On generalized harmonic numbers, Tornheim double series and linear Euler sums *

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Abstract

In this paper, direct links between generalized harmonic numbers, linear Euler sums and Tornheim double series are established in a more perspicuous manner than is found in existing literature. The high point of the paper is the discovery of certain combinations of Euler sums that are reducible to Riemann zeta values.

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1   Introduction

1.1   Generalized harmonic numbers and linear Euler sums

Generalized harmonic numbers have a long history, having been studied since the time of Euler. The rth generalized harmonic number of order n, denoted by $H_{r,n}$ in this paper, is defined by

$$H_{r,n} = \sum_{s=1}^{r} \frac{1}{s^n} \,,$$

where $H_{r,1} = H_r$ is the rth harmonic number and $H_{0,n} = 0$. The generalized harmonic number converges to the Riemann zeta function, $\zeta(n)$:

$$\lim_{r \to \infty} H_{r,n} = \zeta(n), \quad \mathcal{R}[n] > 1 \,, \tag{1.1}$$

since $\zeta(n) = \sum_{s=1}^{\infty} s^{-n}$.

Of particular interest in the study of harmonic numbers is the evaluation of infinite series involving the generalized harmonic numbers, especially linear Euler sums of the type

$$E(m,n) = \sum_{\nu=1}^{\infty} \frac{H_{\nu,m}}{\nu^n} \,.$$

The linear sums can be evaluated in terms of zeta values in the following cases: $m = 1$, $m = n$, $m + n$ odd and $m + n = 6$ (with $n > 1$), (see [1]).

Evaluation of Euler sums, E(m,n) of odd weight, $m+n$ in terms of ζ values can be accomplished through Theorem 3.1 of [1].

As for the case $m = 1$, we have

THEOREM 1.1 (Euler). *For $n - 1 \in \mathbb{Z}^+$ holds*

$$2E(1, n) = 2\sum_{\nu=1}^{\infty} \frac{H_\nu}{\nu^n} = (n + 2)\zeta(n + 1) - \sum_{j=1}^{n-2} \zeta(j + 1)\zeta(n - j).$$

1.2 Tornheim double series and relation to linear Euler sums

Tornheim double series, T(r,s,t), is defined by

$$T(r, s, t) = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{\mu^r \nu^s (\mu + \nu)^t}$$

and named after Leonard Tornheim who made a systematic and extended study of the series in a 1950 paper, [2]. $T(r, s, t)$ has the following basic properties [5]:

$$T(r, s, t) = T(s, r, t), \tag{1.2a}$$

$T(r, s, t)$ is finite if and only if $r + t > 1$, $s + t > 1$ and $r + s + t > 2$, (1.2b)

$$T(r, s, 0) = \zeta(r)\zeta(s), \tag{1.2c}$$

$$T(r, 0, t) + T(t, 0, r) = \zeta(r)\zeta(t) - \zeta(r + t), \quad r \geq 2 \tag{1.2d}$$

and

$$T(r, s - 1, t + 1) + T(r - 1, s, t + 1) = T(r, s, t), \quad r \geq 1, s \geq 1. \tag{1.2e}$$

In light of (1.1), the useful identity

$$\sum_{\nu=1}^{N} \frac{1}{(\mu + \nu)^t} = H_{N+\mu,t} - H_{\mu,t}, \tag{1.3}$$

leads to

$$\sum_{\nu=1}^{\infty} \frac{1}{(\mu + \nu)^t} = \zeta(t) - H_{\mu,t}, \tag{1.4}$$

which establishes the link between the Hurwitz zeta function, $\zeta(t, \mu)$, the Riemann zeta function and the generalized harmonic numbers (see also equation (1.19) of [9]) as

$$\zeta(t, \mu) = \zeta(t) - H_{\mu-1,t}, \tag{1.5}$$

since

$$\zeta(t,\mu) = \sum_{\nu=0}^{\infty} \frac{1}{(\mu+\nu)^t}\,.$$

The identity (1.4) also brings out the direct connection between the linear Euler sums and the Tornheim double series, namely,

$$E(n,m) = \zeta(n)\zeta(m) - T(m,0,n), \quad n > 1, m > 1\,. \tag{1.6}$$

Differentiating the identity

$$\frac{1}{\nu} - \frac{1}{\mu+\nu} = \frac{\mu}{\nu(\mu+\nu)}\,, \tag{1.7}$$

$n-1$ times with respect to ν gives

$$\frac{1}{\nu^n} - \frac{1}{(\mu+\nu)^n} = \sum_{p=0}^{n-1} \frac{\mu}{\nu^{p+1}(\mu+\nu)^{n-p}}\,, \quad n \in \mathbb{N}_0\,,$$

from which, by summing over ν, employing (1.3), we obtain

$$H_{N,n} - H_{N+\mu,n} + H_{\mu,n} = \sum_{p=0}^{n-1} \sum_{\nu=1}^{N} \frac{\mu}{\nu^{p+1}(\mu+\nu)^{n-p}}\,,$$

and hence, in the limit $N \to \infty$ we have (see also [7] and [8] for alternative derivations of the particular case of $n = 1$)

$$H_{\mu,n} = \sum_{p=0}^{n-1} \sum_{\nu=1}^{\infty} \frac{\mu}{\nu^{p+1}(\mu+\nu)^{n-p}}\,, \quad n \in \mathbb{Z}^+\,,$$

which, for $n, r - 1 \in \mathbb{Z}^+$, gives the interesting relation

$$\sum_{\mu=1}^{\infty} \frac{H_{\mu,n}}{\mu^r} = \sum_{p=0}^{n-1} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{\mu^{r-1}\nu^{p+1}(\mu+\nu)^{n-p}}\,,$$

that is,

$$E(n,r) = \sum_{p=0}^{n-1} T(r-1, p+1, n-p)\,,$$

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or equivalently,

$$E(n,r) = \sum_{p=1}^{n} T(r-1, n-p+1, p)\,,$$

which shows that any linear Euler sum can be decomposed into a linear combination of Tornheim double series. In particular,

$$E(1,r) = T(r-1,1,1), \quad r > 1\,.$$

Throughout this paper we shall make frequent tacit use of the following index shift identity:

$$\sum_{i=u-b}^{u-a} f(u-i) \equiv \sum_{i=a}^{b} f(i)\,. \tag{1.8}$$

2 Generalized harmonic numbers and summation of series

In this section we discuss the evaluation of certain sums in terms of the Riemann zeta function and the generalized harmonic numbers.

LEMMA 2.1. *Let a, c and f be arbitrary functions such that $a \neq 0$, $c \neq 0$ and $af = c + a$, then for $m \in \mathbb{Z}^{+}$ holds*

$$af^{m} = af + \sum_{i=1}^{m-1} cf^{i}\,,$$

or, equivalently, using the index shift identity,

$$af^{m} = af + \sum_{i=0}^{m-2} cf^{m-i-1}\,.$$

The Lemma is easily proved by the application of mathematical induction on m.

Choosing $a = -1/(\mu+\nu)$, $c = 1/\nu$ and $f = -\mu/\nu$ in Lemma 2.1 gives the partial fraction decomposition

$$(-1)^{m-1}\frac{\mu^{m}}{\nu^{m}(\mu+\nu)} = \frac{\mu}{\nu(\mu+\nu)} + \sum_{i=1}^{m-1}(-1)^{i}\frac{\mu^{i}}{\nu^{i+1}}\,, \tag{2.1}$$

which, after n times differentiation with respect to μ, yields, for $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$, the identity

$$(-1)^{m-1} \sum_{p=0}^{n} \left\{ (-1)^p \binom{m}{p} \frac{\mu^{m-p}}{\nu^m (\mu+\nu)^{n-p+1}} \right\}$$

$$\tag{2.2}$$

$$= -\frac{1}{(\mu+\nu)^{n+1}} + (-1)^n \sum_{i=n}^{m-1} \left\{ (-1)^i \binom{i}{n} \frac{\mu^{i-n}}{\nu^{i+1}} \right\},$$

from which upon summing from $\nu = 1$ to $\nu = N$ we have

THEOREM 2.2. *For $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$, $\mu \in \mathbb{C} \backslash \mathbb{Z}^-$ holds*

$$(-1)^{m-1} \sum_{p=0}^{n} \left\{ (-1)^p \binom{m}{p} \sum_{\nu=1}^{N} \frac{\mu^{m-p}}{\nu^m (\mu+\nu)^{n-p+1}} \right\}$$

$$= H_{\mu,n+1} - H_{N+\mu,n+1} + (-1)^n \sum_{i=n}^{m-1} \left\{ (-1)^i \binom{i}{n} \mu^{i-n} H_{N,i+1} \right\},$$

which in the limit $N \to \infty$ gives

COROLLARY 2.3. *For $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$, $\mu \in \mathbb{C} \backslash \mathbb{Z}^-$ holds*

$$(-1)^{m-1} \sum_{p=0}^{n} \left\{ (-1)^p \binom{m}{p} \sum_{\nu=1}^{\infty} \frac{\mu^{m-p}}{\nu^m (\mu+\nu)^{n-p+1}} \right\}$$

$$= H_{\mu,n+1} + (-1)^n \sum_{i=n+1}^{m-1} \left\{ (-1)^i \binom{i}{n} \mu^{i-n} \zeta(i+1) \right\}.$$

In particular, for $m, n \in \mathbb{Z}^+$ and $\mu \in \mathbb{C} \backslash \mathbb{Z}^-$, we have

$$(-1)^{m-1} \sum_{\nu=1}^{\infty} \frac{\mu^m}{\nu^m (\mu+\nu)} = H_\mu + \sum_{i=1}^{m-1} (-1)^i \mu^i \zeta(i+1), \tag{2.3}$$

$$\sum_{p=0}^{n} \left\{ (-1)^p \binom{n}{p} \sum_{\nu=1}^{\infty} \frac{\mu^p}{\nu^n (\mu+\nu)^{p+1}} \right\} = \zeta(n+1) - H_{\mu,n+1}$$

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and

$$\sum_{p=1}^{n}\left\{(-1)^{p-1}\binom{n}{p}\sum_{\nu=1}^{\infty}\frac{\mu^p}{\nu^n(\mu+\nu)^p}\right\}=H_{\mu,n}\,.$$

$m=1$ in (2.3) gives the beautiful formula

$$\sum_{\nu=1}^{\infty}\frac{\mu}{\nu(\mu+\nu)}=H_\mu\,,\quad \mu\in\mathbb{C}\backslash\mathbb{Z}^-\,, \tag{2.4}$$

which was also derived in [7] and [8].

Differentiating 2.1 n times with respect to ν, we obtain, for $m\in\mathbb{Z}^+$ and $n\in\mathbb{N}_0$, the identity

$$(-1)^m\sum_{p=0}^{n}\left\{\binom{m+p-1}{p}\frac{\mu^{m+1}}{\nu^{m+p}(\mu+\nu)^{n-p+1}}\right\}$$

$$\tag{2.5}$$

$$=\frac{\mu}{(\mu+\nu)^{n+1}}+\sum_{i=1}^{m}\left\{(-1)^i\binom{i+n-1}{n}\frac{\mu^i}{\nu^{i+n}}\right\}\,,$$

from which upon summing from $\nu=1$ to $\nu=N$ we have

THEOREM 2.4. *For $m\in\mathbb{Z}^+$, $\mu\in\mathbb{C}\backslash\mathbb{Z}^-$ and $n\in\mathbb{N}_0$ holds*

$$(-1)^m\sum_{p=0}^{n}\left\{\binom{m+p-1}{p}\sum_{\nu=1}^{N}\frac{\mu^{m+1}}{\nu^{m+p}(\mu+\nu)^{n-p+1}}\right\}$$

$$=\mu H_{N+\mu,n+1}-\mu H_{N,n+1}$$

$$-\mu H_{\mu,n+1}$$

$$+\sum_{i=2}^{m}(-1)^i\binom{i+n-1}{n}\mu^i H_{N,i+n}\,,$$

which in the limit $N\to\infty$ gives

COROLLARY 2.5. *For $m\in\mathbb{Z}^+$, $\mu\in\mathbb{C}\backslash\mathbb{Z}^-$ and $n\in\mathbb{N}_0$ holds*

$$(-1)^m\sum_{p=0}^{n}\left\{\binom{m+p-1}{p}\sum_{\nu=1}^{\infty}\frac{\mu^{m+1}}{\nu^{m+p}(\mu+\nu)^{n-p+1}}\right\}$$

$$=-\mu H_{\mu,n+1}+\sum_{i=2}^{m}(-1)^i\binom{i+n-1}{n}\mu^i\zeta(n+i)\,.$$

Using $a = 1/\mu\nu$, $c = -1/\left(\mu(\mu+\nu)\right)$ and $f = \mu/(\mu+\nu)$ in Lemma 2.1 gives the identity

$$\frac{\mu^m}{\nu(\mu+\nu)^m} = \frac{1}{\nu} - \sum_{i=1}^{m} \frac{\mu^{i-1}}{(\mu+\nu)^i},$$

which, after n differentiations with respect to ν, gives

$$\sum_{p=0}^{n} \binom{m+n-p-1}{m-1} \frac{\mu^m}{\nu^{p+1}(\mu+\nu)^{m+n-p}}$$

$$= \frac{1}{\nu^{n+1}} - \sum_{i=1}^{m} \binom{i+n-1}{i-1} \frac{\mu^{i-1}}{(\mu+\nu)^{i+n}},$$

from which we get, after summing over ν,

THEOREM 2.6. *For $m \in \mathbb{Z}^+$, $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$ and $n \in \mathbb{N}_0$ holds*

$$\sum_{p=0}^{n} \left\{ \binom{m+n-p-1}{m-1} \sum_{\nu=1}^{N} \frac{\mu^m}{\nu^{p+1}(\mu+\nu)^{m+n-p}} \right\}$$

$$= H_{N,n+1} - \sum_{i=1}^{m} \binom{i+n-1}{i-1} \mu^{i-1} H_{N+\mu,i+n} + \sum_{i=1}^{m} \binom{i+n-1}{i-1} \mu^{i-1} H_{\mu,i+n},$$

which in the limit $N \to \infty$ gives

COROLLARY 2.7. *For $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$ and $m, n \in \mathbb{N}_0$ holds*

$$\sum_{p=0}^{n} \left\{ \binom{m+p}{m} \sum_{\nu=1}^{\infty} \frac{\mu^{m+1}}{\nu^{n-p+1}(\mu+\nu)^{m+p+1}} \right\}$$

$$= \sum_{i=n}^{m+n} \binom{i}{i-n} \mu^{i-n} H_{\mu,i+1} - \sum_{i=n+1}^{m+n} \binom{i}{i-n} \mu^{i-n} \zeta(i+1).$$

In particular, for $m \in \mathbb{N}_0$ and $\mu \in \mathbb{C}\backslash\mathbb{Z}^-$, we have

$$\sum_{\nu=1}^{\infty} \frac{\mu^{m+1}}{\nu(\mu+\nu)^{m+1}} = H_\mu - \sum_{i=1}^{m} \mu^i \zeta(i+1) + \sum_{i=1}^{m} \mu^i H_{\mu,i+1}.$$

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3 Functional relations for the Tornheim double series

Writing (1.7) as

$$\frac{1}{\mu+\nu} + \frac{\mu}{\nu(\mu+\nu)} = \frac{1}{\nu},$$

dividing through by $\mu^r\nu^{s-1}(\mu+\nu)^t$ and summing over μ and ν gives

$$T(r,s-1,t+1) + T(r-1,s,t+1) = T(r,s,t), \qquad (3.1)$$

which is property (1.2e) of Tornheim series.

Dividing through the identity of Corollary 2.3 by μ^r and summing over μ we obtain

THEOREM 3.1. *For $m, r-1 \in \mathbb{Z}^+$, $n \in \mathbb{N}_0$ and $r > m - n$ holds*

$$(-1)^{m-1}\sum_{p=0}^{n}\left\{(-1)^p\binom{m}{p}T(r-m+p,m,n-p+1)\right\}$$

$$= E(n+1,r) + (-1)^n\sum_{i=n+1}^{m-1}\left\{(-1)^i\binom{i}{n}\zeta(i+1)\zeta(r-i+n)\right\}.$$

Setting $n = 0$ in the identity of Theorem 3.1 and using Theorem 1.1 to write $E(1,r)$ in zeta values we have

COROLLARY 3.2. *For $m, r-1 \in \mathbb{Z}^+$ and $r > m$ holds*

$$(-1)^{m-1}T(r-m,m,1)$$

$$= \frac{1}{2}(r+2)\zeta(r+1) - \frac{1}{2}\sum_{i=1}^{r-2}\zeta(r-i)\zeta(i+1)$$

$$+ \sum_{i=1}^{m-1}(-1)^i\zeta(r-i)\zeta(i+1).$$

Setting $n = 1$ in the identity of Theorem 3.1 and employing the above corollary gives

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COROLLARY 3.3. *For $r - 1 \in \mathbb{Z}^+$, $m \in \mathbb{N}_0$ and $r > m - 1$ holds*

$$(-1)^{m-1}T(r - m, m, 2)$$

$$= E(2, r) + \frac{m}{2}(r + 3)\zeta(r + 2) - \frac{m}{2}\sum_{i=1}^{r-1}\zeta(r - i + 1)\zeta(i + 1)$$

$$+ m\sum_{i=1}^{m-1}(-1)^i\zeta(r - i + 1)\zeta(i + 1) - \sum_{i=2}^{m-1}(-1)^i i\zeta(r - i + 1)\zeta(i + 1)\,.$$

Setting $r = 4$ in Corollary 3.3, noting that $E(2, 4) = \zeta(3)^2 - \zeta(6)/3$ (see [1], page 16), and using $m = 0$, $m = 1$ and $m = 2$, respectively, we have

$$T(4, 0, 2) = \frac{\zeta(6)}{3} + \zeta(4)\zeta(2) - \zeta(3)^2\,,$$

$$T(3, 1, 2) = \frac{\zeta(3)^2}{2} + \frac{19}{6}\zeta(6) - 2\zeta(4)\zeta(2)$$

and

$$T(2, 2, 2) = -\frac{20}{3}\zeta(6) + 4\zeta(4)\zeta(2)\,. \tag{3.2}$$

Similarly, setting $r = 2$ in Corollary 3.3, noting that $E(2, 2) = \zeta(2)^2/2 + \zeta(4)/2$, (see (4.2)), and using $m = 0$, $m = 1$, respectively, we have

$$T(2, 0, 2) = \frac{\zeta(2)^2}{2} - \frac{1}{2}\zeta(4)$$

and

$$T(1, 1, 2) = -\zeta(2)^2 + 3\zeta(4)\,. \tag{3.3}$$

Setting $r = s$ in (3.1) and using the reflection property gives

$$2T(s, s - 1, t + 1) = T(s, s, t)\,. \tag{3.4}$$

In particular, we have

$$2T(s, s - 1, 1) = T(s, s, 0) = \zeta(s)^2\,, \quad s - 1 \in \mathbb{Z}^+$$

and

$$2T(s, s - 1, 3) = T(s, s, 2)\,, \quad s \in \mathbb{Z}^+\,,$$

from which we get

$$2T(1,0,3) = -\zeta(2)^2 + 3\zeta(4)$$

and

$$2T(2,1,3) = -\frac{20}{3}\zeta(6) + 4\zeta(4)\zeta(2)\,,$$

after using (3.3) and (3.2).

Setting $r = n + 1$ in the identity of Theorem 3.1 we obtain

COROLLARY 3.4. *For $m, n \in \mathbb{Z}^+$ and $2n > m - 1$ holds*

$$(-1)^{m-1}2\sum_{p=0}^{n}\left\{(-1)^p\binom{m}{p}T(n-m+p+1,m,n-p+1)\right\}$$

$$= \zeta(n+1)^2 + \zeta(2n+2)$$

$$+ (-1)^n 2\sum_{i=n+1}^{m-1}\left\{(-1)^i\binom{i}{n}\zeta(i+1)\zeta(2n-i+1)\right\}.$$

Setting $m = n + 1$ in Corollary 3.4, substituting n for $n + 1$ and utilizing the index shift identity (1.8) gives, for $n - 1 \in \mathbb{Z}^+$,

$$2\sum_{p=1}^{n}(-1)^{p-1}\binom{n}{p}T(n-p,n,p) = \zeta(n)^2 + \zeta(2n)\,,$$

from which, with the aide of Corollary 4.2 and after some manipulation, we get,

$$\sum_{p=1}^{2n-1}(-1)^{p-1}\binom{2n}{p}T(2n-p,2n,p) = \zeta(2n)^2\,, \quad n \in \mathbb{Z}^+$$

and

$$\sum_{p=1}^{2n}(-1)^{p-1}\binom{2n+1}{p}T(2n-p+1,2n+1,p)$$

$$= \zeta\left(2(2n+1)\right)\,, \quad n \in \mathbb{Z}^+\,.$$

In particular,

$$2T(1,2,1) = \zeta(2)^2\,.$$

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Dividing the identity of Corollary 2.5 by μ^r, and summing over μ, we obtain

THEOREM 3.5. *For $m \in \mathbb{Z}^+$, $r - 2 \in \mathbb{Z}^+$ and $n \in \mathbb{N}_0$ holds*

$$(-1)^m \sum_{p=0}^{n} \binom{m+p-1}{p} T(r-m-1, m+p, n-p+1)$$

$$= -E(n+1, r-1) + \sum_{i=2}^{m} (-1)^i \binom{i+n-1}{n} \zeta(n+i)\zeta(r-i).$$

COROLLARY 3.6. *For $m, n \in \mathbb{Z}^+$ holds*

$$(-1)^m 2 \sum_{p=0}^{n} \binom{m+p-1}{p} T(n-m+1, m+p, n-p+1)$$

$$= -\zeta(n+1)^2 - \zeta(2n+2) + 2 \sum_{i=2}^{m} (-1)^i \binom{i+n-1}{n} \zeta(n+i)\zeta(r-i).$$

In particular,

$$2 \sum_{p=1}^{n} T(n-1, n-p+1, p) = \zeta(n)^2 + \zeta(2n), \quad n-1 \in \mathbb{Z}^+.$$

Dividing through the identity of Corollary 2.7 by μ^r and summing over μ gives

THEOREM 3.7. *For $r - 1 \in \mathbb{Z}^+$, $m, n \in \mathbb{N}_0$ and $r > m+1$ holds*

$$\sum_{p=0}^{n} \binom{m+p}{m} T(r-m-1, n-p+1, m+p+1)$$

$$= \sum_{i=n}^{m+n} \binom{i}{i-n} E(i+1, r-i+n) + \sum_{i=n+1}^{m+n} \binom{i}{i-n} \zeta(i+1)\zeta(r-i+n).$$

4   Evaluation of Tornheim double series

4.1   Euler-Zagier double zeta function

Before discussing the general Tornheim double series for finite r, s and t, we first consider the double series $T(r,0,t)$ and $T(0,s,t)$, with $r+t>2$ or $s+t>2$ and $t>0$.

According to (1.6), when r and t are positive integers greater than unity, then

$$T(r,0,t) = \zeta(r)\zeta(t) - E(t,r)\,. \tag{4.1}$$

Thus, reduction of $T(r,0,t)$ to ζ values is possible when r and t are of different parity, in view of Theorem 3.1 of [1], and also when $r=t$ or $r+t=6$.

Using, in (4.1), the symmetry property of linear Euler sums,

$$E(m,n) + E(n,m) = \zeta(m+n) + \zeta(m)\zeta(n)\,, \quad [1,\ 6]\,, \tag{4.2}$$

we have

THEOREM 4.1.

$$T(0,s,t) + T(0,t,s) = \zeta(s)\zeta(t) - \zeta(s+t)\,, \quad s-1,\, t-1 \in \mathbb{Z}^+\,.$$

COROLLARY 4.2.

$$2\,T(0,s,s) = \zeta(s)^2 - \zeta(2s)\,, \quad s-1 \in \mathbb{Z}^+\,.$$

$T(0,0,t)$ was evaluated, in [2], as

THEOREM 4.3.

$$T(0,0,t) = \zeta(t-1) - \zeta(t)\,, \quad t>2\,.$$

Here we give a different derivation as follows.

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Proof. From (1.3), we have

$$
\sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\frac{1}{(\mu+\nu)^t} = \sum_{\mu=1}^{N}(H_{N+\mu,t}-H_{\mu,t})
$$

$$
= \sum_{\mu=1}^{N}H_{N+\mu,t} - \sum_{\mu=1}^{N}H_{\mu,t}
$$

$$
= \sum_{\mu=N+1}^{2N}H_{\mu,t} - \sum_{\mu=1}^{N}H_{\mu,t}
$$

$$
= \sum_{\mu=1}^{2N}H_{\mu,t} - 2\sum_{\mu=1}^{N}H_{\mu,t}
$$

$$
= 2NH_{2N,t} + H_{2N,t} - H_{2N,t-1}
$$
$$
- 2NH_{N,t} - 2H_{N,t} + 2H_{N,t-1},
$$

and the result follows on taking limit $N \to \infty$. $\qquad\square$

Note that in the final step of the above proof, we used

$$
\sum_{r=1}^{N}H_{r,n} = (N+1)H_{N,n} - NH_{N,n-1}, \quad \text{(identity 3.1 of [6])}.
$$

4.2   Evaluation of the general Tornheim double series

Consider the following identity (equation (2.4) of [3], slightly rewritten)

$$
\frac{1}{\mu^r\nu^s(\mu+\nu)^t} = \sum_{i=0}^{s-2}\binom{t+i-1}{t-1}\frac{(-1)^i}{\nu^{s-i}\mu^{t+r+i}} + \sum_{i=0}^{t-2}\binom{s+i-1}{s-1}\frac{(-1)^s}{\nu^{s+r+i}(\mu+\nu)^{t-i}}
$$
$$
+ (-1)^{s-1}\binom{s+t-2}{s-1}\frac{1}{\mu^{r+s+t-1}}\frac{\mu}{\nu(\mu+\nu)},
$$

which is valid for $r \in \mathbb{N}_0$ and $s,t \in \mathbb{Z}^+$. Summing over μ and ν and taking into cognizance identities (1.4) and (2.4), we obtain

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THEOREM 4.4. *For $r \in \mathbb{N}_0$, $s, t \in \mathbb{Z}^+$, $r+s>1$ and $r+t>1$ holds*

$$T(r,s,t) = \sum_{i=0}^{s-2} (-1)^i \binom{t+i-1}{t-1} \zeta(s-i)\zeta(r+t+i)$$

$$+ (-1)^s \sum_{i=0}^{t-2} \binom{s+i-1}{s-1} \zeta(t-i)\zeta(r+s+i)$$

$$- (-1)^s \sum_{i=0}^{t-2} \left\{ \binom{s+i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,t-i}}{\mu^{r+s+i}} \right\}$$

$$- (-1)^s \binom{s+t-2}{t-1} \sum_{\mu=1}^{\infty} \frac{H_\mu}{\mu^{r+s+t-1}}.$$

COROLLARY 4.5.

$$T(r,s,1) = \frac{(-1)^{s-1}}{2} \left[(r+s+2)\zeta(r+s+1) - \sum_{i=1-r}^{s-2} \zeta(s-i)\zeta(r+i+1) \right]$$

$$+ \sum_{i=0}^{s-2} (-1)^i \zeta(s-i)\zeta(r+i+1).$$

In particular, we have the beautiful and well-known result

$$T(1,1,1) = \sum_{\mu=1}^{\infty} \frac{H_\mu}{\mu^2} = 2\zeta(3).$$

We see immediately from Theorem 4.4 that due to the presence of the Euler sum $E(t-i, i+r+s)$, of weight $w = r+s+t$, complete reduction of $T(r,s,t)$ to ζ values is achieved, in general, if w is a positive odd integer or if $t = 1$.

Using the index shift identity (1.8), the identity of Theorem 4.4 can also

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be written as

$$T(r,s,t) = (-1)^t \sum_{i=t-s+2}^{t} (-1)^i \binom{2t-i-1}{t-1} \zeta(s-t+i)\zeta(r+2t-i)$$

$$+ (-1)^s \sum_{i=2}^{t} \binom{s+t-i-1}{s-1} \zeta(i)\zeta(r+s+t-i) \qquad (4.3)$$

$$- (-1)^s \sum_{i=1}^{t} \binom{s+t-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{r+s+t-i}},$$

giving, in particular, for $s \in \mathbb{Z}^+$ and $r+s > 1$,

$$(-1)^{s-1} T(r,s,s) = -\sum_{i=2}^{s} \binom{2s-i-1}{s-1} \left((-1)^i + 1\right) \zeta(i)\zeta(r+2s-i)$$

$$+ \sum_{i=1}^{s} \binom{2s-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{r+2s-i}}$$

$$(4.4)$$

$$= -2 \sum_{i=1}^{\lfloor s/2 \rfloor} \binom{2s-2i-1}{s-1} \zeta(2i)\zeta(r+2s-2i)$$

$$+ \sum_{i=1}^{s} \binom{2s-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{r+2s-i}}.$$

5   On linear Euler sums

Using the reflection symmetry (1.2a) of the Tornheim double series in the identity of Theorem 4.4 and setting $r = s+1$ to ensure that r and s have

different parity, we obtain

$$
\sum_{i=1}^{t-1}\left[\binom{i+s-1}{s}\frac{2s+i-1}{i+s-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu,t-i+1}}{\mu^{2s+i}}\right]+\frac{2s+t-1}{s+t-1}\binom{s+t-1}{t-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu}}{\mu^{2s+t}}
$$

$$
=\sum_{i=1}^{t-1}\binom{i+s-1}{s}\frac{2s+i-1}{i+s-1}\zeta(t-i+1)\zeta(2s+i)
$$

$$
+(-1)^{s+1}\sum_{i=0}^{s-1}(-1)^{i}\binom{i+t-1}{t-1}\frac{2i+t-1}{i+t-1}\zeta(s-i+1)\zeta(s+i+t)\,.
$$

$$(5.1)$$

5.1 Variants of Euler formula for $E(1, 2s+1)$

On setting $t = 1$ in (5.1), we obtain

THEOREM 5.1. *For $s \in \mathbb{Z}^{+}$ holds*

$$
2(-1)^{s+1}\sum_{\mu=1}^{\infty}\frac{H_{\mu}}{\mu^{2s+1}}=\zeta(s+1)^{2}+2\sum_{i=1}^{s-1}(-1)^{i}\zeta(s-i+1)\zeta(s+i+1)\,,
$$

or equivalently, using the index shift identity,

$$
2\sum_{\mu=1}^{\infty}\frac{H_{\mu}}{\mu^{2s+1}}=(-1)^{s-1}\zeta(s+1)^{2}+2\sum_{i=0}^{s-2}(-1)^{i}\zeta(i+2)\zeta(2s-i)\,. \qquad (5.2)
$$

Dividing through (2.3) by μ^{m+1} and summing over μ gives

$$
(-1)^{m-1}\sum_{\nu=1}^{\infty}\left\{\frac{1}{\nu^{m}}\sum_{\mu=1}^{\infty}\frac{H_{\mu}}{\mu(\mu+\nu)}\right\}=\sum_{\mu=1}^{\infty}\frac{H_{\mu}}{\mu^{m+1}}+\sum_{i=1}^{m-1}(-1)^{i}\zeta(i+1)\zeta(m-i+1)\,,
$$

that is

$$
(-1)^{m-1}\sum_{\nu=1}^{\infty}\frac{H_{\nu}}{\nu^{m+1}}=\sum_{\nu=1}^{\infty}\frac{H_{\nu}}{\nu^{m+1}}+\sum_{i=1}^{m-1}(-1)^{i}\zeta(i+1)\zeta(m-i+1)\,,
$$

from which, by setting $m = 2s$ and shifting the summation index in the second sum of the right hand side, we obtain

THEOREM 5.2. *For $s \in \mathbb{Z}^+$ holds*

$$2 \sum_{\nu=1}^{\infty} \frac{H_\nu}{\nu^{2s+1}} = \sum_{i=0}^{2s-2} (-1)^i \zeta(i+2)\zeta(2s-i)\,, \qquad (5.3)$$

a result that was derived first in [7] and later in [10].

Note that the identities (5.2) and (5.3) are equivalent since

$$(-1)^{s-1}\zeta(s+1)^2 + 2\sum_{i=0}^{s-2} (-1)^i \zeta(i+2)\zeta(2s-i)$$

$$= \sum_{i=0}^{s-2} (-1)^i \zeta(i+2)\zeta(2s-i)$$

$$+ \sum_{i=0}^{s-1} (-1)^i \zeta(i+2)\zeta(2s-i)$$

which is equivalent to the sum on right side of (5.3).

Setting $n = 2s + 1$ in the identity of Theorem 1.1 and eliminating $E(1, 2s + 1)$ between the resulting identity and (5.3), we obtain, after a little manipulation,

THEOREM 5.3 (Euler)**.**

$$(2s+1)\zeta(2s) = 2\sum_{i=1}^{s-1} \zeta(2s-2i)\zeta(2i)\,, \quad s-1 \in \mathbb{Z}^+\,,$$

the well-known zeta function relation due to Euler.

5.2 Certain combinations of linear Euler sums that evaluate to zeta values

Researchers have noted that linear Euler sums of even weight are probably not reducible to zeta values alone [7, 4, 1]. In a 1998 paper [1], Flajolet and Salvy gave a couple of examples of linear combinations of Euler sums of even weight, expressed in terms of the Riemann zeta function. Such evaluations are also found in [5, 11, 12]. In this section we discover certain combinations

of linear Euler sums that evaluate to zeta values. The Flajolet and Salvy relations and those of the other authors are particular cases of one of the formulas derived here, Theorem 5.9 to be precise.

Tornheim proved that (equation (8) of [2])

$$T(1, 1, s) = (s+1)\zeta(s+2) - \sum_{i=2}^{s} \zeta(i)\zeta(s-i+2), \quad s \in \mathbb{Z}^+,$$

which, in view of (4.3), gives

THEOREM 5.4. *For $s \in \mathbb{Z}^+$ holds*

$$\sum_{i=1}^{s} \left[\sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{s-i+2}} \right] = \sum_{i=1}^{s} E(i, s-i+2) = (s+1)\zeta(s+2).$$

Setting $r = 0$ in (4.4) and using Corollary 4.2, we obtain

$$2\sum_{i=1}^{s} \binom{2s-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{2s-i}}$$

$$= 4 \sum_{i=1}^{\lfloor s/2 \rfloor} \binom{2s-2i-1}{s-1} \zeta(2i)\zeta(2s-2i)$$

$$+ (-1)^{s-1} \left(\zeta(s)^2 - \zeta(2s) \right),$$

from which we get

THEOREM 5.5. *For $s - 1 \in \mathbb{Z}^+$ holds*

$$2\sum_{i=2}^{s-1} \binom{2s-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{2s-i}}$$

$$= 4 \sum_{i=1}^{\lfloor s/2 \rfloor} \binom{2s-2i-1}{s-1} \zeta(2i)\zeta(2s-2i)$$

$$- 2\binom{2s-2}{s-1} \sum_{i=2}^{s} (-1)^i \zeta(i)\zeta(2s-i)$$

$$- (-1)^{s-1}\zeta(s)^2 \left[\binom{2(s-1)}{(s-1)} + (-1)^{s-1} - 1 \right]$$

$$- (-1)^{s-1}\zeta(2s) \left[(-1)^{s-1} + 1 \right],$$

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after using Theorem 1.1 to write $E(1, 2s - 1)$ and using also the fact that $2E(s, s) = \zeta(s)^2 + \zeta(2s)$.

COROLLARY 5.6.

$$7 \sum_{\mu=1}^{\infty} \frac{H_{\mu,2}}{\mu^8} + 3 \sum_{\mu=1}^{\infty} \frac{H_{\mu,3}}{\mu^7} + \sum_{\mu=1}^{\infty} \frac{H_{\mu,4}}{\mu^6}$$

$$= -12\zeta(4)\zeta(6) + 14\zeta(3)\zeta(7) + 7\zeta(5)^2 - \frac{\zeta(10)}{5}.$$

COROLLARY 5.7.

$$126 \sum_{\mu=1}^{\infty} \frac{H_{\mu,2}}{\mu^{10}} + 56 \sum_{\mu=1}^{\infty} \frac{H_{\mu,3}}{\mu^9} + 21 \sum_{\mu=1}^{\infty} \frac{H_{\mu,4}}{\mu^8} + 6 \sum_{\mu=1}^{\infty} \frac{H_{\mu,5}}{\mu^7}$$

$$= -210\zeta(4)\zeta(8) - 125\zeta(6)^2 + 252\zeta(3)\zeta(9) + 252\zeta(5)\zeta(7).$$

It was proved in [5] that

$$T(s, s, s) = \frac{4}{1 + 2(-1)^s} \sum_{i=0}^{\lfloor s/2 \rfloor} \binom{2s - 2i - 1}{s - 1} \zeta(2i)\zeta(3s - 2i). \qquad (5.4)$$

Setting $r = s$ in (4.4) and equating with (5.4) gives

$$\sum_{i=1}^{s} \binom{2s - i - 1}{s - 1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{3s-i}}$$

$$= \binom{2s - 1}{s - 1} \frac{2\zeta(3s)}{2 + (-1)^s}$$

$$+ \frac{2}{2(-1)^s + 1} \sum_{i=1}^{\lfloor s/2 \rfloor} \binom{2s - 2i - 1}{s - 1} \zeta(2i)\zeta(3s - 2i),$$

from which we get

THEOREM 5.8. *For $s \in \mathbb{Z}^+$ holds*

$$\sum_{i=2}^{s} \binom{2s-i-1}{s-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{3s-i}}$$

$$= \binom{2s-1}{s-1} \frac{2\zeta(3s)}{2+(-1)^s}$$

$$- \binom{2s-2}{s-1} \frac{(3s+1)\zeta(3s)}{2}$$

$$+ \frac{1}{2}\binom{2s-2}{s-1} \sum_{i=1}^{3s-3} \zeta(3s-i-1)\zeta(i+1)$$

$$+ \frac{2}{2(-1)^s+1} \sum_{i=1}^{\lfloor s/2 \rfloor} \binom{2s-2i-1}{s-1} \zeta(2i)\zeta(3s-2i).$$

When $t > 1$ in (5.1) and we use Theorem 1.1 to write $E(1, 2s+t)$ we have

THEOREM 5.9. *For $s, t-1 \in \mathbb{Z}^+$ holds*

$$\sum_{i=1}^{t-1} \left[\binom{s+i-1}{s} \frac{2s+i-1}{s+i-1} \sum_{\mu=1}^{\infty} \frac{H_{\mu,t-i+1}}{\mu^{2s+i}} \right]$$

$$= \sum_{i=1}^{t-1} \binom{s+i-1}{s} \frac{2s+i-1}{s+i-1} \zeta(t-i+1)\zeta(2s+i)$$

$$+ (-1)^{s+1} \sum_{i=0}^{s-1} (-1)^i \binom{t+i-1}{t-1} \frac{t+2i-1}{t+i-1} \zeta(s-i+1)\zeta(s+i+t)$$

$$+ \frac{1}{2} \frac{(2s+t-1)}{(s+t-1)} \binom{s+t-1}{t-1} \sum_{i=1}^{2s+t-2} \zeta(i+1)\zeta(2s+t-i)$$

$$- \frac{1}{2} \frac{(2s+t-1)(2s+t+2)}{(s+t-1)} \binom{s+t-1}{t-1} \zeta(2s+t+1).$$

$$(5.5)$$

or, equivalently,

$$\sum_{i=2}^{t}\left[\binom{s+t-i}{s}\frac{2s+t-i}{s+t-i}\sum_{\mu=1}^{\infty}\frac{H_{\mu,i}}{\mu^{2s+t-i+1}}\right]$$

$$= (-1)^{t-s}\sum_{i=t-s+2}^{t+1}(-1)^{i}\binom{2t-i}{t-1}\frac{3t-2i+1}{2t-i}\zeta(s-t+i)\zeta(s+2t-i+1)$$

$$+ \sum_{i=2}^{t}\binom{s+t-i}{s}\frac{2s+t-i}{s+t-i}\zeta(i)\zeta(2s+t-i+1)$$

$$+ \frac{1}{2}\frac{(2s+t-1)}{(s+t-1)}\binom{s+t-1}{t-1}\sum_{i=1}^{2s+t-2}\zeta(i+1)\zeta(2s+t-i)$$

$$- \frac{1}{2}\frac{(2s+t-1)(2s+t+2)}{(s+t-1)}\binom{s+t-1}{t-1}\zeta(2s+t+1).$$

$$(5.6)$$

If we set $t = 3$ in identity (5.5) we obtain

COROLLARY 5.10. *For $s \in \mathbb{Z}^{+}$ holds*

$$2\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^{2s+1}} + (2s+1)\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^{2s+2}}$$

$$= 2\zeta(3)\zeta(2s+1)$$

$$+ \frac{(-1)^{s+1}}{2}(s+1)^{2}\zeta(s+2)^{2}$$

$$+ (-1)^{s+1}\sum_{i=1}^{s-1}(-1)^{i}(s-i+1)(s+i+1)\zeta(s-i+2)\zeta(s+i+2).$$

In particular (see also [1], page 23),

$$2\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^{5}} + 5\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^{6}} = -\frac{9}{2}\zeta(4)^{2} + 10\zeta(3)\zeta(5)$$

and

$$2\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^{7}} + 7\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^{8}} = -15\zeta(4)\zeta(6) + 14\zeta(3)\zeta(7) + 8\zeta(5)^{2}. \qquad (5.7)$$

Combining (5.7) and the identity of Corollary 5.6 we get

$$\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^7}+\sum_{\mu=1}^{\infty}\frac{H_{\mu,4}}{\mu^6}=3\zeta(4)\zeta(6)-\zeta(5)^2-\frac{\zeta(10)}{5}.$$

Setting $t=s-1$ in (5.6) yields

COROLLARY 5.11. *For $s-2\in\mathbb{Z}^+$ holds*

$$\sum_{i=2}^{s-1}\left\{\binom{2s-i-1}{s}\frac{3s-i-1}{2s-i-1}\sum_{\mu=1}^{\infty}\frac{H_{\mu,i}}{\mu^{3s-i}}\right\}$$

$$=\sum_{i=2}^{s+1}(-1)^i\binom{2s-i-1}{s-2}\frac{3s-2i}{2s-i-1}\zeta(i)\zeta(3s-i)$$

$$+\sum_{i=2}^{s-1}\binom{2s-i-1}{s}\frac{3s-i-1}{2s-i-1}\zeta(i)\zeta(3s-i)$$

$$+\frac{1}{4}\frac{(3s-2)}{(s-1)}\binom{2s-2}{s-2}\sum_{i=2}^{3s-2}\zeta(i)\zeta(3s-i)$$

$$-\frac{1}{4}\frac{(3s-2)(3s+1)}{(s-1)}\binom{2s-2}{s-2}\zeta(3s).$$

In particular,

$$9\sum_{\mu=1}^{\infty}\frac{H_{\mu,2}}{\mu^{10}}+2\sum_{\mu=1}^{\infty}\frac{H_{\mu,3}}{\mu^9}=50\zeta(2)\zeta(10)+18\zeta(3)\zeta(9)$$

$$+29\zeta(4)\zeta(8)+24\zeta(5)\zeta(7)$$
$$+\frac{25}{2}\zeta(6)^2-\frac{325}{2}\zeta(12).$$

Setting $t=s$ in (5.6) gives

COROLLARY 5.12. *For $s - 1 \in \mathbb{Z}^+$ holds*

$$\sum_{i=2}^{s} \left\{ \binom{2s - i}{s} \frac{3s - i}{2s - i} \sum_{\mu=1}^{\infty} \frac{H_{\mu,i}}{\mu^{3s-i+1}} \right\}$$

$$= \sum_{i=2}^{s+1} (-1)^i \binom{2s - i}{s - 1} \frac{3s - 2i + 1}{2s - i} \zeta(i)\zeta(3s - i + 1)$$

$$+ \sum_{i=2}^{s} \binom{2s - i}{s} \frac{3s - i}{2s - i} \zeta(i)\zeta(3s - i + 1)$$

$$+ \frac{1}{2} \frac{(3s - 1)}{(2s - 1)} \binom{2s - 1}{s - 1} \sum_{i=2}^{3s-1} \zeta(i)\zeta(3s - i + 1)$$

$$- \frac{1}{2} \frac{(3s - 1)(3s + 2)}{(2s - 1)} \binom{2s - 1}{s - 1} \zeta(3s + 1).$$

6 Conclusion

Salient aspects of the relationship between generalized harmonic numbers, Euler sums and Tornheim series were discussed in this paper. In particular, we derived certain combinations of linear Euler sums that evaluate to zeta values.

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