-1} \sum_{\nu=1}^{\infty} \frac{\mu}{\nu^{p+1}(\mu+\nu)^{n-p}} \, ,$$

which gives the interesting relation

$$\sum_{\mu=1}^{\infty} \frac{H_{\mu,n}}{\mu^r} = \sum_{p=0}^{n-1} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{\mu^{r-1}\nu^{p+1}(\mu+\nu)^{n-p}} \, ,$$

that is,

$$E(n,r) = \sum_{p=0}^{n-1} T(r-1, p+1, n-p) \, ,$$

or equivalently,

$$\boxed{E(n,r) = \sum_{p=1}^{n} T(r-1, n-p+1, p)}\,.$$

In particular,

$$E(1,r) = T(r-1,1,1), \quad r > 1\,. \tag{1.8}$$

Throughout this paper we shall make frequent tacit use of the following index shift identity:

$$\sum_{i=u-b}^{u-a} f(u-i) \equiv \sum_{i=a}^{b} f(i)\,. \tag{1.9}$$

## 2 Generalized harmonic numbers and summation of series

In this section we discuss the evaluation of certain sums in terms of the Riemann zeta function and the generalized harmonic numbers.

**LEMMA 2.1.** *Let $a$, $c$ and $f$ be arbitrary functions such that $a \neq 0$, $c \neq 0$ and $af = c + a$, then for $m \in \mathbb{Z}^+$ holds*

$$af^m = af + \sum_{i=1}^{m-1} cf^i\,,$$

or, equivalently, using the index shift identity,

$$af^m = af + \sum_{i=0}^{m-2} cf^{m-i-1}\,.$$

The Lemma is easily proved by the application of mathematical induction on $m$.

Choosing $a = -1/(\mu+\nu)$, $c = 1/\nu$ and $f = -\mu/\nu$ in Lemma 2.1 gives the partial fraction decomposition

$$(-1)^{m-1}\frac{\mu^m}{\nu^m(\mu+\nu)} = \frac{\mu}{\nu(\mu+\nu)} + \sum_{i=1}^{m-1}(-1)^i\frac{\mu^i}{\nu^{i+1}}\,, \tag{2.1}$$

which, after $n$ times differentiation with respect to $\mu$, yields, for $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$, the identity

$$(-1)^{m-1} \sum_{p=0}^{n} \left\{ (-1)^p \binom{m}{p} \frac{\mu^{m-p}}{\nu^m (\mu+\nu)^{n-p+1}} \right\}$$

$$= -\frac{1}{(\mu+\nu)^{n+1}} + (-1)^n \sum_{i=n}^{m-1} \left\{ (-1)^i \binom{i}{n} \frac{\mu^{i-n}}{\nu^{i+1}} \right\}, \tag{2.2}$$

from which upon summing from $\nu = 1$ to $\nu = N$ we have

**THEOREM 2.2.** *For $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$, $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$ holds*

$$(-1)^{m-1} \sum_{p=0}^{n} \left\{ (-1)^p \binom{m}{p} \sum_{\nu=1}^{N} \frac{\mu^{m-p}}{\nu^m (\mu+\nu)^{n-p+1}} \right\}$$

$$= H_{\mu,n+1} - H_{N+\mu,n+1} + (-1)^n \sum_{i=n}^{m-1} \left\{ (-1)^i \binom{i}{n} \mu^{i-n} H_{N,i+1} \right\},$$

which in the limit $N \to \infty$ gives

**COROLLARY 2.3.** *For $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$, $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$ holds*

$$(-1)^{m-1} \sum_{p=0}^{n} \left\{ (-1)^p \binom{m}{p} \sum_{\nu=1}^{\infty} \frac{\mu^{m-p}}{\nu^m (\mu+\nu)^{n-p+1}} \right\}$$

$$= H_{\mu,n+1} + (-1)^n \sum_{i=n+1}^{m-1} \left\{ (-1)^i \binom{i}{n} \mu^{i-n} \zeta(i+1) \right\}.$$

*In particular, for $m, n \in \mathbb{Z}^+$ and $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$, we have*

$$(-1)^{m-1} \sum_{\nu=1}^{\infty} \frac{\mu^m}{\nu^m(\mu+\nu)} = H_\mu + \sum_{i=1}^{m-1} (-1)^i \mu^i \zeta(i+1), \tag{2.3}$$

$$\sum_{p=0}^{n} \left\{ (-1)^p \binom{n}{p} \sum_{\nu=1}^{\infty} \frac{\mu^p}{\nu^n (\mu+\nu)^{p+1}} \right\} = \zeta(n+1) - H_{\mu,n+1}$$

*and*

$$\sum_{p=1}^{n}\left\{(-1)^{p-1}\binom{n}{p}\sum_{\nu=1}^{\infty}\frac{\mu^p}{\nu^n(\mu+\nu)^p}\right\}=H_{\mu,n}\,.$$

$m=1$ in (2.3) gives the beautiful formula

$$\sum_{\nu=1}^{\infty}\frac{\mu}{\nu(\mu+\nu)}=H_{\mu}\,,\quad \mu\in\mathbb{C}\backslash\mathbb{Z}^-\,, \tag{2.4}$$

which was also derived in [7] and [8].

Differentiating 2.1 $n$ times with respect to $\nu$, we obtain, for $m\in\mathbb{Z}^+$ and $n\in\mathbb{N}_0$, the identity

$$(-1)^m\sum_{p=0}^{n}\left\{\binom{m+p-1}{p}\frac{\mu^{m+1}}{\nu^{m+p}(\mu+\nu)^{n-p+1}}\right\}$$
$$=\frac{\mu}{(\mu+\nu)^{n+1}}+\sum_{i=1}^{m}\left\{(-1)^i\binom{i+n-1}{n}\frac{\mu^i}{\nu^{i+n}}\right\}, \tag{2.5}$$

from which upon summing from $\nu=1$ to $\nu=N$ we have

**THEOREM 2.4.** *For $m\in\mathbb{Z}^+$, $\mu\in\mathbb{C}\backslash\mathbb{Z}^-$ and $n\in\mathbb{N}_0$ holds*

$$\begin{aligned}(-1)^m\sum_{p=0}^{n}&\left\{\binom{m+p-1}{p}\sum_{\nu=1}^{N}\frac{\mu^{m+1}}{\nu^{m+p}(\mu+\nu)^{n-p+1}}\right\}\\ &=\mu H_{N+\mu,n+1}-\mu H_{N,n+1}\\ &\quad-\mu H_{\mu,n+1}\\ &\quad\quad+\sum_{i=2}^{m}(-1)^i\binom{i+n-1}{n}\mu^i H_{N,i+n}\,,\end{aligned}$$

which in the limit $N\to\infty$ gives

**COROLLARY 2.5.** *For $m\in\mathbb{Z}^+$, $\mu\in\mathbb{C}\backslash\mathbb{Z}^-$ and $n\in\mathbb{N}_0$ holds*

$$(-1)^m\sum_{p=0}^{n}\left\{\binom{m+p-1}{p}\sum_{\nu=1}^{\infty}\frac{\mu^{m+1}}{\nu^{m+p}(\mu+\nu)^{n-p+1}}\right\}$$

$$=-\mu H_{\mu,n+1}+\sum_{i=2}^{m}(-1)^i\binom{i+n-1}{n}\mu^i\zeta(n+i)\,.$$

Using $a = 1/\mu\nu$, $c = -1/(\mu(\mu+\nu))$ and $f = \mu/(\mu+\nu)$ in Lemma 2.1 gives the identity

$$\frac{\mu^m}{\nu(\mu+\nu)^m} = \frac{1}{\nu} - \sum_{i=1}^{m} \frac{\mu^{i-1}}{(\mu+\nu)^i}\,,$$

which, after $n$ differentiations with respect to $\nu$, gives

$$\sum_{p=0}^{n} \binom{m+n-p-1}{m-1} \frac{\mu^m}{\nu^{p+1}(\mu+\nu)^{m+n-p}}$$
$$= \frac{1}{\nu^{n+1}} - \sum_{i=1}^{m} \binom{i+n-1}{i-1} \frac{\mu^{i-1}}{(\mu+\nu)^{i+n}}\,,$$

from which we get, after summing over $\nu$,

**THEOREM 2.6.** *For $m \in \mathbb{Z}^+$, $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$ and $n \in \mathbb{N}_0$ holds*

$$\sum_{p=0}^{n} \left\{ \binom{m+n-p-1}{m-1} \sum_{\nu=1}^{N} \frac{\mu^m}{\nu^{p+1}(\mu+\nu)^{m+n-p}} \right\}$$
$$= H_{N,n+1} - \sum_{i=1}^{m} \binom{i+n-1}{i-1} \mu^{i-1} H_{N+\mu,i+n} + \sum_{i=1}^{m} \binom{i+n-1}{i-1} \mu^{i-1} H_{\mu,i+n}\,,$$

which in the limit $N \to \infty$ gives

**COROLLARY 2.7.** *For $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$ and $m, n \in \mathbb{N}_0$ holds*

$$\sum_{p=0}^{n} \left\{ \binom{m+p}{m} \sum_{\nu=1}^{N} \frac{\mu^{m+1}}{\nu^{n-p+1}(\mu+\nu)^{m+p+1}} \right\}$$
$$= \sum_{i=n}^{m+n} \binom{i}{i-n} \mu^{i-n} H_{\mu,i+1} - \sum_{i=n+1}^{m+n} \binom{i}{i-n} \mu^{i-n} \zeta(i+1)\,.$$

In particular, for $m \in \mathbb{N}_0$ and $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$, we have

$$\sum_{\nu=1}^{\infty} \frac{\mu^{m+1}}{\nu(\mu+\nu)^{m+1}} = H_\mu - \sum_{i=1}^{m} \mu^i \zeta(i+1) + \sum_{i=1}^{m} \mu^i H_{\mu,i+1}\,.$$

## 3 Functional relations for the Tornheim double series

Dividing through the identity of Corollary 2.3 by $\mu^r$ and summing over $\mu$ we obtain

**THEOREM 3.1.** *For $m, r-1 \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$ holds*

$$(-1)^{m-1}\sum_{p=0}^{n}\left\{(-1)^p\binom{m}{p}T(r-m+p,m,n-p+1)\right\}$$
$$=E(n+1,r)+(-1)^n\sum_{i=n+1}^{m-1}\left\{(-1)^i\binom{i}{n}\zeta(i+1)\zeta(r-i+n)\right\}.$$

**COROLLARY 3.2.** *For $m, n \in \mathbb{Z}^+$ holds*

$$(-1)^{m-1}2\sum_{p=0}^{n}\left\{(-1)^p\binom{m}{p}T(n-m+p+1,m,n-p+1)\right\}$$
$$=\zeta(n+1)^2+\zeta(2n+2)$$
$$+(-1)^n2\sum_{i=n+1}^{m-1}\left\{(-1)^i\binom{i}{n}\zeta(i+1)\zeta(2n-i+1)\right\}.$$

Setting $m=n$ in Corollary 3.2, substituting $n$ for $n+1$ and utilizing the index shift identity (1.9) gives, for $n-1 \in \mathbb{Z}^+$,

$$2\sum_{p=1}^{n}(-1)^{p-1}\binom{n}{p}T(n-p,n,p)=\zeta(n)^2+\zeta(2n)\,,$$

from which, with the aide of Corollary 4.2 and after some manipulation, we get,

$$\sum_{p=1}^{2n-1}(-1)^{p-1}\binom{2n}{p}T(2n-p,2n,p)=\zeta(2n)^2\,,\quad n\in\mathbb{Z}^+$$

and

$$\sum_{p=1}^{2n}(-1)^{p-1}\binom{2n+1}{p}T(2n-p+1,2n+1,p)$$
$$=\zeta\left(2(2n+1)\right)^2\,,\quad n\in\mathbb{Z}^+\,.$$

In particular,

$$2T(1,2,1) = \zeta(2)^2\,.$$

Dividing the identity of Corollary 2.5 by $\mu^r$, and summing over $\mu$, we obtain

**THEOREM 3.3.** *For $m \in \mathbb{Z}^+$, $r-2 \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$ holds*

$$(-1)^m \sum_{p=0}^{n} \binom{m+p-1}{p} T(r-m-1, m+p, n-p+1)$$
$$= -E(n+1, r-1) + \sum_{i=2}^{m} (-1)^i \binom{i+n-1}{n} \zeta(n+i)\zeta(r-i)\,.$$

**COROLLARY 3.4.** *For $m, n \in \mathbb{Z}^+$ holds*

$$(-1)^m 2 \sum_{p=0}^{n} \binom{m+p-1}{p} T(n-m+1, m+p, n-p+1)$$
$$= -\zeta(n+1)^2 - \zeta(2n+2) + 2\sum_{i=2}^{m} (-1)^i \binom{i+n-1}{n} \zeta(n+i)\zeta(r-i)\,.$$

In particular,

$$2\sum_{p=1}^{n} T(n-1, n-p+1, p) = \zeta(n)^2 + \zeta(2n)\,.$$

Dividing through the identity of Corollary 2.7 by $\mu^r$ and summing over $\mu$ gives

**THEOREM 3.5.**

$$\sum_{p=0}^{n} \binom{m+p}{m} T(r-m-1, n-p+1, m+p+1)$$
$$= \sum_{i=n}^{m+n} \binom{i}{i-n} E(i+1, r-i+n) + \sum_{i=n+1}^{m+n} \binom{i}{i-n} \zeta(i+1)\zeta(r-i+n)\,.$$

# 4 Evaluation of Tornheim double series

## 4.1 Euler-Zagier double zeta function

Before discussing the general Tornheim double series for finite $r$, $s$ and $t$, we first consider the double series $T(r,0,t)$ and $T(0,s,t)$, with $r+t>2$ or $s+t>2$ and $t>0$.

According to (1.6), when $r$ and $t$ are positive integers greater than unity, then

$$T(r,0,t) = \zeta(r)\zeta(t) - E(t,r)\,. \tag{4.1}$$

Thus, reduction of $T(r,0,t)$ to $\zeta$ values is possible when $r$ and $t$ are of different parity, in view of Theorem 3.1 of [1], and also when $r=t$ or $r+t=6$.

Using, in (4.1), the symmetry property of linear Euler sums,

$$E(m,n) + E(n,m) = \zeta(m+n) + \zeta(m)\zeta(n)\,, \quad [1,\ 6]\,,$$

we have

**THEOREM 4.1.**

$$T(0,s,t) + T(0,t,s) = \zeta(s)\zeta(t) - \zeta(s+t)\,, \quad s-1,\, t-1 \in \mathbb{Z}^+\,.$$

**COROLLARY 4.2.**

$$2\,T(0,s,s) = \zeta(s)^2 - \zeta(2s)\,, \quad s-1 \in \mathbb{Z}^+\,.$$

$T(0,0,t)$ is evaluated as

**THEOREM 4.3.**

$$T(0,0,t) = \zeta(t-1) - \zeta(t)\,, \quad t>2\,,$$

which was derived in [2]. Here we give a different derivation as follows.

*Proof.* From (1.3), we have

$$\begin{aligned}
\sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\frac{1}{(\mu+\nu)^t} &= \sum_{\mu=1}^{N}\left(H_{N+\mu,t}-H_{\mu,t}\right)\\
&= \sum_{\mu=1}^{N}H_{N+\mu,t}-\sum_{\mu=1}^{N}H_{\mu,t}\\
&= \sum_{\mu=N+1}^{2N}H_{\mu,t}-\sum_{\mu=1}^{N}H_{\mu,t}\\
&= \sum_{\mu=1}^{2N}H_{\mu,t}-2\sum_{\mu=1}^{N}H_{\mu,t}\\
&= 2NH_{2N,t}+H_{2N,t}-H_{2N,t-1}\\
&\qquad -2NH_{N,t}-2H_{N,t}+2H_{N,t-1}\,,
\end{aligned}$$

and the result follows on taking limit $N\to\infty$. □

Note that in the final step of the above proof, we used

$$\sum_{r=1}^{N}H_{r,n}=(N+1)H_{N,n}-NH_{N,n-1}\,,\quad \text{(identity 3.1 of [6])}\,.$$

## 4.2 The general Tornheim double series

Consider the following identity (equation (2.4) of [3], slightly rewritten)

$$\begin{aligned}
\frac{1}{\mu^r\nu^s(\mu+\nu)^t} &= \sum_{i=0}^{s-2}\binom{t+i-1}{t-1}\frac{(-1)^i}{\nu^{s-i}\mu^{t+r+i}}+\sum_{i=0}^{t-2}\binom{s+i-1}{s-1}\frac{(-1)^s}{\nu^{s+r+i}(\mu+\nu)^{t-i}}\\
&\quad +(-1)^{s-1}\binom{s+t-2}{s-1}\frac{1}{\mu^{r+s+t-1}}\frac{\mu}{\nu(\mu+\nu)}\,,
\end{aligned}$$

which is valid for $r\in\mathbb{N}_0$ and $s,t\in\mathbb{Z}^+$. Summing over $\mu$ and $\nu$ and taking into cognizance identities (1.4) and (2.4), we obtain

**THEOREM 4.4.** *For $r \in \mathbb{N}_0$, $s, t \in \mathbb{Z}^+$, $r+s>1$ and $r+t>1$ holds*

$$\begin{aligned} T(r,s,t) &= \sum_{i=0}^{s-2} (-1)^i \binom{t+i-1}{t-1} \zeta(s-i)\zeta(r+t+i) \\ &+ (-1)^s \sum_{i=0}^{t-2} \binom{s+i-1}{s-1} \zeta(t-i)\zeta(r+s+i) \\ &- (-1)^s \sum_{i=0}^{t-2} \left\{ \binom{s+i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,t-i}}{\mu^{r+s+i}} \right\} \\ &- (-1)^s \binom{s+t-2}{t-1} \sum_{\mu=1}^{\infty} \frac{H_\mu}{\mu^{r+s+t-1}} \,. \end{aligned}$$

**COROLLARY 4.5.**

$$\begin{aligned} T(r,s,1) &= \frac{(-1)^{s-1}}{2} \left[ (r+s+2)\zeta(r+s+1) - \sum_{i=1-r}^{s-2} \zeta(s-i)\zeta(r+i+1) \right] \\ &+ \sum_{i=0}^{s-2} (-1)^i \zeta(s-i)\zeta(r+i+1) \,. \end{aligned}$$

*In particular, we have the beautiful and well-known result*

$$T(1,1,1) = \sum_{\mu=1}^{\infty} \frac{H_\mu}{\mu^2} = 2\zeta(3) \,.$$

We see immediately from Theorem 4.4 that due to the presence of the Euler sum $E(t-i, i+r+s)$, of weight $w = r+s+t$, complete reduction of $T(r,s,t)$ to $\zeta$ values is achieved, in general, if $w$ is a positive odd integer or if $t = 1$.

Using the index shift identity (1.9), the identity of Theorem 4.4 can also be written as

$$
\begin{aligned}
T(r,s,t) = (-1)^t \sum_{i=t-s+2}^{t} (-1)^i \binom{2t-i-1}{t-1} \zeta(s-t+i)\zeta(r+2t-i) \\
+ (-1)^s \sum_{i=2}^{t} \binom{s+t-i-1}{s-1} \zeta(i)\zeta(r+s+t-i) \\
- (-1)^s \sum_{i=1}^{t} \binom{s+t-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{r+s+t-i}} ,
\end{aligned}
\tag{4.2}
$$

giving, in particular, for $s \in \mathbb{Z}^+$ and $r+s>1$,

$$
\begin{aligned}
(-1)^{s-1} T(r,s,s) &= -\sum_{i=2}^{s} \binom{2s-i-1}{s-1} \left((-1)^i + 1\right) \zeta(i)\zeta(r+2s-i) \\
&\quad + \sum_{i=1}^{s} \binom{2s-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{r+2s-i}} \\
&= -2 \sum_{i=1}^{\lfloor s/2 \rfloor} \binom{2s-2i-1}{s-1} \zeta(2i)\zeta(r+2s-2i) \\
&\quad + \sum_{i=1}^{s} \binom{2s-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{r+2s-i}} .
\end{aligned}
\tag{4.3}
$$

# 5 On linear Euler sums

Using the reflection symmetry (1.2a) of the Tornheim double series in the identity of Theorem 4.4 and setting $r = s+1$ to ensure that $r$ and $s$ have different parity, we obtain

$$\sum_{i=1}^{t-1}\left[\binom{i+s-1}{s}\frac{2s+i-1}{i+s-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu,t-i+1}}{\mu^{2s+i}}\right]+\frac{2s+t-1}{s+t-1}\binom{s+t-1}{t-1}\sum_{\mu=1}^{\infty}\frac{H_\mu}{\mu^{2s+t}}$$
$$=\sum_{i=1}^{t-1}\binom{i+s-1}{s}\frac{2s+i-1}{i+s-1}\zeta(t-i+1)\zeta(2s+i)$$
$$+(-1)^{s+1}\sum_{i=0}^{s-1}(-1)^i\binom{i+t-1}{t-1}\frac{2i+t-1}{i+t-1}\zeta(s-i+1)\zeta(s+i+t)\,. \tag{5.1}$$

## 5.1 Variants of Euler formula for $E(1,2s+1)$

On setting $t=1$ in (5.1), we obtain

**THEOREM 5.1.** *For $s\in\mathbb{Z}^+$holds*

$$2(-1)^{s+1}\sum_{\mu=1}^{\infty}\frac{H_\mu}{\mu^{2s+1}}=\zeta(s+1)^2+2\sum_{i=1}^{s-1}(-1)^i\zeta(s-i+1)\zeta(s+i+1)\,,$$

*or equivalently, using the index shift identity,*

$$2\sum_{\mu=1}^{\infty}\frac{H_\mu}{\mu^{2s+1}}=(-1)^{s-1}\zeta(s+1)^2+2\sum_{i=0}^{s-2}(-1)^i\zeta(i+2)\zeta(2s-i)\,. \tag{5.2}$$

Dividing through (2.3) by $\mu^{m+1}$ and summing over $\mu$ gives

$$(-1)^{m-1}\sum_{\nu=1}^{\infty}\left\{\frac{1}{\nu^m}\sum_{\mu=1}^{\infty}\frac{H_\mu}{\mu(\mu+\nu)}\right\}=\sum_{\mu=1}^{\infty}\frac{H_\mu}{\mu^{m+1}}+\sum_{i=1}^{m-1}(-1)^i\zeta(i+1)\zeta(m-i+1)\,,$$

that is

$$(-1)^{m-1}\sum_{\nu=1}^{\infty}\frac{H_\nu}{\nu^{m+1}}=\sum_{\nu=1}^{\infty}\frac{H_\nu}{\nu^{m+1}}+\sum_{i=1}^{m-1}(-1)^i\zeta(i+1)\zeta(m-i+1)\,,$$

from which, by setting $m=2s$ and shifting the summation index in the second sum of the right hand side, we obtain

**THEOREM 5.2.** *For $s \in \mathbb{Z}^+$ holds*

$$2\sum_{\nu=1}^{\infty}\frac{H_\nu}{\nu^{2s+1}} = \sum_{i=0}^{2s-2}(-1)^i\zeta(i+2)\zeta(2s-i)\,, \tag{5.3}$$

*a result that was derived first in [7] and later in [10].*

Note that the identities (5.2) and (5.3) are equivalent since

$$\begin{aligned}(-1)^{s-1}\zeta(s+1)^2 &+ 2\sum_{i=0}^{s-2}(-1)^i\zeta(i+2)\zeta(2s-i)\\ &= \sum_{i=0}^{s-2}(-1)^i\zeta(i+2)\zeta(2s-i)\\ &\quad+\sum_{i=0}^{s-1}(-1)^i\zeta(i+2)\zeta(2s-i)\end{aligned}$$

which is equivalent to the sum on right side of (5.3).

## 5.2 Certain combinations of linear Euler sums that evaluate to zeta values

Researchers have noted that linear Euler sums of even weight are probably not reducible to zeta values alone [7, 4, 1]. In a 1998 paper [1], Flajolet and Salvy gave a couple of examples of linear combinations of Euler sums of even weight, expressed in terms of the Riemann zeta function. Such evaluations are also found in [5, 11, 12]. In this section we discover certain combinations of linear Euler sums that evaluate to zeta values. The Flajolet and Salvy relations and those of the other authors are particular cases of one of the formulas derived here, Theorem 5.8 to be precise.

Tornheim proved that (equation (8) of [2])

$$T(1,1,s) = (s+1)\zeta(s+2) - \sum_{i=2}^{s}\zeta(i)\zeta(s-i+2), \quad s\in\mathbb{Z}^+\,,$$

which, in view of (4.2), gives

**THEOREM 5.3.** *For $s \in \mathbb{Z}^+$ holds*

$$\sum_{i=1}^{s}\left[\sum_{\mu=1}^{\infty}\frac{H_{\mu,i}}{\mu^{s-i+2}}\right] = \sum_{i=1}^{s} E(i, s-i+2) = (s+1)\zeta(s+2)\,.$$

Setting $r = 0$ in (4.3) and using Corollary 4.2, we obtain

$$\begin{aligned}
&2\sum_{i=1}^{s}\binom{2s-i-1}{s-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu,i}}{\mu^{2s-i}} \\
&\quad = 4\sum_{i=1}^{\lfloor s/2\rfloor}\binom{2s-2i-1}{s-1}\zeta(2i)\zeta(2s-2i) \\
&\qquad + (-1)^{s-1}\left(\zeta(s)^2 - \zeta(2s)\right)\,,
\end{aligned}$$

from which we get

**THEOREM 5.4.** *For $s-1 \in \mathbb{Z}^+$ holds*

$$\begin{aligned}
&2\sum_{i=2}^{s-1}\binom{2s-i-1}{s-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu,i}}{\mu^{2s-i}} \\
&\quad = 4\sum_{i=1}^{\lfloor s/2\rfloor}\binom{2s-2i-1}{s-1}\zeta(2i)\zeta(2s-2i) \\
&\qquad - 2\binom{2s-2}{s-1}\sum_{i=2}^{s}(-1)^i\zeta(i)\zeta(2s-i) \\
&\qquad - (-1)^{s-1}\zeta(s)^2\left[\binom{2(s-1)}{(s-1)} + (-1)^{s-1} - 1\right] \\
&\qquad - (-1)^{s-1}\zeta(2s)\left[(-1)^{s-1} + 1\right]\,,
\end{aligned}$$

*after using Theorem 1.1 to write $E(1, 2s-1)$ and using also the fact that $2E(s,s) = \zeta(s)^2 + \zeta(2s)$.*

**COROLLARY 5.5.**

$$\begin{aligned}
&7\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^8} + 3\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^7} + \sum_{\mu=1}^{\infty}\frac{H_{\mu,4}}{\mu^6} \\
&\qquad = -12\zeta(4)\zeta(6) + 14\zeta(3)\zeta(7) + 7\zeta(5)^2 - \frac{\zeta(10)}{5}\,.
\end{aligned}$$

**COROLLARY 5.6.**

$$126\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^{10}}+56\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^{9}}+21\sum_{\mu=1}^{\infty}\frac{H_{\mu,4}}{\mu^{8}}+6\sum_{\mu=1}^{\infty}\frac{H_{\mu,5}}{\mu^{7}}$$
$$=-210\zeta(4)\zeta(8)-125\zeta(6)^2+252\zeta(3)\zeta(9)+252\zeta(5)\zeta(7)\,.$$

It was proved in [5] that

$$T(s,s,s)=\frac{4}{1+2(-1)^s}\sum_{i=0}^{\lfloor s/2\rfloor}\binom{2s-2i-1}{s-1}\zeta(2i)\zeta(3s-2i)\,. \qquad (5.4)$$

Setting $r=s$ in (4.3) and equating with (5.4) gives

$$\sum_{i=1}^{s}\binom{2s-i-1}{s-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu,i}}{\mu^{3s-i}}$$
$$=\binom{2s-1}{s-1}\frac{2\zeta(3s)}{2+(-1)^s}$$
$$+\frac{2}{2(-1)^s+1}\sum_{i=1}^{\lfloor s/2\rfloor}\binom{2s-2i-1}{s-1}\zeta(2i)\zeta(3s-2i)\,,$$

from which we get

**THEOREM 5.7.** *For $s\in\mathbb{Z}^+$ holds*

$$\sum_{i=2}^{s}\binom{2s-i-1}{s-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu,i}}{\mu^{3s-i}}$$
$$=\binom{2s-1}{s-1}\frac{2\zeta(3s)}{2+(-1)^s}$$
$$-\binom{2s-2}{s-1}\frac{(3s+1)\zeta(3s)}{2}$$
$$+\frac{1}{2}\binom{2s-2}{s-1}\sum_{i=1}^{3s-3}\zeta(3s-i-1)\zeta(i+1)$$
$$+\frac{2}{2(-1)^s+1}\sum_{i=1}^{\lfloor s/2\rfloor}\binom{2s-2i-1}{s-1}\zeta(2i)\zeta(3s-2i)\,.$$

When $t>1$ in (5.1) and we use Theorem 1.1 to write $E(1,2s+t)$ we have

**THEOREM 5.8.** *For $s$, $t-1\in\mathbb{Z}^+$ holds*

$$\sum_{i=1}^{t-1}\left[\binom{s+i-1}{s}\frac{2s+i-1}{s+i-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu,t-i+1}}{\mu^{2s+i}}\right]$$
$$=\sum_{i=1}^{t-1}\binom{s+i-1}{s}\frac{2s+i-1}{s+i-1}\zeta(t-i+1)\zeta(2s+i)$$
$$+(-1)^{s+1}\sum_{i=0}^{s-1}(-1)^i\binom{t+i-1}{t-1}\frac{t+2i-1}{t+i-1}\zeta(s-i+1)\zeta(s+i+t)$$
$$+\frac{1}{2}\frac{(2s+t-1)}{(s+t-1)}\binom{s+t-1}{t-1}\sum_{i=1}^{2s+t-2}\zeta(i+1)\zeta(2s+t-i)$$
$$-\frac{1}{2}\frac{(2s+t-1)(2s+t+2)}{(s+t-1)}\binom{s+t-1}{t-1}\zeta(2s+t+1)\,.$$

**COROLLARY 5.9.**

$$2\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^{2s+1}}+(2s+1)\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^{2s+2}}$$
$$=2\zeta(3)\zeta(2s+1)$$
$$+\frac{(-1)^{s+1}}{2}(s+1)^2\zeta(s+2)^2$$
$$+(-1)^{s+1}\sum_{i=1}^{s-1}(-1)^i(s-i+1)(s+i+1)\zeta(s-i+2)\zeta(s+i+2)\,.$$

*In particular (see also [1], page 23),*

$$2\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^{5}}+5\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^{6}}=-\frac{9}{2}\zeta(4)^2+10\zeta(3)\zeta(5)$$

*and*

$$2\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^{7}}+7\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^{8}}=-15\zeta(4)\zeta(6)+14\zeta(3)\zeta(7)+8\zeta(5)^2\,.$$

# 6 Conclusion

Salient aspects of the relationship between generalized harmonic numbers, Euler sums and Tornheim series were discussed in this paper. In particular, we derived certain combinations of linear Euler sums that evaluate to zeta values.

# References


[1] P. Flajolet and B. Salvy, Euler Sums and Contour Integral Representations, *Experimental Mathematics*, 7 (1):15–35, 1998.

[2] L. Tornheim, Harmonic double series, *American Journal of Mathematics*, 72:303–314, 1950.

[3] M. V. Subbarao and R. Sitaramachandrarao, On some infinite series of L. J. Mordell and their analogues, *Pacific Journal of Mathematics*, 119 (1):245–255, 1985.

[4] D. Borwein, J. M. Borwein and R. Girgensohn, Explicit evaluation of Euler sums, *Proceedings of the Edingburgh Mathematical Society*, 38 (2):277–294, 1995.

[5] J. G. Huard, K. S. Williams and Z. Nan-Yue, On Tornheim's double series, *Acta Arithmetica*, LXXV.2:105–117, 1996.

[6] K. Adegoke and O. Layeni, Finite and Infinite summation identities involving the generalized harmonic numbers, *Preprint available at http://arxiv.org/abs/1508.07345*, 2015.

[7] C. Georghiou and A. N. Philippou, Harmonic sums and the zeta function, *Fibonacci Quarterly*, (1):29–36, 1983.

[8] A. Basu and T. M. Apostol, A new method for investigating Euler sums, *The Ramanujan Journal*, 4:397–419, 2000.

[9] V. S. Adamchik and H. M. Srivastava, Some series of the $\zeta$ and related functions, *Analysis*, 31:131–144, 1998.

[10] K. Boyadzhiev, Evaluation of Euler-Zagier sums, *International Journal of Mathematics and Mathematical Sciences*, 27:407–412, 2001.

[11] O. Espinosa and V. H. Moll, The evaluation of Tornheim double sums, Part 1, *Journal of Number Theory*, 116:200 – 229, 2006.

[12] O. Espinosa and V. H. Moll, The evaluation of Tornheim double sums, Part 2, *Preprint available at* ***http://arxiv.org/pdf/0811.0557.pdf***, 2008.