The Right Scalene Triangle Problem

This paper solves the problem of the right scalene triangle through a general sequential solution. A simplified solution is also presented.

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1. The Problem

The right scalene triangle problem is the problem of finding a method of calculating the value of the sides of any right scalene triangle such that the following two conditions are satisfied:

Condition 1
a, b and c are integers

Condition 2
$b = a + 1$.

where a is the smallest cathetus (smallest side), b is the largest cathetus (medium side) and c is the hypotenuse (largest side) of the triangle. In the next section I present the solution to this problem which requires very simple mathematics. It is worthwhile to emphasize that the number of right scalene triangles that satisfy the above conditions is infinite. Consequently, I shall find the first 11 solutions to illustrate the method.

2. The Solution

We start by applying the Pythagoras theorem

$$c = \sqrt{a^2 + b^2} \tag{2.1}$$

According to condition 2 the following relation must be satisfied

$$b = a + 1 \tag{2.2}$$

The combination the above two equations yields

$$c = \sqrt{a^2 + (a+1)^2} \tag{2.3}$$

Which may be written as

$$c = \sqrt{a^2+a^2+2a+1} = \sqrt{2a^2+2a+1} \tag{2.4}$$

The above equation may be written as follows

$$2a^2+2a+1-c^2 = 0 \tag{2.5}$$

Thus we need to find the roots of a quadratic equation. We do this through the quadratic equation formula

$$a = -\frac{B}{2A} \pm \frac{\sqrt{B^2-4ac}}{2A} \tag{2.6}$$

Where the coefficients A, B and C, in this case, are given by

$$A \equiv 2 \tag{2.7 a}$$
$$B \equiv 2 \tag{2.7 b}$$
$$C \equiv 1-c^2 \tag{2.7 c}$$

Substituting the values of A, B and C into equation (2.6), with the second side of equations (2.7 a), (2.7 b) and (2.7 c), respectively, we get

$$a = -\frac{1}{2} \pm \frac{1}{2}\sqrt{2c^2-1} \tag{2.8}$$

Since the smallest side, a, of the triangle must be positive, we write the solution as

$$a = \frac{1}{2}\left(\sqrt{2c^2-1}-1\right) \tag{2.9}$$

This equation give us the value of the smallest side, *a,* of a right scalene triangle as a function of the hypotenuse, c. We shall use this equation later on to calculate the smallest side of the triangle. But first, we have to devise a method of calculating the hypotenuse.

Next, from trial and error we find the smallest two or three triangles that comply with the conditions given in section 1. These triangles are

Triangle 1 $$\sqrt{3^2+4^2} = 5 \tag{2.10}$$

Triangle 2 $$\sqrt{20^2+21^2} = 29 \tag{2.11}$$

Triangle 3 (*) $$\sqrt{119^2+120^2} = 169 \tag{2.12}$$

(*) An important point to make is that, if you wish, you could use this method to calculate the sides of triangle 2 or 3 onwards. However, because the first ratio is 5.8 and not 5.827 586 207 (or 5.828 427 103 which will lead us to the correct value of a in a smaller number of iterations) you will need to start with a relatively small hypotenuse value and then iterate by increasing the value of the hypotenuse by 1 each time until you get an integer value of side a. For example, if we use the ratio $x/29 \approx 5.8$ to get the sides of triangle 3, we would get $x = 168.2$. Then we use formula (2.9) starting with 168. This value, however, will not produce an integer value of a, then we discard the value 168 and increase it by 1. This will give

169. This time formula (2.9) will produce an integer value for the smallest side (a =119). This means that the value of the hypotenuse for triangle 3 is 169.

Next, we calculate the ratio between the hypotenuse of triangle 2 to the hypotenuse of triangle 1. This gives

Hypotenuse of Triangle 2 divided hypotenuse of Triangle 1

$$\frac{29}{5} = 5.8 \tag{2.13}$$

Now, we calculate the same ratio but, this time, between triangles 3 and 2. This yields

Hypotenuse of Triangle 3 divided hypotenuse of Triangle 2

$$\frac{169}{29} = 5.827\ 586\ 207 \tag{2.14}$$

Calculation of Triangle 4 (the fourth smallest triangle)

Based on the above last ratio, we extrapolate this result by predicting the hypotenuse of the fourth smallest triangle (triangle 4) as follows

Hypotenuse of Triangle 4 divided hypotenuse of Triangle 3

$$\frac{x}{169} \approx 5.827\ 586\ 207 \tag{2.15}$$

Hence

Estimation

$$\begin{gathered} x = 5.827\ 586\ 207 \times 169 \\ x \approx 984.862 \end{gathered} \tag{2.16}$$

Now we round off this number to get

This should be the exact value of the hypotenuse of triangle 4

$$x = 985 \tag{2.17}$$

Then, we introduce this value into equation (2.9) to get the value of the smallest side of the right scalene triangle we are looking for

$$a = \frac{1}{2}\left(\sqrt{2 \times 985^2 - 1} - 1\right) = 696 \tag{2.18}$$

Then, because of condition 2, the value of the other cathetus of this triangle must be 697. We may verify this result before continuing

Triangle 4

$$\sqrt{696^2 + 697^2} = 985 \tag{2.19}$$

Therefore, the right scalene triangle whose sides are 696, 697 and 985 satisfies the two conditions given above. This is the fourth solution to the problem.

Calculation of Triangle 5 (the fifth smallest triangle)

To calculate the fifth smallest triangle, we calculate the ratio of the hypotenuse of triangle 4 to the hypotenuse of triangle 3. This ratio is

Hypotenuse of Triangle 4 divided hypotenuse of Triangle 3

$$\frac{985}{169} = 5.827\ 402\ 367 \tag{2.14}$$

Now, based on this ratio, we predict the hypotenuse of the fifth smallest triangle (triangle 5) by assuming that the hypothenuse x of triangle 5 divided by the hypotenuse of triangle 4 has, approximately, the same value as the one given by the last ratio (equation 2.14).

Hypotenuse of Triangle 4 divided hypotenuse of Triangle 3

$$\frac{x}{985} = 5.828\ 402\ 367 \tag{2.20}$$

Hence

Estimation

$$\begin{aligned} x &= 5.828\ 402\ 367 \times 985 \\ x &\approx 5740.976 \end{aligned} \tag{2.21}$$

We round off this number to get an integer value

This should be the exact value of the hypotenuse of triangle 5

$$x = 5741 \tag{2.22}$$

Then, we introduce this value into equation (2.9) to get the value of the smallest side of the right scalene triangle

$$a = \frac{1}{2}\left(\sqrt{2 \times 5741^2 - 1} - 1\right) = 4059 \tag{2.23}$$

Then, according to condition 2, the value of the medium side of this triangle must be 4060. We may verify this result before continuing

Triangle 5

$$\sqrt{4059^2 + 4060^2} = 5741 \tag{2.24}$$

Therefore, the right scalene triangle whose sides are 4059, 4060 and 5741 satisfies the two conditions given above. This is the fifth solution to the problem.

Calculation of Triangle 6 (the sixth smallest triangle)

To calculate the sixth smallest triangle, we calculate the ratio of the hypotenuse of triangle 5 to the hypotenuse of triangle 4. This gives. This ratio is

Hypotenuse of Triangle 5 divided hypotenuse of Triangle 4

$$\frac{5741}{985} = 5.828\ 426\ 396 \tag{2.25}$$

Now, based on this ratio, we predict the hypotenuse of the sixth smallest triangle (triangle 6) by assuming that the hypothenuse x of triangle 6 divided by the hypotenuse of triangle 5 has, approximately, the same value as the one given by the last ratio

Hypotenuse of Triangle 5 divided hypotenuse of Triangle 4

$$\frac{x}{5741} = 5.828\,426\,396 \tag{2.26}$$

Hence

Estimation

$$x = 5.828\,426\,396 \times 5741$$
$$x \approx 33460.99594 \tag{2.27}$$

Now we round off this number to get

This should be the exact value of the hypotenuse of triangle 6

$$x = 33461 \tag{2.28}$$

Then, we introduce this value into equation (2.9) to obtain the value of the smallest side of the sixth right scalene triangle

$$a = \frac{1}{2}\left(\sqrt{2 \times 33461^2 - 1} - 1\right) = 23660 \tag{2.29}$$

Then, according to condition 2, the value of the other cathetus of the triangle must be 23661. We may verify this result before continuing

Triangle 6

$$\sqrt{23660^2 + 23661^2} = 33461 \tag{2.30}$$

Therefore, the right scalene triangle whose sides are 23660, 23661 and 33461 satisfies the two conditions given above.

Calculation of Triangle 7 (the seventh smallest triangle)

We calculate the ratio of the hypotenuse of triangle 6 to the hypotenuse of triangle 5. This gives

Hypotenuse of Triangle 6 divided hypotenuse of Triangle 5

$$\frac{33461}{5741} \approx 5.828\,427\,103 \tag{2.31}$$

Because this ratio converges around the above number $(5.828\,427\,103)$, we realize that the above ratio may be used to compute the hypotenuse from the first triangle onwards in a sequential way. This way we may now compute the hypotenuse for any right scalene triangle (sequentially) so that it complies with the above two conditions. This is the simplified method and its results are given in table 1.

TRIANGLE NUMBER	HYPOTENUSE FOR TRIANGLE n (real number)	HYPOTENUSE FOR TRIANGLE n (Rounded off to the closest integer)
2	$5.828\ 427\ 103 \times 5 \approx 29.142\ 135$	29
3	$5.828\ 427\ 103 \times 29 \approx 169.024\ 386$	169
4	$5.828\ 427\ 103 \times 169 \approx 985.004$	985
5	$5.828\ 427\ 103 \times 985 \approx 5741.000\ 7$	5741
6	$5.828\ 427\ 103 \times 5741 \approx 33461$	33 461
7	$5.828\ 427\ 103 \times 33461 \approx 195024.999$	195 025
8	$5.828\ 427\ 103 \times 195\ 025 \approx 1\ 136\ 688.996$	1 136 689
9	$5.828\ 427 \times 1\ 136\ 689 \approx 6\ 625\ 108.975$	6 625 109
10	$5.828\ 427\ 103 \times 6\ 625\ 109 \approx 38\ 613\ 964.86$	38 613 965

TABLE 1: Simplified method. Calculation of the hypotenuse for nine right scalene triangles.

The dimensions of the first 11 right scalene triangles that comply with the conditions outlined in section 1 are given in table 2.

TRIANGLE NUMBER	SMALLEST CATHETUS	LARGEST CATHETUS	HYPOTENUSE
1	3	4	5
2	20	21	29
3	119	120	169
4	696	697	985
5	4059	4060	5741
6	23660	23661	33461
7	137 903	137 904	195 025
8	803 760	803 761	1 136 689
9	4 684 659	4 684 660	6 625 109
10	27 304 196	27 304 197	38 613 965
11	159 140 519	159 140 520	225 058 681

TABLE 2: Values of each side of the first 11 right scalene triangles that comply with the conditions given in section 1.

3. Conclusions

The method presented in this paper works satisfactorily for calculating any right scalene triangle in sequential order. Because the hypotenuse ratio seems to converge to the number 5.828 427 103, we were able to simplified the method quite a bit. However, if this convergence were to change for larger triangles, then we would need to use the complete method as per calculations of triangles 4, 5 and 6 given in section 2.

4. Nomenclature

a = smallest cathetus (smallest side) of the right scalene triangle
b = largest cathetus (medium side) of the right scalene triangle
c = hypotenuse of the right scalene triangle
x = variable used to denote the hypotenuse of a right triangle
A = coefficient of the quadratic equation
B = coefficient of the quadratic equation
C = coefficient of the quadratic equation

Notes

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