Smarandache-lattice and algorithms

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Abstract

In this paper we introduced algorithms for constructing Smarandache-lattice from the Boolean algebra through Atomic lattice, weakly atomic modular lattice, Normal ideals, Minimal subspaces, Structural matrix algebra, Residuated lattice. We also obtained algorithms for Smarandache-lattice from the Boolean algebra. For basic concept we refer to Gratzer [3].

Keywords: Smarandache-lattice, Lattice, Boolean algebra.

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1 Introduction

In this paper we have introduced algorithms to construct Smarandache-lattice. Smarandache-lattice is one the Smarandache-2-Algebraic Structure. By [7] Smarandache \(n\)-structure on a set \(S\) means a weak structure \(\{w_0\}\) on \(S\) such that there exists a chain of proper subsets \(P_{n-1} < P_{n-2} < \cdots < P_2 < P_1 < S\), where ‘\(<\)’ means ‘included in’, whose corresponding structures verify the inverse chain \(\{w_{n-1}\} > \{w_{n-2}\} > \cdots > \{w_2\} > \{w_1\} > \{w_0\}\), where ‘\(>\)’ signifies ‘strictly stronger’ (i.e., structure satisfying more axioms)By proper subset of a set \(S\), we mean a subset \(P\) of \(S\), different from the empty set, from the original set \(S\), and from the idempotent elements if any. And by structure on \(S\) we mean the strongest possible structure \(\{w\}\) on \(S\) under the given operation(s). As a particular case, a Smarandache 2-algebraic structure (two levels only of structures in algebra) on a set \(S\), is a weak structure \(\{w_0\}\) on \(S\) such that there exists a proper subset \(P\) of \(S\), which is embedded with a stronger structure \(\{w_1\}\).

Example: Semi lattice \(<\) Lattice \(<\) Boolean algebra.

2 Preliminaries

**Definition 2.1.** The Lattice \(L\) is called complemented Lattice. If \(L\) has a greatest element and least element and each element has at least one complement; that is, for \(b \in L\), there exists \(a \in L\) such that \(a \lor b = 1\), \(a \land b = 0\).

**Definition 2.2.** The Smarandache-lattice is defined to be a lattice \(S\), such that a proper subset of \(S\), is a Boolean algebra (with respect to with same induced operations). By proper subset we understand a set included in \(S\), different from the empty set, from the unit element if any, and from \(S\).

**Definition 2.3** (Alternative Definition 2.2). If there exists a non empty set \(L\) which is a Boolean algebra such that its Superset \(S\) of \(L\) is a Lattice with respect same induced operations. Then \(S\) is called Smarandache-lattice.

**Definition 2.4.** A Residuated lattice is an algebraic structure \((R, \land, \lor, \to, \otimes, \oplus, 0, 1)\) such that

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(i) \((R, \land, \lor, \to, \otimes, \oplus, 1, 30)\) is bounded lattice with least element \(1\) and greatest element \(30\).

(ii) \((R, \otimes, 30)\) is Commutative monoid where \(30\) is a unit element.

(iii) \(a \ast b \leq c\) if and only if \(a \leq b \to c\).

**Definition 2.5.** Let \((L, \land, \lor, 0, 1)\) be a Boolean algebra. A subset \(I\) of \(L\) is called an ideal of \(B\) if

(i) \(0 \in I\).

(ii) \(a, b \in I \Rightarrow a \lor b \in I\).

(iii) \(a \in I\) and \(b \leq a \Rightarrow b \in I\).

**Definition 2.6.** Given an element \(a\) of a Boolean algebra (or other poset) \(A\), recall that \(a\) is atomic in \(A\) if \(a\) is minimal among non-trivial (non-bottom) elements of \(A\). That is, given any \(b \in A\) such that \(b \leq a\), either \(b = 0\) or \(b = a\). A Boolean algebra \(A\) is atomic if we have \(b = \bigvee i a_i\) for every \(b \in A\), where \(\{a_i\}_I\) is some set of atoms in \(A\).

**Definition 2.7.** Boolean algebra is a distributive lattice which satisfies lattices whose congruences form a Boolean algebra.

(i) Involution: \((a')' = a\).

(ii) Complements: \(a \lor a' = 1\) and \(a \land a' = 0\).

(iii) Identities: \(a \land 1 = a\) and \(a \lor 0 = a\), \(a \lor 1 = 1\) and \(a \land 0 = 0\).

(iv) De Morgan’s laws: \((a \land b)' = a' \lor b', (a \lor b)' = a' \land b'\).

### 3 Characteristics

#### 3.1 Atomic lattice: Algorithm-3.1

Peter Crawley has introduced the notion, “Lattices whose congruence’s form a Boolean algebra 1960. In [6] it has been proved that \(S\) is an arbitrary lattice, \(L\) is a Boolean algebra if and only if for each proper quotient \(a/b\) of \(S\) there exists a finite chain \(a = x_0 > x_1 > \cdots > x_k = b\) such that each \(c_i−1/c_i\) is minimal. We have proved that Boolean algebra itself is a atomic lattice \((L = A_0)\), and hence every element of \(L\) is join of atoms \(c_i−1/c_i\) generated by minimal quotients \(x_i/y_j\), we must have \(c_i−1/c_i = x_i/x_j \in S\). The union of atomic lattice is called as a Lattice at the same time the intersection of atomic Lattice is non-zero unique set included in a lattice. By Gratze [3], \(S\) is a Lattice by definition \(S\) is a Smarandache-lattice. According to this hypothesis, we have to write an Algorithm for constructing a Smarandache-lattice from the Boolean algebra as follows.

Step 1: Consider a Boolean algebra \(L\).

Step 2: Let \(L = A_0\).

Step 3: Let \(A_i = \theta_{c_i−1/c_i}, i = 1, 2, \ldots\) be supersets of \(\theta_{c_0/c_1}\).

Step 4: Let \(S = \bigcup_{i=1}^{K} \theta_{c_i−1/c_i}\).

Step 5: Choose sets \(A_j\) from \(A_j's\) subject to for all \(a, b \in S\).

A Boolean algebra \(A\) is atomic if for every \(b \in A\) such that \(b = \bigvee i a_i\), \(a_i\) is some set of atoms in \(A\).

Step 6: Verify that \(\cap A_j = \theta_{c_0/c_1} \cap \theta_{c_1/c_2} \cap \theta_{c_2/c_3} \cap \theta_{c_3/c_4} \cdots \cap \theta_{c_{k−1}/c_k} = \theta_{c_0/c_1} \neq \{0\} \subset S\).

Step 7: If Step (6) is a true, then we write \(S\) is a Smarandache-lattice.
3.2 Weakly atomic modular lattice: Algorithm-3.2

Peter Crawly has introduced the notion, “Lattices whose congruence’s form a Boolean algebra 1960. In [6] it has been proved that $S$ be a weakly atomic modular lattice. Then $\theta(L)$ is a Boolean algebra if and only if every quotient of $L$ is finite dimensional. We have proved $L$ be a weakly atomic modular lattice itself Boolean algebra ($L = M_0$). The union of weakly atomic modular Lattice called as a Lattice at the same time the intersection of weakly atomic modular Lattice is non-zero unique set included in a Lattice. By Gratzer [3], $S$ is a lattice by definition $S$ is a Smarandache-lattice According to this hypothesis, we have to write an Algorithm for constructing a Smarandache-lattice from the Boolean algebra as follows.

Step 1: Consider a Boolean algebra $L$.
Step 2: Let $L = M_0$.
Step 3: Let $M_i, i = 0, 1, 2 \ldots$ be supersets of $M_0$.
Step 4: Let $S = \cup M_i$.
Step 5: Choose sets $M_j$ from $M_i$s subject to for all $a, b \in S$, $(a')' = a, a \lor a' = 1$ and $a \land a' = 0, a \lor 1 = a$ and $a \lor 0 = a, a \lor 1 = 1$ and $a \land 0 = 0, (a \land b)' = a' \lor b', (a \lor b)' = a' \land b'$.
Step 6: Verify that for every $\cap M_j = M_0 \neq \{0\} \subset S$.
Step 7: If step (6) is a true, then we write $S$ is a Smarandache-lattice.

3.3 Normal ideals: Algorithm-3.3

In [4], it has been proved that NI is a Normal ideals itself complete semi-Lattice(Boolean algebra). The union of Normal ideals called as a Lattice at the same time the intersection of Normal ideals contained in all other nonzero normal ideals of Lattice. By Gratzer [3], $S$ is a lattice by definition $S$ is a Smarandache-lattice.

According to this hypothesis, we have to write an Algorithm for constructing a Smarandache-lattice from the Boolean algebra as follows.

Step 1: Consider a Boolean algebra $L$.
Step 2: Let $L = I_0$.
Step 3: Let $I_i, i = 0, 1, 2, \ldots$ be supersets of $I_0$.
Step 4: Let $S = \cup I_i$.
Step 5: Choose sets $I_j$ from $I_i$’s, subject to for all $a, b \in S$.
   (i) $0 \in I$
   (ii) $a, b \in I \Rightarrow a \lor b \in I$
   (iii) $a \in I$ and $b \leq a \Rightarrow b \in I$.
Step 6: Verify that for every $\cap I_j = I_0 \neq \{0\} \subset S$.
Step 7: If Step (6) is a true, then we write $L$ is a Smarandache-lattice.

3.4 Minimal subspaces: Algorithm 3.4

In 2013, Emira, Barker George Philip have introduced the notion of a Lattice to be a Boolean algebra. Emira, Barker George Philip in their paper [1] have proved, if the Lattice $L$ of subspaces of a structural algebra is complemented then the complement $W$ is unique. Suppose $V \in S, V$ is a sum of minimal subspaces, each of which is in other irreducible subspaces then $V$ has complement in $S$. $L$ is a Boolean algebra if and only if there is no chain of non zero irreducible elements. We have proved $V_0$ be a Minimal subspaces itself Boolean algebra. The union of Minimal subspaces called as a Lattice at the same time the intersection of Minimal subspaces is nonzero unique set included in a Lattice. By Gratzer, [3], $S$ is a lattice by definition $S$ is a Smarandache-lattice. According to this hypothesis, we have to write an Algorithm for constructing a Smarandache-lattice from the Boolean algebra as follows.
Step 1: Consider a Boolean algebra $L$.

Step 2: Let $L = V_0$.

Step 3: Let $V_i$, $i = 0, 1, 2, \ldots$ be supersets of $V_0$.

Step 4: Let $S = \cup V = (U_i \cap V_i)$.

Step 5: Choose sets $V_j$ from $V_i$ subject to for all $B_1, B_2 \in S$ such that $B_1 = B \cap V$, $B_2 = B / B_1 \Leftrightarrow \text{span} B_2 \in S$, $V$ has a complement in $S$ where $B$ is a basis for $S$. Each $U_j$ has a complement $W_j$ now suppose $V$ is the sum of minimal subspaces

$V = U_1 + U_2 + \cdots + U_S$,

$W = W_1 \cap W_2 \cap \cdots \cap W_S \in S$

$U \cap W = U_1 + \cdots + U_S \cap W \subseteq (U_1 \cap W_1) + \cdots + (U_S \cap W_S)$

$V + W = V + (W_1 \cap \cdots \cap W_S) \supseteq (U_1 + W_1) \cap \cdots \cap (U_S + W_S) = \mathbb{F}^n$.

Step 6: $\cap V_j = V_0 \neq \{0\} \subset S$.

Step 7: If step (6) is a true, then we write $S$ is a Smarandache-lattice.

### 3.5 Point lattice: Algorithm-3.5

In 2013, Emira, Barker George Philip have introduced the notion of a Lattice to be a Boolean algebra. Akkurt, Mustafa, Emira, Barker George Philip in their paper [1] have proved, If the Lattice $L$ of subspaces of a structural algebra is complemented then the complement $W$ is unique, where $W = V_1 + V_2 + \cdots + V_k$ is the collection of the irreducible subspaces contained in $W$. Let $M_n(F, \rho)$ be structural matrix algebra with $L = \text{Lat}(M_n(F, \rho))$ its lattice. $L$ is Boolean algebra if and only if $L$ is an atomic lattice. We have proved $P_0$ be a Point Lattice itself Boolean algebra. The union of Point Lattice is called as a Lattice at the same time the intersection of Point Lattice is nonzero unique set included in a Lattice. By Gratzer [3], $S$ is a Lattice by definition $S$ is a Smarandache-lattice.

According to this hypothesis, we have to write an Algorithm for constructing a Smarandache-lattice from the Boolean algebra as follows.

Step 1: Consider a Boolean algebra $L$.

Step 2: Let $L = P_0$ point lattice.

Step 3: Let $P_i$, $i = 0, 1, 2, \ldots$ be super sets of $P_0$.

Step 4: Let $S = \cup P_i$.

Step 5: Choose sets $P_j$ from $P_i$ subject to for all $P_1, P_2 \in L$ such that $P_1 = B \cap P$, $P_2 = B / B_1 \Leftrightarrow \text{span} B_2 \in S$, $V$ has a complement in $L$, where $S$ is a basis for $L$ each $U_j$ has a complement $W_j$ now suppose $V$ is the sum of minimal.

Step 6: $W = \cap P_j = P_0 \neq \{0\} \subset S$.

Step 7: If step (6) is a true, then we write $S$ is a Smarandache-lattice.

### 3.6 Residuated lattice: Algorithm-3.6

$L = \{1, 30\}$ is a Boolean algebra with respect to $(L, \lor, \land, 1, 30)$ [5]. We have proved that all axioms are satisfied for Boolean algebra and this Boolean algebra itself is a Residuated Lattice. The union of Residuated Lattice is called as a Lattice at the same time the intersection of Residuated lattices is a unique nonzero set included in Lattice. By Gratzer [3], $S$ is a Lattice by definition $S$ is a Smarandache-lattice. According to this hypothesis, we have to write an Algorithm for constructing a Smarandache-lattice from the Boolean algebra as follows.

Step 1: Consider a nonempty Set $L = \{1, 30\}$.

Step 2: Verify that $L = \{1, 30\}$ is a Boolean algebra with respect to $\land, \lor$.

For, check the following conditions
(i) Associative Law: For any $a, b, c \in L$, $a \lor (b \lor c) = (a \lor b) \lor c \lor$ is defined as follows:

\[
\begin{align*}
1 \lor (1 \lor 1) &= 1 \lor 1 = 1 \in L \\
(1 \lor 1) \lor 1 &= 1 \lor 1 = 1 \in L \\
1 \lor (1 \lor 1) &= (1 \lor 1) \lor 1 \\
30 \lor (30 \lor 30) &= 30 \lor 30 = 30 \in L \\
(30 \lor 30) \lor 30 &= 30 \lor 30 = 30 \in L \\
30 \lor (30 \lor 30) &= (30 \lor 30) \lor 30 \\
1 \lor (30 \lor 30) &= 1 \lor 30 = 30 \in L \\
(1 \lor 30) \lor 30 &= 30 \lor 30 = 30 \in L \\
1 \lor (30 \lor 30) &= (1 \lor 30) \lor 30 \\
1 \lor (30 \lor 1) &= 1 \lor 30 = 30 \in L \\
(1 \lor 30) \lor 1 &= 1 \lor 1 = 30 \in L \\
1 \lor (30 \lor 1) &= (1 \lor 30) \lor 1 \\
30 \lor (30 \lor 30) &= 30 \lor 30 = 30 \in L \\
(30 \lor 30) \lor 30 &= 30 \lor 30 = 30 \in L \\
30 \lor (30 \lor 30) &= (30 \lor 30) \lor 30
\end{align*}
\]

\[
\begin{align*}
\land \text{ is defined as follows:} \\
1 \land (30 \land 30) &= 1 \land 30 = 1 \in L \\
(1 \land 30) \land 30 &= 1 \land 30 = 1 \in L \\
1 \land (30 \land 30) &= (1 \land 30) \land 30 \\
1 \land (30 \land 1) &= 1 \land 1 = 1 \in L \\
(1 \land 30) \land 1 &= 1 \land 1 = 1 \in L \\
1 \land (30 \land 1) &= (1 \land 30) \land 1
\end{align*}
\]

(ii) Commutative law: For any $a, b \in L$, $(a \lor b) = (b \lor a)$

\[
\begin{align*}
1 \lor 1 &= 1 \lor 1 = 1 \in L \\
30 \lor 30 &= 30 \lor 30 = 30 \in L \\
1 \lor 30 &= 30 \lor 1 = 30 \in L
\end{align*}
\]

(iii) Distributive law: For all $a, b, c \in L$

\[
\begin{align*}
a \lor (b \land c) &= (a \lor b) \land (a \lor c) \\
a \land (b \lor c) &= (a \land b) \lor (a \land c)
\end{align*}
\]

\[
\begin{align*}
1 \lor (30 \land 30) &= (1 \lor 30) \land (1 \lor 30) = 30 \in L \\
1 \lor (1 \land 1) &= (1 \lor 1) \land (1 \lor 1) = 1 \in L \\
30 \lor (30 \land 30) &= (30 \lor 30) \land (30 \lor 30) = 30 \in L \\
30 \lor (1 \land 30) &= (30 \lor 1) \land (30 \lor 30) = 30 \in L \\
1 \land (30 \lor 30) &= (1 \land 30) \lor (1 \land 30) = 1 \in L \\
1 \land (1 \lor 1) &= (1 \land 1) \lor (1 \land 1) = 1 \in L \\
30 \land (30 \lor 30) &= (30 \land 30) \lor (30 \land 30) = 30 \in L \\
30 \land (1 \lor 30) &= (30 \land 1) \lor (30 \land 30) = 30 \in L
\end{align*}
\]
(iv) Identity element there exists identity 1 (‘0’ element) for ∨ and 30 (‘1’ element) for ∧

For any \( a \in L(\land 1) = a \), \( a \land 30 = a \)
For \( 1 \in L(1 \lor 1) = 1 \), \( 1 \land 30 = 1 \in L \)
For \( 30 \in L(1 \lor 30) = 30 \), \( 30 \land 30 = 30 \in L \)

(v) Complement every element of \( L \) has a complement with in \( L \) there exists \( a' \) is the complement of \( a \) then \( a \in L \), \( a \lor a' = 30 \), \( a \land a' = 1 \), \( 1' = 30 \).

(vi) Idempotent Laws: For any \( a \in L \), \( a \lor a = a \), \( a \land a = a \), \( 1 \land 1 = 1 \), \( 1 \lor 1 = 1 \), \( 30 \lor 30 = 30 \), \( 30 \land 30 = 30 \)

(vii) Null Law: For any \( a \in L \), \( a \lor 1 = 1 \), \( a \land 0 = 0 \), \( 0 \) element is 1 and 1 element is 30.

\( 30 \lor 1 = 30 \), \( 30 \land 30 = 30 \), \( 1 \land 1 = 1 \), \( 30 \land 1 = 1 \).

(viii) Absorption Law: For any \( a, b \in L \), \( a \lor (a \land b) = a \), \( a \land (a \lor b) = a \)
\[
30 \land (30 \lor 1) = 30, 30 \lor (30 \land 1) = 30 \\
1 \land (1 \lor 30) = 1, 1 \lor (1 \land 30) = 1 \\
\]

(ix) De-Morgan’s Law: For any \( a, b \in L \), \( (a \lor b)' = a' \land b' \), \( (a \land b)' = a' \lor b' \).
\[
(1 \lor 30)' = 30' = 1 \\
1' \land 30' = 30 \land 1 = 1 \\
(1 \lor 30)' = 1' \land 30' = 30 \\
(1 \land 30)' = 1' = 30 \\
1' \lor 30' = 30 \lor 1 = 30 \\
(1 \land 30)' = 1' \lor 30' \\
\]

(x) Involution Law: \( a \in L \), \( (a)' = a \), \( (1)' = 30' = 1 \) and \( (30)' = 1' = 30 \).

\( L = \{1, 30\} \) satisfies all the conditions of Boolean algebra.

Hence \( L = (L, \land, \lor, 1, 30') \) is a Boolean algebra.

Step 3: Let \( L = R_0 \) be a Residuated lattice. Let \( R_0 = L = \{1, 30\} \).

Step 4: Consider super sets \( R_i; i = 0, 2, 3 \) of \( R_0 \).
\[
R_0 = \{1, 30\}, \\
R_1 = \{1, 2, 15, 30\}, \\
R_2 = \{1, 2, 3, 10, 15, 30\}, \\
R_3 = \{1, 2, 3, 5, 6, 10, 15, 30\}. \\
\]

Step 5:
\[
S = \bigcup_{i=0}^{3} R_i, \\
S = R_0 \cup R_1 \cup R_2 \cup R_3 \\
S = \{1, 30\} \cup \{1, 2, 15, 30\} \cup \{1, 2, 3, 10, 15, 30\} \cup \{1, 2, 3, 5, 6, 10, 15, 30\} \\
= \{1, 2, 3, 5, 6, 10, 15, 30\} \supseteq L \\
\]

Step 6: A Residuated lattice is an algebraic structure \((R, \land, \lor, \to, \otimes, 0, 1)\) such that

(i) \((R, \land, \lor, \to, \otimes, 1, 30)\) is bounded lattice with least element 1 and greatest element 30.

(ii) \((R, \otimes, 30)\) is Commutative monoid where 30 is a unit element.

(iii) \(a \otimes b \leq c\) if and only if \(a \leq b \to c\) for all \(a, b, c \in R\).

Step 7: \((R, \otimes, \otimes, \to, 1, 30)\) is a Residuated Lattice. \(\otimes, \otimes, \to\) is defined as follows.
(i) \( a \otimes b = \text{GLB}\{a, b\} \).
\[
1 \otimes 1 = 1, \quad 1 \otimes 30 = 1, \quad 30 \otimes 1 = 1 \quad \text{and} \quad 30 \otimes 30 = 30.
\]
(ii) \( a \oplus b = \text{LUB}\{a, b\} \).
\[
1 \oplus 1 = 1, \quad 1 \oplus 30 = 30, \quad 30 \oplus 1 = 30 \quad \text{and} \quad 30 \oplus 30 = 30.
\]
(iii) \( a \rightarrow b = a' \oplus b \).
\[
1 \rightarrow 1 = 30, \quad 1 \rightarrow 30 = 30, \quad 30 \rightarrow 1 = 1, \quad 30 \rightarrow 30 = 30.
\]
Hence \( R_0 \) satisfies required condition. We observe that \( a \ast b \leq c \) if and only if \( a \leq b \rightarrow c \) for all \( a, b, c \in R \).
Hence for \( R_1 = \{1, 2, 15, 30\} \) and \( S = \{1, 2, 3, 5, 6, 10, 15, 30\} \).

Step 8: Verify \( \cap R_j = \{1, 30\} \cap \{1, 2, 15, 30\} \cap \{1, 2, 3, 10, 15, 30\} \cap \{1, 2, 3, 5, 6, 10, 15, 30\} = R_0 \subseteq S \).

Step 9: If the Step 8 is true, then write \( S \) is Smarandache-lattice

4 Conclusion

In this paper we have to study Algorithm for construct a Smarandache-lattice from the Boolean algebra by an algorithmic approach through its substructures and smarandache lattice has been introduced in some applications.

References


[8] [www.gallup.unm.edu/Smarandache/algebra.htm](http://www.gallup.unm.edu/Smarandache/algebra.htm).

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