Smarandache – R-Module and BF-Algebras

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Abstract. In this paper we introduced Smarandache – 2 – algebraic structure of R-Module namely Smarandache – R-Module. A Smarandache – 2 – algebraic structure on a set N means a weak algebraic structure A₀ on N such that there exist a proper subset M of N, which is embedded with a stronger algebraic structure A₁, stronger algebraic structure means satisfying more axioms, by proper subset one understands a subset different from the empty set, from the unit element if any, from the whole set. We define Smarandache-R-Module and obtain some of its characterization through S-Algebra and BF Algebras.

Keyword: R-Module, S-algebra, Smarandache – R-Module, BF-Algebras

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1. Introduction

New notions are introduced in algebra to study more about the congruence in number theory by Florentin Smarandache [2]. By a proper subset of a set A, we consider a set P included in A and different from A, different from the empty set, and from the unit element in A – if any they rank the algebraic structures using an order relationship. The algebraic structures S₁ << S₂ if both are defined on the same set :: all S₁ laws are also S₂ laws; all axioms of S₁ law are accomplished by the corresponding S₂ law; S₂ law strictly accomplishes more axioms than S₁ laws, or in other words S₂ laws has more laws than $S₁$.

For example: semi group << monoid << group << ring << field, or Semi group << commutative semi group, ring << unitary ring, etc. they define a General special structure to be a structure SM on a set A, different from a structure SN, such that a proper subset of A is an SN structure, where SM << SN.

Definition 1. Let R be a module, called R-module. If R is said to be smarandache – R – module. Then there exist a proper subset A of R which is an S-algebra with respect to the same induced operations of R.
Definition 2. The B-algebra is an algebra \((A;*,0)\) of type \((2,0)\) (i.e., a nonempty set \(A\) with a binary operation \(*\) and a constant 0) satisfying the following axioms:

(B1) \(x*x = 0\),
(B2) \(x*0 = x\),
(B) \((x*y)*z = x*(z*(0*y))\).

In BH-algebras, which are a generalization of BCK/BCI/B-algebras. An algebra \((A;*,0)\) of type \((2,0)\) is a BH-algebra if it obeys (B1), (B2), and (BH) \(x*y = 0\) and \(y*x = 0\) imply \(x = y\). In a BG-algebra is an algebra \((A;*,0)\) of type \((2,0)\) satisfying (B1), (B2), and (BG) \(x = (x*y)*(0*y)\).

Definition 3. A BF-algebra is an algebra \((A;*,0)\) of type \((2,0)\) satisfying (Bl), (B2), and the following axiom:
(BF) \(0*(x*y) = y*x\).

Theorem 1. Let \(R\) be a smarandache-R-module, if there exists a proper subset \(A\) of \(R\) in which satisfies (B1) to (B6) then the following axioms are true.
(a) \(0*(0*x) = x\) for all \(x \in A\);
(b) if \(0*x = 0*y\), then \(x = y\) for any \(x, y \in A\);
(c) if \(x*y = 0\), then \(y*x = 0\) for any \(x, y \in A\).

Proof. Let \(R\) be a smarandache-R-module. Then by definition there exists a proper subset \(A\) of \(R\) which is an algebra. By hypothesis \(A\) holds for (B1) to (B6) then \(A\) is BF-algebras. Let \(A\) be a BF-algebra and \(x \in A\). By (BF) and (B2) we obtain \(0*(0*x) = x*0 = x\), that is, (a) holds. Also (b) follows from (a). Let now, \(y \in A\) and \(x*y = 0\). Then \(0 - 0*0 = 0*(x*y) - y*x\). This gives (c).

Definition 4. A BF-algebra is called a BF-algebra (resp. a BF2-algebra) if it obeys (BG) (resp. (BH)).

Theorem 2. Let \(R\) be a smarandache-R-module, if there exists a proper subset \(A\) of \(R\) in which satisfies (B1) to (B6). Then the algebra \(A = (A;*,0)\) of type \((2,0)\) is a BF-algebra if and only if it obeys the laws (Bl), (BF), and (BG).

Proof. Let \(R\) be a smarandache-R-module. Then by definition there exists a proper subset \(A\) of \(R\) which is an algebra. By hypothesis \(A\) holds for (B1) to (B6) then \(A\) is BF-algebras. Suppose that (B1), (BF), and (BG) are valid in \(A\). Let \(x \in A\). Substituting \(y = x\), (BG) becomes \(x = (x*x)*(0*x)\). Hence applying (B1) and (BF) we conclude that \(x = 0*0 = 0*(x*y)\). Consequently, (B2) holds. Therefore \(A\) is a BF-algebra. The converse is obvious.

Theorem 3. Let \(R\) be a smarandache-R-module, if there exists a proper subset \(A\) of \(R\) in which satisfies (B1) to (B6). Then \(A\) is a BF2-algebra if and only if \(A\) satisfies (B2), (BF); and the following axiom:
(BH') \(x*y = 0 \iff x = y\).
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**Proof.** Let R be a smarandache-R-module. Then by definition there exists a proper subset A of R which is an algebra. By hypothesis A holds for (B1) to (B6) then A is BF-algebras. Let A be a BF₂-algebra. By definition, (B2) and (BF) are valid in A. Suppose that \( x * y = 0 \) for \( x, y \in A \). *Proof 1(c) yields \( y * x = 0 \). From (BH) we see that \( x = y \). If \( x = y \), then \( x * y = 0 \) by (Bl). Thus (BH') holds in A.

Let now A satisfies (B2), (BF), and (BH'). (BH') implies (Bl) and (BH). Therefore A is BF₃-algebra.

**Theorem 4.** Let R be a smarandache-R-module, if there exists a proper subset A of R in which satisfies (B1) to (B6) then the following statements are equivalent:

(a) A is a BF₂-algebra;
(b) \( x = [x * (0 * y)] * y \) for all \( x, y \in A \);
(c) \( x = y * [(0 * x) * (0 * y)] \) for all \( x, y \in A \).

**Proof.** Let R be a smarandache-R-module. Then by definition there exists a proper subset A of R which is an algebra. By hypothesis A holds for (B1) to (B6) then A is BF-algebras. (a) \( \Rightarrow \) (b): Let A be a BF₁-algebra and \( x, y \in A \). To obtain (b), substitute \( 0 * y \) for \( y \) in (BG) and then use Theorem 1(a).

(b) \( \Rightarrow \) (c): We conclude from (b) that \( 0 * x = [(0 * x) * (0 * y)] * y \). Hence \( 0 * (0 * x) = y * [(0 * x) * (0 * y)] \) by (BF). But \( 0 * (0 * x) = x \), and we have (c).

(c) \( \Rightarrow \) (a): Let (c) hold. (BF) clearly forces

\[ 0 * x = [(0 * x) * (0 * y)] * y. \]

Using (1) with \( x = 0 * a \) and \( y = 0 * b \) we have \( 0 * (0 * a) = [(0 * (0 * a)) * (0 * (0 * b))] * (0 * b) \).

Hence applying Theorem 1(a), we deduce that \( a = (a * b) * (0 * b) \). Consequently, A is a BF₃-algebra.

**Theorem 5.** Let R be a smarandache-R-module, if there exists a proper subset A of R in which satisfies (B1) to (B6) then the following statements are true:

(a) A is a BG-algebra;
(b) For \( x, y \in A \), \( x * y = 0 \) implies \( x = y \);
(c) The right cancellation law holds in A. i.e., If \( x * y = z * y \), then \( x = z \) for any \( x, y, z \in A \);
(d) The left cancellation law holds in A. i.e., if \( y * x = y * z \), then \( x = z \) for any \( x, y, z \in A \).

**Proof.** Let R be a smarandache-R-module. Then by definition there exists a proper subset A of R which is an algebra. By hypothesis A holds for (B1) to (B6) then A is BF-algebras. (a) is a direct consequence of the definitions.

(b): Let \( x, y \in A \) and \( x * y = 0 \). By (BG), \( x = (x * y) * (0 * y) = 0 * (0 * y) \). From Theorem 1(a) we conclude that \( x = y \).

(c) is obvious, since the right cancellation law holds in every BG-algebra.

(d) Follows from (c) and (BF).

**Definition 5.** A subset I of A is called an ideal of A if it satisfies:

(I1) \( 0 \in I \);
(I2) \( x * y \in I \) and \( y \in I \) imply \( x \in I \) for any \( x, y \in A \).
We say that an ideal \( I \) of \( A \) is **normal** if for any \( x,y,z \in A, x * y \in I \) implies \((z * x) * (z * y) \in I\).

An ideal \( I \) of \( A \) is said to be **proper** if \( I \neq A \).

**Theorem 6.** Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which satisfies (B1) to (B6) and **let \( I \) be a normal ideal of a BF-algebra \( A \).** then the following statements are true:

(a) \( x \in I \Rightarrow 0 * x \in I \),
(b) \( x * y \in I \Rightarrow y * x \in I \).

**Proof.** Let \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (B1) to (B6) then \( A \) is BF-algebras

(a) Let \( x \in I \). Then \( x = x * 0 \in I \). Since \( I \) is normal, \((0 * x) * (0 * 0) \in I \).
Hence \( 0 * x \in I \).
(b) Let \( x * y \in F \) By (a), \( 0 * (x * y) \in I \) Applying (BF) we have \( y * x \in I \).

**Definition 6.** A nonempty subset \( N \) of \( A \) is called a **subalgebra** of \( A \) if for any \( x, y \in N \). It is easy to see that if \( N \) is a subalgebra of \( A \), then \( 0 \in N \).

**Theorem 7.** Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which satisfies (B1) to (B6) and **let \( N \) be a subalgebra of \( A \).** If it satisfies \( x * y \in N \), then \( y * x \in N \).

**Proof.** Let \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (B1) to (B6) then \( A \) is BF-algebras.

Let \( x * y \in N \). By (BF), \( y * x = 0 * (x * y) \). Since \( 0 \in N \) and \( x * y \in N \), we see that \( 0 * (x * y) \in N \). Consequently, \( y * x \in N \).

**Theorem 8.** Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which satisfies (B1) to (B6) then \( I \) is a subalgebra of \( A \) satisfying the following condition:

\[(NI) \text{ if } x \in A \text{ and } y \in I, \text{ then } x * (x * y) \in I.\]

**Proof.** Let \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (B1) to (B6) then \( A \) is BF-algebras.

Let \( x \in A \) and \( y \in I \). Theorem 3(a) shows that \( 0 * y \in I \). Since \( I \) is normal, we conclude that \((x*0) * (x*y) \in I \), i.e., \( x * (x * y) \in I \). Thus (NI) holds. Let now \( x,y \in I \). Therefore \( x * (x * y) \in I \). By Theorem 3(b), \( (x * y) * x \in I \). From the definition of ideal we have \( x * y \in I \). Thus \( I \) is a subalgebra satisfying (NI).

**Theorem 9.** Let \( R \) be a smarandache-R-module, if there exists a proper subset \( A \) of \( R \) in which satisfies (B1) to (B6) then \( N \) is a normal subalgebra of \( A \) if and only if \( N \) is a normal ideal.

**Proof.** Let \( R \) be a smarandache-R-module. Then by definition there exists a proper subset \( A \) of \( R \) which is an algebra. By hypothesis \( A \) holds for (B1) to (B6) then \( A \) is BF-algebras.

Let \( N \) be a normal subalgebra of \( A \). Clearly, \( 0 \in N \). Suppose that \( x * y \in N \) and \( y \in N \). Then \( 0 * y \in N \). Since \( N \) is a subalgebra, we have \( (x * y) * (0 * y) \in N \). But \( (x * y) * (0 * y) \in N \).
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* \((0 \ast y) = x\), because every B-algebra satisfies (BG). Therefore \(x \in N\), and thus \(N\) is an ideal. Let now \(x, y, z \in A\) and \(x \ast (y) \in N\). By (NS), \((z \ast x) \ast (z \ast y) \in N\). Consequently, \(N\) is normal. The converse follows from Theorem 8. Hence the Proof.

REFERENCES

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