Neutrosophic Set – A Generalization of the Intuitionistic Fuzzy Set

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Abstract—In this paper one generalizes the intuitionistic fuzzy set (IFS), paraconsistent set, and intuitionistic set to the neutrosophic set (NS). Many examples are presented. Distinctions between NS and IFS are underlined.

Index Terms—Intuitionistic Fuzzy Set, Paraconsistent Set, Intuitionistic Set, Neutrosophic Set, Non-standard Analysis, Philosophy

I. INTRODUCTION

One first presents the evolution of sets from fuzzy set to neutrosophic set. Then one introduces the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where \([0, 1]^T\) is the non-standard unit interval, and thus one defines the neutrosophic set. One gives examples from mathematics, physics, philosophy, and applications of the neutrosophic set. Afterwards, one introduces the neutrosophic set operations (complement, union, difference, Cartesian product, inclusion, and n-ary relationship), some generalizations and comments on them, and finally the distinctions between the neutrosophic set and the intuitionistic fuzzy set.

II. SHORT HISTORY

The fuzzy set (FS) was introduced by L. Zadeh in 1965, where each element had a degree of membership.

The intuitionistic fuzzy set (IFS) on a universe \(X\) was introduced by K. Atanassov in 1983 as a generalization of FS, where besides the degree of membership \(\mu_A(x) \in [0, 1]\) of each element \(x \in X\) to a set \(A\) there is considered a degree of non-membership \(\nu_A(x) \in [0, 1]\), but such that

\[ \forall x \in X, \mu_A(x) + \nu_A(x) \leq 1. \]  

According to Deschrijver & Kerre (2003) the vague set defined by Gau and Buehrer (1993) was proven by Bistune & Burillo (1996) to be the same as IFS.

Goguen (1967) defined the \(L\)-fuzzy Set in \(X\) as a mapping \(X \rightarrow L\) such that \((L, \leq, \cdot)\) is a complete lattice, where \(L = \{(x_1, x_2) \in [0, 1]^2; x_1 + x_2 \leq 1\}\) and \((x_1, x_2) \leq (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \leq y_2\). The interval-valued fuzzy set (IVFS) apparently first studied by Sambuc (1975), which were called by Deng (1989) grey sets, and IFS are specific kinds of L-fuzzy sets.

According to Cornelis et al. (2003), Gehrke et al. (1996) stated that “Many people believe that assigning an exact number to an expert’s opinion is too restrictive, and the assignment of an interval of values is more realistic”, which is somehow similar with the imprecise probability theory where instead of a crisp probability one has an interval (upper and lower) probabilities as in Walley (1991).

Atanassov (1999) defined the interval-valued intuitionistic fuzzy set (IVIFS) on a universe \(X\) as an object \(A\) such that:

\[ A = \{(x, \mu_A(x), \nu_A(x)), x \in X\}, \]

with \(\mu_A, \nu_A \rightarrow \text{Interval}[0, 1]\) and \(\mu_A, \nu_A \rightarrow \text{Interval}[0, 1]\)

\[ \forall x \in X, \sup \mu_A(x) + \sup \nu_A(x) \leq 1. \]

Belnap (1977) defined a four-valued logic, with truth (T), false (F), unknown (U), and contradiction (C). He used a billatice where the four components were inter-related.

In 1995, starting from philosophy (when I fretted to distinguish between absolute truth and relative truth or between absolute falsehood and relative falsehood in logics, and respectively between absolute membership and relative membership or absolute non-membership and relative non-membership in set theory) I began to use the non-standard analysis. Also, inspired from the sport games (winning, defeating, or tight scores), from votes (pro, contra, null/black votes), from positive/negative/zero numbers, from yes/no/NA, from decision making and control theory (making a decision, not making, or hesitating), from accepted/rejected/pending, etc. and guided by the fact that the law of excluded middle did not work any longer in the modern logics, I combined the non-standard analysis with a tri-component logic/set/probability theory and with philosophy (I was excited by paradoxism in science and arts and letters, as well as by paraconsistent and incompleteness in knowledge). How to deal with all of them at once, is it possible to unity them?

I proposed the term “neutrosophic” because "neutrosophic" etymologically comes from "neutro-sophy" [French neutre < Latin neuter, neutral, and Greek sophia, skill/wisdom] which means knowledge of neutral thought, and this third/neural represents the main distinction between "fuzzy" and "intuitionistic fuzzy" logic/set, i.e. the included middle component (Lupasco-Nicolescu's logic in philosophy), i.e. the neutral/indeterminate/unknown part (besides the "truth"/"membership" and "falsehood"/"non-membership" components that both appear in fuzzy logic/set). See the...
III. DEFINITION OF NEUTROSOPHIC SET

Let T, I, F be real standard or non-standard subsets of \( \mathbb{I} \), with \( \sup T = t_{\text{sup}}, \inf T = t_{\text{inf}} \), \( \sup I = i_{\text{sup}}, \inf I = i_{\text{inf}} \), \( \sup F = f_{\text{sup}}, \inf F = f_{\text{inf}} \), and \( n_{\text{sup}} = t_{\text{sup}} + i_{\text{sup}} + f_{\text{sup}}, \) \( n_{\text{inf}} = t_{\text{inf}} + i_{\text{inf}} + f_{\text{inf}} \).

T, I, F are called neutrosophic components.

Let U be a universe of discourse, and M a set included in U. An element \( x \) from U is noted with respect to the set M as \( x(T, I, F) \) and belongs to M in the following way:

- it is \( t\% \) true in the set, \( i\% \) indeterminate (unknown if it is) in the set, and \( f\% \) false, where \( t \) varies in T, \( i \) varies in I, \( f \) varies in F.

IV. GENERAL EXAMPLES

Let A, B, and C be three neutrosophic sets.

One can say, by language abuse, that any element neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between 0 and 1 or even less than 0 or greater than 1.

Thus: \( x(0.5,0.2,0.3) \) belongs to A (which means, with a probability of 50% \( x \) is in A, with a probability of 50% \( x \) is not in A, and the rest is undecided); or \( y(0,0,1) \) belongs to A (which normally means \( y \) is not for sure in A); or \( z(0,1,0) \) belongs to A (which means one does know absolutely nothing about z's affiliation with A); here 0.5+0.2+0.3=1; thus A is a NS and an IFS too. More general, \( y(0.20-0.30), (0.40-0.45), (0.50-0.51), (0.20, 0.24, 0.28) \) belongs to the set B, which means:

- with a probability in between 20-30% \( y \) is in B (one cannot find an exact approximation because of various sources used);
- with a probability of 20% or 24% or 28% \( y \) is not in B;
- the indeterminacy related to the appurtenance of \( y \) to B is in between 40-45% or between 50-51% (limits included);

The subsets representing the appurtenance, indeterminacy, and falsity may overlap, and \( n_{\text{sup}} = 0.30 + 0.51 + 0.28 > 1 \) in this case; then B is a NS but is not an IFS; we can call it paraconsistent set (from paraconsistent logic, which deals with paraconsistent information).

Or, another example, say the element \( z(0.1, 0.3, 0.4) \) belongs to the set C, and here 0.1+0.3+0.4<1; then B is a NS but is not an IFS; we can call it intuitionistic set (from intuitionistic logic, which deals with incomplete information).

Remarkably, in the same NS one can have elements which have paraconsistent information (sum of components >1), others incomplete information (sum of components < 1), others consistent information (in the case when the sum of components = 1), and others interval-valued components (with no restriction on their superior or inferior sums).

V. PHYSICS EXAMPLES

For example the Schrödinger's Cat Theory says that the quantum state of a photon can basically be in more than one place in the same time, which translated to the neutrosophic set means that an element (quantum state) belongs and does not belong to a set (one place) in the same time; or an element (quantum state) belongs to two different sets (two different places) in the same time. It is a question of "alternative worlds" theory very well represented by the neutrosophic set theory.

In Schrödinger's Equation on the behavior of electromagnetic waves and "matter waves" in quantum theory, the wave function \( \psi \) which describes the superposition of possible states may be simulated by a neutrosophic function, i.e. a function whose values are not unique for each argument from the domain of definition (the vertical line test fails, intersecting the graph in more points).

Don't we better describe, using the attribute "neutrosophic" than "fuzzy" or any others, a quantum particle that neither exists nor non-exists?

How to describe a particle \( \xi \) in the infinite micro-universe that belongs to two distinct places \( P_1 \) and \( P_2 \) in the same time? \( \xi \in P_1 \) and \( \xi \notin P_1 \) as a true contradiction, or \( \xi \in P_1 \) and \( \xi \notin P_1 \).

VI. PHILOSOPHICAL EXAMPLES

Or, how to calculate the truth-value of Zen (in Japanese) / Chan (in Chinese) doctrine philosophical proposition: the present is eternal and comprises in itself the past and the future?

In Eastern Philosophy the contradictory utterances form the core of the Taoism and Zen/Chan (which emerged from Buddhism and Taoism) doctrines.

How to judge the truth-value of a metaphor, or of an ambiguous statement, or of a social phenomenon which is positive from a standpoint and negative from another standpoint? There are many ways to construct them, in terms of the practical problem we need to simulate or approach. Below there are mentioned the easiest ones.

VII. APPLICATION

A cloud is a neutrosophic set, because its borders are ambiguous, and each element (water drop) belongs with a neutrosophic probability to the set (e.g. there are a kind of separated water drops, around a compact mass of water drops, that we don't know how to consider them: in or out of the cloud).

Also, we are not sure where the cloud ends nor where it
begins, neither if some elements are or are not in the set. That's why the percent of indeterminacy is required and the neutrosophic probability (using subsets - not numbers - as components) should be used for better modeling: it is a more organic, smooth, and especially accurate estimation. Indeterminacy is the zone of ignorance of a proposition's value, between truth and falsehood.

VIII. OPERATIONS WITH CLASSICAL SETS

We need to present these set operations in order to be able to introduce the neutrosophic connectors. Let $S_1$ and $S_2$ be two (unidimensional) real standard or non-standard subsets included in the non-standard interval $[0, \infty)$ then one defines:

1. **Addition of classical Sets:**
   \[ S_1 + S_2 = \{x|x = s_1 + s_2, \text{ where } s_1 \in S_1 \text{ and } s_2 \in S_2\}, \]
   with $\inf(S_1 + S_2) = \inf S_1 + \inf S_2$, $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$;
   and, as some particular cases, we have
   \[ \{a\} + S_2 = \{x|x = a + s_2, \text{ where } s_2 \in S_2\}, \]
   with $\inf(\{a\} + S_2) = a + \inf S_2$, $\sup(\{a\} + S_2) = a + \sup S_2$.

2. **Subtraction of classical Sets:**
   \[ S_1 - S_2 = \{x|x = s_1 - s_2, \text{ where } s_1 \in S_1 \text{ and } s_2 \in S_2\}, \]
   with $\inf(S_1 - S_2) = \inf S_1 - \sup S_2$, $\sup(S_1 - S_2) = \sup S_1 - \inf S_2$;
   and, as some particular cases, we have
   \[ \{a\} - S_2 = \{x|x = a - s_2, \text{ where } s_2 \in S_2\}, \]
   with $\inf(\{a\} - S_2) = a - \sup S_2$, $\sup(\{a\} - S_2) = a - \inf S_2$; also
   \[ \{1\} - S_2 = \{x|x = 1 - s_2, \text{ where } s_2 \in S_2\}, \]
   with $\inf(\{1\} - S_2) = 1 - \sup S_2$, $\sup(\{1\} - S_2) = 1 - \inf S_2$.

3. **Multiplication of classical Sets:**
   \[ S_1 \cdot S_2 = \{x|x = s_1 \cdot s_2, \text{ where } s_1 \in S_1 \text{ and } s_2 \in S_2\}, \]
   with $\inf(S_1 \cdot S_2) = \inf S_1 \cdot \sup S_2$, $\sup(S_1 \cdot S_2) = \sup S_1 \cdot \inf S_2$;
   and, as some particular cases, we have
   \[ \{a\} \cdot S_2 = \{x|x = a \cdot s_2, \text{ where } s_2 \in S_2\}, \]
   with $\inf(\{a\} \cdot S_2) = a \cdot \inf S_2$, $\sup(\{a\} \cdot S_2) = a \cdot \sup S_2$; also
   \[ \{1\} \cdot S_2 = \{x|x = 1 \cdot s_2, \text{ where } s_2 \in S_2\}, \]
   with $\inf(\{1\} \cdot S_2) = 1 \cdot \inf S_2$, $\sup(\{1\} \cdot S_2) = 1 \cdot \sup S_2$.

4. **Division of a classical Set by a Number:**
   Let $k \in \mathbb{R}^*$, then $S_1 / k = \{x|x = s_1 / k, \text{ where } s_1 \in S_1\}$.

IX. NEUTROSOPHIC SET OPERATORS

One notes, with respect to the sets $A$ and $B$ over the universe $U$,
   \[ x = x(T_1, I_1, F_1) \in A \text{ and } x = x(T_2, I_2, F_2) \in B, \]
   by mentioning $x$'s neutrosophic membership, indeterminacy, and non-membership respectively.

And, similarly, $y = y(T, I, F) \in B$.

If, after calculations, in the below operations one obtains values $< 0$ or $> 1$, then one replaces them with $0$ or $1$ respectively.

1. **Complement of $A$:**
   If $x(T_1, I_1, F_1) \in A$, then $x(\{1\}^c - T_1, \{1\}^c - I_1, \{1\}^c - F_1) \in C(A)$.

2. **Intersection:**
   If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$, then $x(T_1 \cap T_2, I_1 \cap I_2, F_1 \cap F_2) \in A \cap B$.

3. **Union:**
   If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$, then $x(T_1 \cup T_2, I_1 \cup I_2, F_1 \cup F_2) \in A \cup B$.

4. **Difference:**
   If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$, then $x(T_1 \setminus T_2, I_1 \setminus I_2, F_1 \setminus F_2) \in A \setminus B$.

5. **Cartesian Product:**
   If $x(T_1, I_1, F_1) \in A$, $y(T, I, F) \in B$, then $x(T_1 \times T, I_1 \times I, F_1 \times F) \in A \times B$.

6. **$M$ is a subset of $N$:**
   If $x(T_1, I_1, F_1) \in M$, $x(T_2, I_2, F_2) \in N$, where $\inf T_1 \leq \inf T_2$, $\sup T_1 \leq \sup T_2$, and $\inf F_1 \geq \inf F_2$, $\sup F_1 \geq \sup F_2$.

7. **Neutrosophic n-ary Relation:**
   Let $A_1, A_2, \ldots, A_n$ be arbitrary non-empty sets.

A Neutrosophic n-ary Relation $R$ on $A_1 \times A_2 \times \ldots \times A_n$ is defined as a subset of the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ such that for each ordered n-tuple $(x_1, x_2, \ldots, x_n)(T, I, F)$, $T$ represents the degree of validity, $I$ the degree of indeterminacy, and $F$ the degree of non-validity respectively of the relation $R$.


X. GENERALIZATIONS AND COMMENTS

From the intuitionistic logic, paraconsistent logic, dialecticism, faillibilism, paradoxes, pseudoparadoxes, and tautologies we transfer the "adjectives" to the sets, i.e. to intuitionistic set (set incompletely known), paraconsistent set, dialectheist set, fallibilist set (each element has a percentage of indeterminacy), paradoxist set (an element may belong and may not belong in the same time to the set), pseudoparadoxist set, and tautologic set respectively.

Hence, the neutrosophic set generalizes:
- the *intuitionistic set*, which supports incomplete set theories (for $0 < n < 1$ and $i = 0$, $0 \leq t, f \leq 1$) and incomplete known elements belonging to a set;
- the *fuzzy set* (for $n = 1$ and $i = 0$, and $0 \leq t, f \leq 1$);
- the *intuitionistic fuzzy set* (for $t+i+f=1$ and $0\leq t, f \leq 1$);
- the *classical set* (for $n = 1$ and $i = 0$, with $t, f$ either 0 or 1);
- the *paraconsistent set* (for $n > 1$ and $i = 0$, with both $t, f < 1$);
there is at least one element $x(T, I, F)$ of a paraconsistent set $M$ which belongs at the same time to $M$ and to its complement set $C(M)$;
- the *fallibilist set* ($i > 0$);
- the *dialethiset set*, which says that the intersection of some disjoint sets is not empty (for $t = f = 1$ and $i = 0$; some
paradoxist sets can be denoted this way too);
every element x(T, I, F) of a dialethist set M belongs at the same
time to M and to its complement set C(M);
- the paradoxist set, each element has a part of
indeterminacy if it is or not in the set (i > 1);
- the pseudoparadoxist set (0 < i < 1, t + f > 1);
- the tautological set (i < 0).

Compared with all other types of sets, in the neutrosophic
set each element has three components which are subsets (not
numbers as in fuzzy set) and considers a subset, similarly to
intuitionistic fuzzy set, of "indeterminacy" - due to unexpected
parameters hidden in some sets, and let the superior limits of
the components to even boil over 1 (overflooded) and the
inferior limits of the components to even freeze under 0
(underdried).

For example: an element in some tautological sets may have
t > 1, called "overincluded". Similarly, an element in a
set may be "overdeterminat" (for i > 1, in some paradoxist
sets), "overexcluded" (for f > 1, in some unconditionally false
appurtenance); or "undertrue" (for t < 0, in some
unconditionally false appurtenances), "underdeterminat" (for
i < 0, in some unconditionally true or false appurtenances),
"underfalse" (for f < 0, in some unconditionally true
appurtenances).

This is because we should make a distinction between
unchannelly true (t > 1, and f < 0 or i < 0) and
conditionally true appurtenances (t ≤ 1, and f ≤ 1 or i ≤ 1).

In a rough set RS, an element on its boundary-line cannot
be classified neither as a member of RS nor of its complement
with certainty. In the neutrosophic set a such element may be
characterized by x(T, I, F), with corresponding set-values for
T, I, F ⊆ {0, 1}.

Compared to Belnap's quadruplet logic, NS and NL do not
use restrictions among the components - and that's why the
NS/NL have a more general form, while the middle component
in NS and NL (the indeterminacy) can be split in
more subcomponents if necessarily in various applications.

XI. DISTINCTIONS BETWEEN NEUTROSOPHIC SET (NS) AND
INTUITIONISTIC FUZZY SET (IFS)

a) Neutrosophic Set can distinguish between absolute
membership (i.e. membership in all possible worlds; we have
extended Leibniz's absolute truth to absolute membership) and
relative membership (membership in at least one world but not
in all), because NS(absolut membership element)=1 while
NS(relative membership element)=1. This has application in
philosophy (see the neutrosophy). That's why the unitary
standard interval [0, 1] used in IFS has been extended to the
unitary non-standard interval ]0, 1[' in NS.

Similar distinctions for absolute or relative non-
membership, and absolute or relative indeterminant
appurtenance are allowed in NS.

b) In NS there is no restriction on T, I, F other than they are
subsets of ]0, 1[', thus: "0 ≤ inf T + inf I + inf F ≤ sup T + sup
I + sup F ≤ 3".

The inequalities (2.1) and (2.4) of IFS are relaxed.

This non-restriction allows paraconsistent, dialetheist, and
incomplete information to be characterized in NS {i.e. the sum
of all three components if they are defined as points, or sum of
superior limits of all three components if they are defined as
subsets can be > 1 (for paraconsistent information coming from
different sources), or < 1 for incomplete information}, while
that information can not be described in IFS because in IFS
the components T (membership), I (indeterminacy), F (non-
membership) are restricted either to t+i+f=1 or to t² + f² ≤ 1, if
T, I, F are all reduced to the points t, i, f respectively, or to sup
T + sup I + sup F = 1 if T, I, F are subsets of ]0, 1['.

Of course, there are cases when paraconsistent and
incomplete informations can be normalized to 1, but this
procedure is not always suitable.

c) Relation (2.3) from interval-valued intuitionistic fuzzy
set is relaxed in NS, i.e. the intervals do not necessarily belong
to Int(0,1) but to [0,1], even more general to ]-0, 1+[

d) In NS the components T, I, F can also be non-standard
subsets included in the unitary non-standard interval ]0, 1[',
not only standard subsets included in the unitary standard
interval [0, 1] as in IFS.

e) NS, like dialethism, can describe paradoxist elements,
NS(paradoxist element) = (1, 1, 1), while IFL can not describe
a paradox because the sum of components should be 1 in IFS.

f) The connectors in IFS are defined with respect to T and
F, i.e. membership and non-membership only (hence the
Indeterminacy is what's left from 1), while in NS they can be
defined with respect to any of them (no restriction).

g) Component "I", indeterminacy, can be split into more
subcomponents in order to better catch the vague information
we work with, and such, for example, one can get more
accurate answers to the Question-Answering Systems initiated
by Zadeh (2003). {In Belnap's four-valued logic (1977)
indeterminacy is split into Uncertainty (U) and Contradiction
(C), but they were inter-related.}

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