Differential Operators in the G8,2 Geometric Algebra, DCGA

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Abstract

This paper introduces the differential operators in the $\mathcal{G}_{8,2}$ Geometric Algebra, called the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA). The differential operators are three x, y, and z-direction bivector-valued differential elements and either the commutator product or the anti-commutator product for multiplication into a geometric entity that represents the function to be differentiated. The general form of a function is limited to a Darboux cyclide implicit surface function. Using the commutator product, entities representing 1st, 2nd, or 3rd order partial derivatives in x, y, and z can be produced. Using the anti-commutator product, entities representing the anti-derivation can be produced from 2-vector quadric surface and 4-vector conic section entities. An operator called the pseudo-integral is defined and has the property of raising the x, y, or z degree of a function represented by an entity, but it does not produce a true integral. The paper concludes by offering some basic relations to limited forms of vector calculus and differential equations that are limited to using Darboux cyclide implicit surface functions. An example is given of entity analysis for extracting the parameters of an ellipsoid entity using the differential operators.

Keywords: conformal geometric algebra, calculus, differential operators

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1 Introduction

This paper[1] introduces the DCGA geometric differential operators, which are the fundamental operators of the DCGA geometric differential calculus on the DCGA geometric entities. The DCGA geometric differential calculus is an algebraic calculus, where the differential operators are algebraic operators within the $\mathcal{G}_{8,2}$ Geometric Algebra, DCGA. The derivative of a DCGA *geometric entity* is a DCGA *geometric derivative entity* that represents the geometry associated with the derivative.

The reader is assumed to be familiar with the $\mathcal{G}_{8,2}$ Geometric Algebra, also called the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA), that is introduced in the earlier paper *G8,2 Geometric Algebra, DCGA*[2] [7] by this author.

2 Geometric Algebra

This section is a review of some Geometric Algebra products, identities, and notations that apply to DCGA operations on DCGA entities. For general introductions to Geometric Algebra, there are many books [8][9][10][4][5][13]. The book [10] is the standard reference that first introduced Geometric Algebra and Geometric Calculus.

1. Revised version v3, *December 18, 2015*, uploaded to http://vixra.org/author/robert_b_easter. This version may be superseded at the above link by newer revised versions.

2. The paper *G8,2 Geometric Algebra, DCGA* (*revised version vA*) was uploaded to http://vixra.org/abs/1508.0086 on *October 1, 2015*. The first version v1 is dated *August 11, 2015*.

2.1 Blades, vectors, and multivectors

A vector $\mathbf{a} = \sum a_i \mathbf{e}_i$ is a linear combination of vector elements $\mathbf{e}_i$. An r-blade $\mathbf{A}_r = \mathbf{A}_{\langle r \rangle} = \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_r$ is the outer product of r linearly independent vectors $\mathbf{a}_i$. An r-vector $A_r = A_{\langle r \rangle}$ is a linear combination of r-blades. A multivector A is a linear combination of blades that may be of different grades. An r-versor, or grade-r versor, is the geometric product of r vectors with inverses [10]. A degenerate multivector D has the property $D \cdot D = 0$ and $D^2 \neq 0$, while a null multivector N has the property $N^2 = 0$.

All of the DCGA entities and operators are even-grade multivectors A_+, which simplifies some of the algebra as shown in the next sections. A DCGA point (§3.1)

$$\mathbf{T} = \mathcal{D}(\mathbf{t}) = \mathcal{C}^1(\mathbf{t}_{\mathcal{E}^1})\mathcal{C}^2(\mathbf{t}_{\mathcal{E}^2}) = \mathcal{C}^1(\mathbf{t}_{\mathcal{E}^1}) \wedge \mathcal{C}^2(\mathbf{t}_{\mathcal{E}^2})$$

is a null bivector with square $\mathbf{T}^2 = 0$. All of the DCGA geometric surface entities B_s are s-vectors of even-grade s that are either a 2-vector $B_2 = \mathbf{\Omega}$, 4-vector B_4, 6-vector B_6, or an 8-vector B_8. The B_4, B_6, and B_8 entities are usually intersection entities of a bivector Darboux cyclide entity $\mathbf{\Omega}$ with one, two, or three bivector-valued spheres $\mathbf{S}$ or planes $\mathbf{\Pi}$. The DCGA operators for rotation

$$R = R_{\mathcal{C}^1} R_{\mathcal{C}^2} = R_{\mathcal{C}^1} \wedge R_{\mathcal{C}^2},$$

translation

$$T = T_{\mathcal{C}^1} T_{\mathcal{C}^2} = T_{\mathcal{C}^1} \wedge T_{\mathcal{C}^2},$$

and dilation

$$D = D_{\mathcal{C}^1} D_{\mathcal{C}^2} = D_{\mathcal{C}^1} \wedge D_{\mathcal{C}^2}$$

are 4-versors with scalar, bivector, and 4-vector parts. The DCGA operators for spherical inversion

$$\mathbf{S} = \mathbf{S}_{\mathcal{C}^1} \mathbf{S}_{\mathcal{C}^2} = \mathbf{S}_{\mathcal{C}^1} \wedge \mathbf{S}_{\mathcal{C}^2}$$

and planar reflection

$$\mathbf{\Pi} = \mathbf{\Pi}_{\mathcal{C}^1} \mathbf{\Pi}_{\mathcal{C}^2} = \mathbf{\Pi}_{\mathcal{C}^1} \wedge \mathbf{\Pi}_{\mathcal{C}^2}$$

are 2-versors and are 2-blades, and they are just the standard DCGA bivector-valued sphere $\mathbf{S}$ and plane $\mathbf{\Pi}$ entities. The DCGA differential elements (§3.3)

$$\begin{aligned} D_x &= 2T_x T_{x^2}^{-1} = 2T_x \times T_{x^2}^{-1} \\ D_y &= 2T_y T_{y^2}^{-1} = 2T_y \times T_{y^2}^{-1} \\ D_z &= 2T_z T_{z^2}^{-1} = 2T_z \times T_{z^2}^{-1} \end{aligned}$$

are degenerate bivectors that are each the product of a degenerate bivector $2T_x$, $2T_y$, or $2T_z$ and a 2-blade versor $T_{x^2}^{-1}$, $T_{y^2}^{-1}$, or $T_{z^2}^{-1}$, respectively. The differential elements should not be confused with the dilator D.

2.2 Geometric product

The geometric product of any two multvectors A and B is written as AB, without any product symbol. The geometric product AB can be expanded as the sum of the anti-symmetric commutator product $A\times B$ and the symmetric anti-commutator product $A\bar{\times}B$ as

$$\begin{aligned} AB &= A\times B + A\bar{\times}B \\ &= \frac{1}{2}(AB-BA)+\frac{1}{2}(AB+BA). \end{aligned}$$

The geometric product AB can be expanded as a sum of grade projections

$$AB = \langle AB\rangle + \langle AB\rangle_1 + \langle AB\rangle_2 + \cdots + \langle AB\rangle_n$$

where $n=p+q+r$ is the number of unit vector elements $\mathbf{e}_i\colon 1\le i\le n$ in the algebra $\mathcal{G}_{p,q,r}$ having p Euclidean $\mathbf{e}_i^2=1\colon 1\le i\le p$, q anti-Euclidean $\mathbf{e}_i^2=-1\colon p+1\le i\le p+q$, and r null $\mathbf{e}_i^2=0\colon p+q+1\le i\le n$ elements.

The geometric product A_rB_s of an r-vector A_r and s-vector B_s can be expanded as a sum of specific grade projections

$$\begin{aligned} A_rB_s &= \sum_{l=0}^{\min(r,s)} \langle A_rB_s\rangle_{r+s-2l} \\ &= \langle A_rB_s\rangle_{r+s} + \langle A_rB_s\rangle_{r+s-2} + \cdots + \langle A_rB_s\rangle_{|r-s|}. \end{aligned}$$

The grades $r+s-2l$ of the terms $\langle A_rB_s\rangle_{r+s-2l}$ differ in steps of two grades $2l$ since the product of any two elements $\mathbf{e}_i\mathbf{e}_j=\mathbf{e}_i\cdot\mathbf{e}_j+\mathbf{e}_i\wedge\mathbf{e}_j$ is either the grade-0 scalar $\mathbf{e}_i\cdot\mathbf{e}_j=\mathrm{signature}(\mathbf{e}_i)\colon i=j$ or the grade-2 bivector $\mathbf{e}_i\wedge\mathbf{e}_j\colon i\neq j$. The minimum-grade term is the inner product $A_r\cdot B_s$ of grade $r+s-2l=|r-s|$ using the maximum l

$$\begin{aligned} \max(l) &= \frac{r+s}{2}-\frac{|r-s|}{2}=\min(r,s) \\ \max(r,s) &= \frac{r+s}{2}+\frac{|r-s|}{2}=r+s-\min(r,s). \end{aligned}$$

The maximum-grade term is the outer product $A_r\wedge B_s$ of grade $r+s-2l=r+s$ using the minimum l, which is $\min(l)=0$. The integer range of l is $0\le l\le\min(r,s)$. Integer values of l between min (l) and max (l) expand other products that are terms of the geometric product A_rB_s.

Let $A_r=\sum_{i=1}^m \mathbf{A}_{\langle r\rangle_i}$ and $B_s=\sum_{j=1}^n \mathbf{B}_{\langle s\rangle_j}$ be linear combinations of m r-blades and n s-blades, respectively. For each geometric product $\mathbf{A}_{\langle r\rangle_i}\mathbf{B}_{\langle s\rangle_j}$ of an r-blade $\mathbf{A}_{\langle r\rangle_i}$ and s-blade $\mathbf{B}_{\langle s\rangle_j}$ that is a term of the geometric product A_rB_s, the further expansion of $\langle\mathbf{A}_{\langle r\rangle_i}\mathbf{B}_{\langle s\rangle_j}\rangle_{r+s-2l}$ is given by a formula called the Expansion of the Geometric Product of Blades (EGPB), which is discussed in [12] and [8]. The EGPB formula will not be discussed here, but it can be used to generate identities and formulas for specific products that may be of interest in a detailed analysis of products that is not undertaken here.

For $r=2$ and $s\in\{2,4,6,8\}$, then $l\in\{0,1,2\}$. For these values of r, s, and l, the geometric product A_2B_s can be expanded into the sum of only three products

$$\begin{aligned}
A_2B_s=\sum_l \langle A_2B_s\rangle_{2+s-2l} &= \langle A_2B_s\rangle_{s+2}+\langle A_2B_s\rangle_s+\langle A_2B_s\rangle_{s-2}\\
A_2\wedge B_s &= \langle A_2B_s\rangle_{s+2}\\
A_2\cdot B_s &= \langle A_2B_s\rangle_{s-2}\\
A_2\times B_s &= \langle A_2B_s\rangle_s\\
A_2\bar{\times}B_s &= A_2\cdot B_s+A_2\wedge B_s.
\end{aligned}$$

The commutator product $B'_s=A_2\times B_s$ holds the grade s and is called the *derivation* of B_s. The derivation may be related to the Lie derivative in Lie Algebra, but this possible relation is not explored in this paper. As a DCGA surface entity $B'_s=A_2\times B_s$, the derivation B'_s can represent a derivative function that can be evaluated as $\mathcal{D}(\mathbf{p})\cdot B'_s$ at a DCGA point $\mathbf{P}=\mathcal{D}(\mathbf{p})$, and B'_s is the surface on which derivatives are zero.

The anti-commutator product $\bar{B}'_s=A_2\bar{\times}B_s$ can be called the *anti-derivation* of B_s. The anti-derivation may be related to the exterior derivative or curl, but this possible relation is not fully explored in this paper. As a DCGA surface entity $\bar{B}'_s=A_2\bar{\times}B_s$, the anti-derivation $\bar{B}'_s$ can represent the surface on which the reciprocal and infinite derivatives exist.

The abstract definitions of the terms *derivation* and *anti-derivation* are given in [2]. These terms are used as alternative terminology to describe the DCGA differential operations $B'_s=D_{\mathbf{n}}\times B_s$ and $\bar{B}'_s=D_{\mathbf{n}}\bar{\times}B_s$, where $D_{\mathbf{n}}$ is a DCGA differential element (§3.3) and $D_{\mathbf{n}}\times$ is a DCGA differential operator (§3.4) on a DCGA surface entity B_s that represents an implicit surface function F. The abstract definitions, propositions, corollaries, and examples for derivations given in [2] may be useful in further studies of the DCGA differential operators.

For $r=2$, the inner product $A_2\cdot B_s=B_s\cdot A_2$ and outer product $A_2\wedge B_s=B_s\wedge A_2$ are both symmetric products and are terms of the symmetric anti-commutator product $A_2\bar{\times}B_s=B_s\bar{\times}A_2$. The 2-vector A_2 can be a linear combination $D_{\mathbf{n}}$ of the DCGA degenerate 2-vector differential elements D_x, D_y, and D_z (§3.3). For $A_2=D_{\mathbf{n}}$ and any DCGA extraction element T_s (§3.2), we find that

$$D_{\mathbf{n}}\cdot T_s = 0.$$

For $A_2=\mathcal{D}_{\mathbf{n}}$ and any DCGA 2-vector surface entity $B_2=\boldsymbol{\Omega}$ that is a linear combination of extraction elements T_s, then

$$\begin{aligned}
D_{\mathbf{n}}\cdot\boldsymbol{\Omega} &= 0\\
\mathcal{D}_{\mathbf{n}}\bar{\times}\boldsymbol{\Omega} &= \mathcal{D}_{\mathbf{n}}\wedge\boldsymbol{\Omega}.
\end{aligned}$$

For a DCGA 2-vector Darboux cyclide surface entity $B_s=B_2=\boldsymbol{\Omega}$, the derivation or derivative entity $\boldsymbol{\Omega}'_2=\mathcal{D}_{\mathbf{n}}\times\boldsymbol{\Omega}$ is a sum of differential extraction elements $\mathcal{D}_{\mathbf{n}}\times T_s$.

Using the derivative property or Jacobi identity of the commutator product, then

$$\begin{aligned}
A_2\times B_4 &= D_{\mathbf{n}}\times(\boldsymbol{\Omega}\wedge\boldsymbol{\Pi})\\
&= (D_{\mathbf{n}}\times\boldsymbol{\Omega})\wedge\boldsymbol{\Pi}+\boldsymbol{\Omega}\wedge(D_{\mathbf{n}}\times\boldsymbol{\Pi}).
\end{aligned}$$

For a plane $\boldsymbol{\Pi}^{||\mathbf{n}}$ that contains the direction $\mathbf{n}$, then $D_{\mathbf{n}}\times\boldsymbol{\Pi}^{||\mathbf{n}}=0$ and

$$D_{\mathbf{n}}\times(\boldsymbol{\Omega}\wedge\boldsymbol{\Pi}^{||\mathbf{n}}) = (D_{\mathbf{n}}\times\boldsymbol{\Omega})\wedge\boldsymbol{\Pi}^{||\mathbf{n}}$$

represents the derivative of a DCGA 4-vector cyclidic section $B_4 = \mathbf{\Omega} \wedge \mathbf{\Pi}^{||\mathbf{n}}$ in the direction $\mathbf{n}$ in the plane $\mathbf{\Pi}^{||\mathbf{n}}$.

2.3 Scalar product

The scalar product $A_r * B_s$ is defined as the grade 0 projection or part $\langle A_r B_s\rangle_0$ of the geometric product

$$A_r * B_s \;=\; \langle A_r B_s\rangle = \langle A_r B_s\rangle_0$$

which is the scalar part of the geometric product. The grade g projection operator $\langle A B\rangle_g$ is introduced in [10]. The scalar product is equal to the inner product or contraction product when $r = s$.

2.4 Inner product

The inner product $A_r \cdot B_s$ is defined as the grade $|s - r|$ projection of the geometric product

$$A_r \cdot B_s \;=\; \langle A_r B_s\rangle_{|s-r|}$$

for $r \neq 0$ and $s \neq 0$. The inner product is defined as zero when $r = 0$ and/or $s = 0$.

The commutation of the inner product can affect the sign and is given by

$$B_s \cdot A_{r\leq s} \;=\; (-1)^{r(s-1)} A_{r\leq s} \cdot B_s.$$

In this paper, $A_r = A_+$ and $B_s = B_+$ are always even-grade vectors, denoted generally by subscript $+$ [10], of grades $r, s \in \{2,4,6,8\}$. In this case, the inner product is commutative

$$B_+ \cdot A_+ \;=\; A_+ \cdot B_+$$

and is always a term of the symmetric anti-commutator product $A_r \bar{\times} B_s$.

2.5 Dot product

The dot product $A_r \bullet B_s$ is defined as the grade $|s - r|$ projection of the geometric product

$$A_r \bullet B_s \;=\; \langle A_r B_s\rangle_{|s-r|}$$

for any r and s. The dot product allows scalar multiplication, while the inner product is defined as zero for any scalar multiplication.

The commutation of the dot product can affect the sign and is given by

$$B_s \bullet A_{r\leq s} \;=\; (-1)^{r(s-1)} A_{r\leq s} \bullet B_s.$$

In this paper, $A_r = A_+$ and $B_s = B_+$ are always even-grade vectors, denoted generally by subscript $+$, of grades $r, s \in \{2, 4, 6, 8\}$. In this case, the dot product is commutative

$$B_+ \bullet A_+ \;=\; A_+ \bullet B_+$$

and is always a term of the symmetric anti-commutator product $A_r \bar{\times} B_s$.

2.6 Outer product

The outer product $A_r \wedge B_s$ is defined as the grade $r + s$ projection of the geometric product

$$A_r \wedge B_s \;=\; \langle A_r B_s\rangle_{r+s}.$$

The outer product has the associative property

$$A \wedge B \wedge C = (A \wedge B) \wedge C = A \wedge (B \wedge C)$$

which the geometric product also has, but which other products do not have in general.

The commutation of the outer product can affect the sign and is given by

$$B_s \wedge A_r = (-1)^{sr} A_r \wedge B_s.$$

In this paper, A_r and B_s are always even-grade vectors, denoted generally by subscript $+$, of grades $r, s \in \{2, 4, 6, 8\}$. In this case, the outer product is commutative

$$B_+ \wedge A_+ = A_+ \wedge B_+$$

and is always a term of the symmetric anti-commutator product $A_r \bar{\times} B_s$.

2.7 Left and right contraction products

Chapter 2, *The Inner Products of Geometric Algebra* by Leo Dorst, in [3] introduces the left and right contraction products, and also the dot product. The concepts associated with contractions and duality operations are clearly explained using these products in [5]. The left and right contraction products are nearly the same as the inner product for the purposes considered in this paper.

The left contraction is defined as

$$A_{r \le s} \rfloor B_s = \langle A_{r \le s} B_s \rangle_{s-r}$$

for $r \le s$. The left contraction is defined as zero when $r > s$.

The right contraction is defined as

$$B_s \lfloor A_{r \le s} = \langle B_s A_{r \le s} \rangle_{s-r}$$

for $r \le s$. The right contraction is defined as zero when $r > s$.

The contractions are also defined for scalars where $r = 0$ and/or $s = 0$, while the similar inner products are defined as zero when $r = 0$ and/or $s = 0$.

The relation between left and right contraction is

$$B_s \lfloor A_{r \le s} = (-1)^{r(s-1)} A_{r \le s} \rfloor B_s$$

which follows from the commutation formula for inner products. For any multivectors A and B,

$$A \rfloor B + A \lfloor B = A * B + A \bullet B.$$

2.8 Symmetric anti-commutator product

The symmetric *anti-commutator product* $\bar{\times}$ of any two multivectors A and B is defined as

$$A \bar{\times} B = \frac{1}{2}(AB + BA)$$

which has the symmetric commutative property

$$B \bar{\times} A = A \bar{\times} B.$$

In this paper, the anti-commutator product $\bar{\times}$ is used to define the *symmetric differential operators* as elements of the DCGA algebra. Also in this paper, $A_r = A_2$ will usually be grade-2 and B_s is always an even-grade vector of grade $s \in \{2,4,6,8\}$. In this case, the anti-commutator product is the sum of the commutative inner and outer products

$$\begin{aligned} A_2 \bar{\times} B_s &= \langle A_2 B_s \rangle_{s-2} + \langle A_2 B_s \rangle_{s+2} \\ A_2 \cdot B_s &= \langle A_2 B_s \rangle_{s-2} \\ A_2 \wedge B_s &= \langle A_2 B_s \rangle_{s+2}. \end{aligned}$$

Also in this case, the other part, of grade s, is leftover as the anti-commutative commutator product

$$A_2 \times B_s = \langle A_2 B_s \rangle_s.$$

These are the only products that have to be considered in this paper. The grade-2 vector A_2 will usually be a bivector-valued differential element $D_{\mathbf{n}}$, and B_s will usually be a DCGA GIPNS 2-vector geometric entity $\mathbf{\Omega}$. The product $A_2 \times B_s$ always has the same grade s as B_s and can produce a *differentiated version* of B_s that, as shown later, can actually represent an exact derivative.

2.9 Anti-symmetric commutator product

The anti-symmetric *commutator product* $\times$ of any two multivectors A and B is defined as

$$A \times B = \frac{1}{2}(AB - BA)$$

which has the anti-symmetric *anti-commutative* property

$$B \times A = -A \times B.$$

In this paper, the commutator product $\times$ is used to define the *anti-symmetric differential operators* as elements of the DCGA algebra.

For any multivectors A, B, and C, the commutator product is *linear*, or *distributive* over addition

$$A \times (B + C) = A \times B + A \times C.$$

For any multivectors A, B, and C, the commutator product has the important *derivative identity* (Equation 1.57 in [10])

$$A \times (BC) = (A \times B)C + B(A \times C)$$

and the *Jacobi identity* (Equation 1.56c in [10])

$$A \times (B \times C) = (A \times B) \times C + B \times (A \times C).$$

As we will see later, if $A = D_{\mathbf{n}_1}$ and $B = D_{\mathbf{n}_2}$ are two *bivector differential elements* that differentiate an even-grade entity $C = \mathbf{\Omega}$ successively in the $\mathbf{n}_2$ and $\mathbf{n}_1$ directions, then any differential element operating on another differential element produces zero $D_{\forall \mathbf{n}_1} \times D_{\forall \mathbf{n}_2} = 0$ and the Jacobi identity reduces to

$$D_{\mathbf{n}_1} \times (D_{\mathbf{n}_2} \times \mathbf{\Omega}) = D_{\mathbf{n}_2} \times (D_{\mathbf{n}_1} \times \mathbf{\Omega})$$

which is the expected result of the form

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

A bivector element $D_{\mathbf{n}}$ that satisfies this result is can be a differential element and can define a differential operator $\partial_{\mathbf{n}} = D_{\mathbf{n}} \times$ that can differentiate, in some direction $\mathbf{n}$, certain other elements or entities C of the algebra that may represent functions that can be differentiated as $\partial_{\mathbf{n}} C = D_{\mathbf{n}} \times C$. Later in this paper, a complete set of bivector differential elements are defined to form differential operators that can differentiate functions that are represented by DCGA geometric entities.

For a bivector $A_2 = \langle A \rangle_2$ and any multivectors B and C, other known identities are (Equations 1.65 and 1.66 in [10])

$$\begin{aligned} A_2 \times (B \cdot C) &= (A_2 \times B) \cdot C + B \cdot (A_2 \times C) \\ A_2 \times (B \wedge C) &= (A_2 \times B) \wedge C + B \wedge (A_2 \times C). \end{aligned}$$

Later in the paper, we will see that when $A_2 = D_{\mathbf{n}}$ is a bivector differential element, and $B_2 = \mathbf{P}$ is a bivector DCGA null point entity, and $C_2 = \mathbf{\Omega}$ is a bivector geometric function entity, then the evaluation or test $B_2 \cdot C_2 = \mathbf{P} \cdot \mathbf{\Omega} = d$ is a scalar d and

$$A_2 \times (B_2 \cdot C_2) = D_{\mathbf{n}} \times d = \frac{1}{2}(D_{\mathbf{n}} d - d D_{\mathbf{n}}) = 0$$

and using the identities above we have

$$\begin{aligned} B_2 \cdot (A_2 \times C_2) &= (B_2 \times A_2) \cdot C_2 \\ \mathbf{P} \cdot (D_{\mathbf{n}} \times \mathbf{\Omega}) &= -(D_{\mathbf{n}} \times \mathbf{P}) \cdot \mathbf{\Omega}. \end{aligned}$$

This result says that we can differentiate the entity and then evaluate $\mathbf{\Omega}$ at point $\mathbf{P}$, or we can differentiate the point $\mathbf{P}$ as $-\partial_{\mathbf{n}} \mathbf{P} = -D_{\mathbf{n}} \times \mathbf{P}$ then evaluate $\mathbf{\Omega}$ using the differential point. Since a surface function F represented by an entity $\mathbf{\Omega}_F$ is for an implicit surface $F = 0$ that is evaluated at a point $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ as $F(\mathbf{p}) = \mathbf{P} \cdot \mathbf{\Omega}_F = \mathcal{D}(\mathbf{p}) \cdot \mathbf{\Omega}_F$, then the minus sign on the differential point $-\partial_{\mathbf{n}} \mathbf{P}$ can sometimes be ignored.

For a bivector $A_2 = \langle A \rangle_2$ and any multivector B that has no vector part $\langle B \rangle_1 = 0$ then (Equation 1.63 in [10])

$$A_2 B = A_2 \cdot B + A_2 \times B + A_2 \wedge B.$$

When $A_2 = \langle A \rangle_2$ is a bivector and B_s is an even-grade s 2, 4, 6, or 8-vector, then the geometric product $A_2 B_s$ is the sum of three possible product terms

$$\begin{aligned} A_2 B_s &= \langle A_2 B_s \rangle_{s-2} + \langle A_2 B_s \rangle_s + \langle A_2 B_s \rangle_{s+2} \\ \langle A_2 B_s \rangle_{s+2} &= A_2 \wedge B_s \\ \langle A_2 B_s \rangle_{s-2} &= A_2 \cdot B_s \\ \langle A_2 B_s \rangle_s &= A_2 \times B_s. \end{aligned}$$

We will only be concerned with these products and the identities associated with them. In DCGA, only even-grade vectors of grades 2, 4, 6, and 8 are used for operators and entities.

The *derivative* identity is named after the form of the product rule for differentiating a product of scalar functions fg in the $\mathbf{n}$-direction of a unit vector $\mathbf{n}$

$$\begin{aligned} \partial_{\mathbf{n}}(fg) &= (\partial_{\mathbf{n}} f) g + f(\partial_{\mathbf{n}} g) \\ &= ((\nabla \cdot \mathbf{n}) f) g + f((\nabla \cdot \mathbf{n}) g) \\ &= (\nabla \cdot \mathbf{n} f) g + f(\nabla \cdot \mathbf{n} g) \\ &= (\nabla f \cdot \mathbf{n}) g + f(\nabla g \cdot \mathbf{n}). \end{aligned}$$

Notice how scalar-valued functions f and g generally commute with all other values and products, but they cannot commute into and out of a differential operator. Scalars and scalar-valued functions usually commute in general within an algebra, but the differential operators are not elements of the algebra and are special symbolic operators. The DCGA differential operators defined later in this paper implement differential operators as elements in the algebra that enforce the non-commutativity with respect to the differential operators.

The gradient operator ∇, also called the vector derivative operator, is a symbolic vector-valued operator

$$\begin{aligned}\nabla &= \partial_x\mathbf{e}_1 + \partial_y\mathbf{e}_2 + \partial_z\mathbf{e}_3\\ &= \frac{\partial}{\partial x}\mathbf{e}_1 + \frac{\partial}{\partial y}\mathbf{e}_2 + \frac{\partial}{\partial z}\mathbf{e}_3\end{aligned}$$

where each differential operator ∂_x, ∂_y, and ∂_z is symbolically handled as a scalar.

In a fixed direction such as the $\mathbf{e}_1$-direction that is conventionally assigned to the x variable or x-axis direction, the product rule reduces to

$$\partial_x(fg) = \frac{\partial(fg)}{\partial x} = \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}.$$

In geometric calculus, multivector-valued functions f and g are generally non-commutative and the differential operation must not commute f and g. The overdot notation

$$\partial_{\mathbf{n}}(fg) = \partial_{\mathbf{n}}\dot{f}g + \partial_{\mathbf{n}}f\dot{g}$$

denotes a function having an overdot is the function that is differentiated by the differential operator $\partial_{\mathbf{n}}$.

2.10 Associative and non-associative products

The geometric product has the general associative property

$$ABC = (AB)C = A(BC).$$

The geometric product is an *associative product*.

The commutator $\times$ and anti-commutator $\bar{\times}$ products generally do not have the associative property and are *non-associative products*. By convention, products are evaluated from the left to the right unless enclosed in parentheses. Therefore, in general

$$\begin{aligned}A\times B\times C &= (A\times B)\times C\\ &\neq A\times(B\times C)\\ A\bar{\times}B\bar{\times}C &= (A\bar{\times}B)\bar{\times}C\\ &\neq A\bar{\times}(B\bar{\times}C).\end{aligned}$$

There may be exceptions to the general case, where particular associative products exist, but these particular exceptions are not considered here.

The non-associative property of the commutator and anti-commutator products is important in this paper. To obtain correct results, it will usually be required to enclose these products in parentheses. Partial derivative computations such as

$$\begin{aligned}\frac{\partial^2 C}{\partial x\partial y} &= D_x\times(D_y\times C) = D_y\times(D_x\times C)\\ &\neq (D_x\times D_y)\times C\end{aligned}$$

will be defined in the next sections.

3 DCGA geometric differential calculus

In this section, a type of differential calculus, which is being called here the *DCGA geometric differential calculus*, is introduced. The DCGA geometric differential calculus is the work of a continued independent research by this author that adds new results to previous work on DCGA [7]. No prior works that offer these specific results were consulted or known to this author at the time of research into this paper. The results presented in this section may be new results that are being introduced for the first time into the literature.

In standard differential calculus, derivatives are defined as certain limits that are evaluated and simplified to obtain expressions that represent tangents to curves and surfaces. Generalizing on the results of limits, the familiar *rules* for differentiation are derived, memorized, and applied to *form* derivatives, without evaluation of limits and without *algebraic operations* for *computing* derivatives.

In the DCGA geometric differential calculus, derivatives of polynomial implicit surface functions $F(x,\,y,\,z)$, which can represent Darboux cyclides, are produced and represented as certain algebraic products of differential operators with geometric entities. The DCGA differential operators are elements of the DCGA algebra that *compute* derivatives, without using the *rules* for differentiation that *form* derivatives.

The DCGA differential operators are of two complementary orthogonal types, which are the DCGA *anti-symmetric differential operators* and the DCGA *symmetric differential operators*. The DCGA anti-symmetric differential operators $D_x\times$, $D_y\times$, and $D_z\times$ are the primary differential operators that correspond to the differential operators D_x, D_y, and D_z of standard differential calculus, and they can be related to the divergence operator (div or $\nabla\cdot$) of vector analysis. The DCGA symmetric differential operators $D_x\bar{\times}$, $D_y\bar{\times}$, and $D_z\bar{\times}$ can be related to the circulation operator (curl or $\nabla\times$) of vector analysis. The sums $D_x = D_x\times + D_x\bar{\times}$, $D_y = D_y\times + D_y\bar{\times}$, and $D_z = D_z\times + D_z\bar{\times}$ of DCGA anti-symmetric and symmetric differential operators can be related to the gradient operator (grad or ∇) of vector analysis. A weighted sum of differential elements $D_{\mathbf{n}} = (n_x D_x + n_y D_y + n_z D_z)$ forms a directional derivative element in a unit direction $\mathbf{n} = n_x\mathbf{e}_1 + n_y\mathbf{e}_2 + n_z\mathbf{e}_3$ and the directional derivation $D_{\mathbf{n}}\times$ and anti-derivation $D_{\mathbf{n}}\bar{\times}$ operators.

The relations to vector analysis operations are not direct since both the DCGA geometric entities and the DCGA differential operators are not vectors. The relations to vector analysis are also limited to using polynomial functions $F(x,\,y,\,z)$ in the general form of Darboux cyclide implicit surface functions. The relations to vector analysis will be considered later as examples, but the derivatives of DCGA geometric entities have a geometrical significance and a geometrical representation that are different than the derivatives of ordinary scalar-valued functions $F(x,\,y,\,z)$ and vector-valued functions $\mathbf{F}(x,y,z)$ in vector analysis.

The DCGA differential operators could also be called DCGA *geometric differential operators*, and when these operators act on DCGA geometric entities they produce derivative entities that could be called DCGA *geometric derivative entities*.

There two types of DCGA geometric derivative entities, which are the DCGA anti-symmetric derivative entities and the DCGA symmetric derivative entities. A DCGA *anti-symmetric derivative entity* $B' = D_{\mathbf{n}}\times B$ is produced by applying a DCGA anti-symmetric differential operator $D_{\mathbf{n}}\times$ to a DCGA geometric entity B. A DCGA *symmetric derivative entity* $\bar{B}' = D_{\mathbf{n}}\bar{\times}B$ is produced by applying a DCGA symmetric differential operator $D_{\mathbf{n}}\bar{\times}$ to a DCGA geometric entity B.

Using abstract terminology for *derivations* and *anti-derivations* [2], B' can be called a *derivation entity*, and $\bar{B}'$ can be called an *anti-derivation entity*. Using more abstract terminology, it may also be appropriate to call these entities function objects, function entities, or *functors* when the entities are seen as representing functions rather than surfaces. If a DCGA geometric derivative entity has zero-points, then they represent an implicit surface of those points.

The intersection of a GIPNS surface B with its *anti-symmetric derivative* surface B' is the set of surface points $\mathbf{P}$ on B where tangents are *parallel* to the direction $\mathbf{n}$ of the derivative, and where $\mathbf{P}\cdot B'=0$. The intersection of a GIPNS quadric surface or conic section B with its *symmetric derivative* surface $\bar{B}'$ is the set of surface points $\mathbf{P}$ on B where tangents are *perpendicular* to the direction $\mathbf{n}$ of the derivative, and where $\mathbf{P}\cdot\bar{B}'=0$ but $\mathbf{P}\cdot B'=\infty$. For any surface B, the surface B' is also being called a 0-derivative surface or derivation surface. While restricted to a quadric surface or conic section B, the surface $\bar{B}'$ is also being called a ∞-derivative surface or anti-derivation surface.

In this paper, the interpretation of the symmetric derivative entity $\bar{B}'$ is limited to the case where B is a quadric surface or conic section entity. The meaning or significance of $\bar{B}'$ is undefined in this paper when B is any entity other than a quadric surface or conic section.

The anti-symmetric derivative entity B' is valid for any DCGA entity B, and its meaning is clearly the derivative surface for all entities that are formed directly from the extraction entities, which are reviewed next. However, if B is a standard(*bi-CGA*) DCGA sphere or plane, then the derivative surface B' may not represent the derivative that is expected since these entities represent the squared implicit surface function F^2, up to a scalar multiple, and then the derivative surface $B'=D_{\mathbf{n}}\times B$ represents the implicit surface function $2F\partial_{\mathbf{n}}F$, not $\partial_{\mathbf{n}}F$ as may be expected.

3.1 DCGA points

The $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) is introduced in [7]. This section defines $\mathcal{G}_{8,2}$ DCGA and gives the definitions of CGA1, CGA2, and DCGA points in $\mathcal{G}_{8,2}$ DCGA.

$\mathcal{G}_{8,2}$ DCGA uses the ten unit vector elements $\mathbf{e}_i$: $1\leq i\leq 10$ with signatures

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i\in\{1,2,3,4,6,7,8,9\} \\ -1 & : i\in\{5,10\}. \end{cases}$$

The elements $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ are the elements of a $\mathcal{G}_3$ Algebra of Physical Space (APS) [9] that is called Euclidean1 and denoted $\mathcal{E}^1$. A Euclidean1 test vector $\mathbf{t}=\mathbf{t}_{\mathcal{E}^1}$ is

$$\mathbf{t}=\mathbf{t}_{\mathcal{E}^1} = x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3.$$

The elements $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{e}_5$ are the elements of a $\mathcal{G}_{4,1}$ Conformal Geometric Algebra (CGA) [15]

$$\begin{aligned} \mathbf{e}_{o1} &= \frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_5) \\ \mathbf{e}_{\infty 1} &= (\mathbf{e}_4+\mathbf{e}_5) \end{aligned}$$

that is called CGA1 and denoted $\mathcal{C}^1$. A CGA1 null point $\mathbf{T}_{\mathcal{C}^1}$ is defined as

$$\mathbf{T}_{\mathcal{C}^1}=\mathcal{C}^1(\mathbf{t}) = \mathbf{t}+\frac{1}{2}\mathbf{t}^2\mathbf{e}_{\infty 1}+\mathbf{e}_{o1}.$$

The elements $\mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$ are the elements of another $\mathcal{G}_3$ APS that is called Euclidean2 and denoted $\mathcal{E}^2$. A Euclidean2 test vector $\mathbf{t}_{\mathcal{E}^2}$ is

$$\mathbf{t}_{\mathcal{E}^2} = x\mathbf{e}_6 + y\mathbf{e}_7 + z\mathbf{e}_8.$$

The elements $\mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_9, \mathbf{e}_{10}$ are the elements of another $\mathcal{G}_{4,1}$ CGA

$$\begin{aligned} \mathbf{e}_{o2} &= \frac{1}{2}(-\mathbf{e}_9 + \mathbf{e}_{10}) \\ \mathbf{e}_{\infty 2} &= (\mathbf{e}_9 + \mathbf{e}_{10}) \end{aligned}$$

that is called CGA2 and denoted $\mathcal{C}^2$. A CGA2 null point $\mathbf{T}_{\mathcal{C}^2}$ is defined as

$$\mathbf{T}_{\mathcal{C}^2} = \mathcal{C}^2(\mathbf{t}_{\mathcal{E}^2}) = \mathbf{t}_{\mathcal{E}^2} + \frac{1}{2}\mathbf{t}_{\mathcal{E}^2}^2 \mathbf{e}_{\infty 2} + \mathbf{e}_{o2}.$$

The DCGA null *point* $\mathbf{T} = \mathbf{T}_{\mathcal{D}}$ is defined as

$$\mathbf{T} = \mathbf{T}_{\mathcal{D}} = \mathcal{D}(\mathbf{t}) = \mathcal{C}^1(\mathbf{t}_{\mathcal{E}^1}) \wedge \mathcal{C}^2(\mathbf{t}_{\mathcal{E}^2})$$

where $\mathbf{t}_{\mathcal{E}^2} = (\mathbf{t}\cdot\mathbf{e}_1)\mathbf{e}_6 + (\mathbf{t}\cdot\mathbf{e}_2)\mathbf{e}_7 + (\mathbf{t}\cdot\mathbf{e}_3)\mathbf{e}_8$. The DCGA points for the origin and infinity are

$$\begin{aligned} \mathbf{e}_o &= \mathbf{e}_{o1} \wedge \mathbf{e}_{o2} \\ \mathbf{e}_\infty &= \mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}. \end{aligned}$$

3.2 DCGA value-extraction elements

The DCGA point *value-extraction elements* or operators are

$$\begin{aligned} T_x &= \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_6) \\ T_y &= \frac{1}{2}(\mathbf{e}_2 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_7) \\ T_z &= \frac{1}{2}(\mathbf{e}_3 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_8) \end{aligned}$$

$$\begin{aligned} T_{xy} &= \frac{1}{2}(\mathbf{e}_7 \wedge \mathbf{e}_1 + \mathbf{e}_6 \wedge \mathbf{e}_2) \\ T_{yz} &= \frac{1}{2}(\mathbf{e}_7 \wedge \mathbf{e}_3 + \mathbf{e}_8 \wedge \mathbf{e}_2) \\ T_{zx} &= \frac{1}{2}(\mathbf{e}_8 \wedge \mathbf{e}_1 + \mathbf{e}_6 \wedge \mathbf{e}_3) \end{aligned}$$

$$\begin{aligned} T_{x^2} &= \mathbf{e}_6 \wedge \mathbf{e}_1 \\ T_{y^2} &= \mathbf{e}_7 \wedge \mathbf{e}_2 \\ T_{z^2} &= \mathbf{e}_8 \wedge \mathbf{e}_3 \end{aligned}$$

$$\begin{aligned} T_{x\mathbf{t}^2} &= (\mathbf{e}_1 \wedge \mathbf{e}_{o2}) + (\mathbf{e}_{o1} \wedge \mathbf{e}_6) \\ T_{y\mathbf{t}^2} &= (\mathbf{e}_2 \wedge \mathbf{e}_{o2}) + (\mathbf{e}_{o1} \wedge \mathbf{e}_7) \\ T_{z\mathbf{t}^2} &= (\mathbf{e}_3 \wedge \mathbf{e}_{o2}) + (\mathbf{e}_{o1} \wedge \mathbf{e}_8) \end{aligned}$$

$$\begin{aligned} T_1 &= -(\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}) = -\mathbf{e}_\infty \\ T_{\mathbf{t}^2} &= -(\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_{\infty 2}) \\ T_{\mathbf{t}^4} &= -4(\mathbf{e}_{o1} \wedge \mathbf{e}_{o2}) = -4\mathbf{e}_o \end{aligned}$$

and these were first introduced in [7], where they are used to define many DCGA GIPNS 2-vector geometric surface entities.

Using the symbolic DCGA null *point* entity $\mathbf{T} = \mathcal{D}(\mathbf{t})$ that represents the symbolic test point $\mathbf{t} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ in Euclidean 3D space, the value s is extracted from any test point $\mathbf{T}$ as $s = \mathbf{T} \cdot T_s$.

The extraction elements T_s are used to define the DCGA GIPNS 2-vector surface entities that can represent implicit surface functions $F(x, y, z)$, where the most general implicit surface is a Darboux cyclide

$$\begin{aligned} F(x,y,z) &= A\mathbf{t}^4 + B\mathbf{t}^2 + \\ &\quad Cx\mathbf{t}^2 + Dy\mathbf{t}^2 + Ez\mathbf{t}^2 + \\ &\quad Fx^2 + Gy^2 + Hz^2 + \\ &\quad Ixy + Jyz + Kzx + \\ &\quad Lx + My + Nz + O. \end{aligned}$$

The vector $\mathbf{t} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ is a test point and the $A...O$ are 15 real scalar constants. The surface is the set of points where $F(x, y, z) = 0$. When some of the constants $A...O$ are zero, the degenerate surfaces include Dupin cyclides, parabolic cyclides, and quadric surfaces.

The DCGA GIPNS 2-vector *Darboux cyclide* surface entity $\mathbf{\Omega}$ represents an implicit surface $F(x, y, z) = 0$, and it is defined as a linear combination of extraction elements as

$$\begin{aligned} \mathbf{\Omega} &= AT_{\mathbf{t}^4} + BT_{\mathbf{t}^2} + \\ &\quad CT_{x\mathbf{t}^2} + DT_{y\mathbf{t}^2} + ET_{z\mathbf{t}^2} + \\ &\quad FT_{x^2} + GT_{y^2} + HT_{z^2} + \\ &\quad IT_{xy} + JT_{yz} + KT_{zx} + \\ &\quad LT_x + MT_y + NT_z + OT_1. \end{aligned}$$

3.3 DCGA differential elements

The extraction elements T_{x^2}, T_{y^2}, and T_{z^2} are the only ones that have inverses

$$\begin{aligned} 1/T_{x^2} = T_{x^2}^{-1} &= \mathbf{e}_1 \wedge \mathbf{e}_6 \\ 1/T_{y^2} = T_{y^2}^{-1} &= \mathbf{e}_2 \wedge \mathbf{e}_7 \\ 1/T_{z^2} = T_{z^2}^{-1} &= \mathbf{e}_3 \wedge \mathbf{e}_8. \end{aligned}$$

These three inverses can be used to define the following three ratios.

The DCGA *differential elements* D_x, D_y, and D_z are defined as

$$\begin{aligned} D_x &= 2T_x/T_{x^2} = 2T_xT_{x^2}^{-1} \\ &= \mathbf{e}_1 \wedge (\mathbf{e}_4 + \mathbf{e}_5) + \mathbf{e}_6 \wedge (\mathbf{e}_9 + \mathbf{e}_{10}) \\ &= \mathbf{e}_1 \wedge \mathbf{e}_{\infty 1} + \mathbf{e}_6 \wedge \mathbf{e}_{\infty 2} \end{aligned}$$

$$\begin{aligned} D_y &= 2T_y/T_{y^2} = 2T_yT_{y^2}^{-1} \\ &= \mathbf{e}_2 \wedge (\mathbf{e}_4 + \mathbf{e}_5) + \mathbf{e}_7 \wedge (\mathbf{e}_9 + \mathbf{e}_{10}) \\ &= \mathbf{e}_2 \wedge \mathbf{e}_{\infty 1} + \mathbf{e}_7 \wedge \mathbf{e}_{\infty 2} \end{aligned}$$

$$\begin{aligned} D_z &= 2T_z/T_{z^2} = 2T_zT_{z^2}^{-1} \\ &= \mathbf{e}_3 \wedge (\mathbf{e}_4 + \mathbf{e}_5) + \mathbf{e}_8 \wedge (\mathbf{e}_9 + \mathbf{e}_{10}) \\ &= \mathbf{e}_3 \wedge \mathbf{e}_{\infty 1} + \mathbf{e}_8 \wedge \mathbf{e}_{\infty 2}. \end{aligned}$$

Using the anti-symmetric *commutator product* $D_{\mathbf{n}} \times T_s$, where $D_{\mathbf{n}}$ is one of the three differential elements D_x, D_y, or D_z and T_s is the extraction element for value s, the product table of all such products $D_{\mathbf{n}} \times T_s$ is computed as shown in Table 1.

$\times$	T_1	T_x	T_y	T_z	T_{x^2}	T_{y^2}	T_{z^2}	T_{xy}	T_{yz}	T_{zx}	$T_{\mathbf{t}^2}$	$T_{x\mathbf{t}^2}$	$T_{y\mathbf{t}^2}$	$T_{z\mathbf{t}^2}$	$T_{\mathbf{t}^4}$
D_x	0	T_1	0	0	$2T_x$	0	0	T_y	0	T_z	$2T_x$	$2T_{x^2}+T_{\mathbf{t}^2}$	$2T_{xy}$	$2T_{zx}$	$4T_{x\mathbf{t}^2}$
D_y	0	0	T_1	0	0	$2T_y$	0	T_x	T_z	0	$2T_y$	$2T_{xy}$	$2T_{y^2}+T_{\mathbf{t}^2}$	$2T_{yz}$	$4T_{y\mathbf{t}^2}$
D_z	0	0	0	T_1	0	0	$2T_z$	0	T_y	T_x	$2T_z$	$2T_{zx}$	$2T_{yz}$	$2T_{z^2}+T_{\mathbf{t}^2}$	$4T_{z\mathbf{t}^2}$

Table 1. Differential operations $D_{\mathbf{n}} \times$ on extraction elements T_s

As shown in Table 1, the operations $D_{\mathbf{n}} \times T_s$ produce the correct *derivative extraction elements* that extract derivative values $\partial_{\mathbf{n}} s = \mathbf{T} \cdot (D_{\mathbf{n}} \times T_s)$ in the direction $\mathbf{n}$.

It can be verified that all inner products of any differential element $D_{\mathbf{n}}$ with any extraction element T_s are equal to zero

$$D_{\mathbf{n}} \cdot T_s \;=\; 0.$$

and then

$$D_{\mathbf{n}} \bar{\times} T_s \;=\; D_{\mathbf{n}} \cdot T_s + D_{\mathbf{n}} \wedge T_s = D_{\mathbf{n}} \wedge T_s.$$

The DCGA GIPNS 2-vector surface entities B_2 are defined as linear combinations of the extraction elements T_s, and any such surface entity B_2 that is operated on by an operator $D_{\mathbf{n}} \times$ is transformed into the *derivative entity* $D_{\mathbf{n}} \times B_2$ that is differentiated in the direction $\mathbf{n} = n_x \mathbf{e}_1 + n_y \mathbf{e}_2 + n_z \mathbf{e}_3$. The operator $D_{\mathbf{n}} \times = (n_x D_x + n_y D_y + n_z D_z) \times$ is a *differential operator* on DCGA entities in the direction $\mathbf{n}$.

3.4 DCGA anti-symmetric differential operators

The conventional x, y, and z *differential operators* $\frac{\partial}{\partial x} = D_x$, $\frac{\partial}{\partial y} = D_y$, and $\frac{\partial}{\partial z} = D_z$ of standard calculus are represented in DCGA by the DCGA x, y, and z *anti-symmetric differential operators* $D_x \times$, $D_y \times$, and $D_z \times$ that are defined as

$$\begin{aligned}
\frac{\partial}{\partial x} = \partial_x = D_x \times \;&=\; (2T_x / T_{x^2}) \times = (2T_x T_{x^2}^{-1}) \times \\
\frac{\partial}{\partial y} = \partial_y = D_y \times \;&=\; (2T_y / T_{y^2}) \times = (2T_y T_{y^2}^{-1}) \times \\
\frac{\partial}{\partial z} = \partial_z = D_z \times \;&=\; (2T_z / T_{z^2}) \times = (2T_z T_{z^2}^{-1}) \times .
\end{aligned}$$

The symbols, such as $\frac{\partial}{\partial x}$ and ∂_x, are alternative notations for the algebraic operators, such as $D_x \times$.

The DCGA *differential elements* D_x, D_y, and D_z are the left-hand side (LHS) operands of the commutator product $\times$, and they are defined as the ratios of certain DCGA point *value-extraction elements* T_s, which were first introduced and defined in [7]. The right-hand side (RHS) operand of one of the DCGA anti-symmetric differential operators should be the DCGA GIPNS entity B, or its dual DCGA GOPNS entity $B^{*\mathcal{D}}$, that is to be differentiated. Swapping the LHS and RHS is anti-commutative such that

$$B \times D_{\mathbf{n}} \;=\; -D_{\mathbf{n}} \times B$$

where $D_{\mathbf{n}}$ is one of the DCGA differential elements D_x, D_y, or D_z, and B is a DCGA GIPNS or GOPNS entity. The vector $\mathbf{n}$ represents the direction of the derivative, where the x, y, z-directions conventionally correspond to the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$-directions, respectively, in a Euclidean 3D space.

Given any DCGA GIPNS 2-vector *surface* entity $\mathbf{\Omega}$ as defined in [7], then the following DCGA GIPNS 2-vector *0-derivative surface* entities $\partial_x\mathbf{\Omega}$, $\partial_y\mathbf{\Omega}$, and $\partial_z\mathbf{\Omega}$ are computed as

$$\begin{aligned}\frac{\partial\mathbf{\Omega}}{\partial x}=\partial_x\mathbf{\Omega}=D_x\times\mathbf{\Omega} &= (2T_x/T_{x^2})\times\mathbf{\Omega}\\ \frac{\partial\mathbf{\Omega}}{\partial y}=\partial_y\mathbf{\Omega}=D_y\times\mathbf{\Omega} &= (2T_y/T_{y^2})\times\mathbf{\Omega}\\ \frac{\partial\mathbf{\Omega}}{\partial z}=\partial_z\mathbf{\Omega}=D_z\times\mathbf{\Omega} &= (2T_z/T_{z^2})\times\mathbf{\Omega}.\end{aligned}$$

The DCGA surface entity $\mathbf{\Omega}$ represents a scalar field or implicit surface function $F(x, y, z)$, which can represent any Darboux cyclide or any degenerate, such as a Dupin cyclide, parabolic cyclide, or quadric surface. The derivative surface entities $\partial_x\mathbf{\Omega}$, $\partial_y\mathbf{\Omega}$, and $\partial_z\mathbf{\Omega}$ represent the partial derivatives of an implicit surface function F in different x, y, and z directions, and they can represent surfaces or curves on which those derivatives are *zero*. At the points where a surface entity intersects its derivative surface entity, the surface tangent has a zero slope or zero derivative relative to the direction in which the derivative was taken, such that the surface tangent is parallel to the direction of the derivative.

Given any DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity $\boldsymbol{\psi}=\mathbf{\Omega}\wedge\mathbf{\Pi}$ as defined in [6], then the following DCGA GIPNS 4-vector *0-derivative curve* entities $\partial_x\boldsymbol{\psi}$, $\partial_y\boldsymbol{\psi}$, and $\partial_z\boldsymbol{\psi}$ are computed as

$$\begin{aligned}\frac{\partial\boldsymbol{\psi}}{\partial x}=\partial_x\boldsymbol{\psi}=D_x\times\boldsymbol{\psi} &= (2T_x/T_{x^2})\times\boldsymbol{\psi}\\ \frac{\partial\boldsymbol{\psi}}{\partial y}=\partial_y\boldsymbol{\psi}=D_y\times\boldsymbol{\psi} &= (2T_y/T_{y^2})\times\boldsymbol{\psi}\\ \frac{\partial\boldsymbol{\psi}}{\partial z}=\partial_z\boldsymbol{\psi}=D_z\times\boldsymbol{\psi} &= (2T_z/T_{z^2})\times\boldsymbol{\psi}.\end{aligned}$$

The derivative entities $\partial_x\boldsymbol{\psi}$, $\partial_y\boldsymbol{\psi}$, and $\partial_z\boldsymbol{\psi}$ can represent other coplanar curves on which the derivatives are *zero*.

The DCGA anti-symmetric differential operators can be applied successively to produce entities representing higher-order mixed partial derivatives of an implicit surface function F that is represented by a geometric entity $\mathbf{\Omega}$. As examples, the following are valid derivative surface entities:

$$\begin{aligned}\frac{\partial^2\mathbf{\Omega}}{\partial x^2} &= D_x\times(D_x\times\mathbf{\Omega})\\ \frac{\partial^2\mathbf{\Omega}}{\partial x\partial y} &= D_x\times(D_y\times\mathbf{\Omega})=D_y\times(D_x\times\mathbf{\Omega})\\ \frac{\partial^3\mathbf{\Omega}}{\partial x^2\partial y} &= D_x\times(D_x\times(D_y\times\mathbf{\Omega}))=D_x\times(D_y\times(D_x\times\mathbf{\Omega}))=D_y\times(D_x\times(D_x\times\mathbf{\Omega})).\end{aligned}$$

The sequence in which the partial derivatives are taken does not affect the result.

Using a symbolic computer algebra system (CAS), such as the *Geometric Algebra Module for Sympy* [1], the DCGA anti-symmetric differential operators can be defined and then tested on any DCGA entity to check that the correct derivative entities are produced. The derivative entities are produced by using the DCGA differential elements with commutator products, without using differentiation rules or limits.

3.5 DCGA anti-symmetric directional derivative operator

The DCGA anti-symmetric differential operators $D_x\times$, $D_y\times$, and $D_z\times$ are *directional derivative* operators in the fixed x ($\mathbf{e}_1$), y ($\mathbf{e}_2$), and z ($\mathbf{e}_3$) directions, respectively. The DCGA anti-symmetric directional derivative operators correspond to the directional derivative operators of vector calculus. The general $\mathbf{n}$-directional derivative operator, in the direction of a Euclidean 3D vector $\mathbf{n}$, can be formed as a weighted sum of these operators.

The DCGA *anti-symmetric directional derivative operator* $D_{\mathbf{n}}\times$ in the direction of a unit vector $\mathbf{n} = n_x\mathbf{e}_1 + n_y\mathbf{e}_2 + n_z\mathbf{e}_3$ can be defined as

$$\partial_{\mathbf{n}} = D_{\mathbf{n}}\times \;\; = \;\; ((\mathbf{n}\cdot\mathbf{e}_1)D_x + (\mathbf{n}\cdot\mathbf{e}_2)D_y + (\mathbf{n}\cdot\mathbf{e}_3)D_z)\times\,.$$

For any DCGA GIPNS entity $\mathbf{\Omega}$ representing an implicit surface function $F(x,y,z)$, the DCGA GIPNS anti-symmetric directional derivative entity $\partial_{\mathbf{n}}\mathbf{\Omega} = D_{\mathbf{n}}\times\mathbf{\Omega}$ can represent the surface or curve, if it exists, on which $\partial_{\mathbf{n}}F = (\nabla\cdot\mathbf{n})F = 0$ in standard calculus. The zero-points on the surface of $\partial_{\mathbf{n}}\mathbf{\Omega}$ that intersect the surface $\mathbf{\Omega}$ are the points where $\mathbf{n}$ is tangent to the surface $\mathbf{\Omega}$.

Any DCGA GOPNS entity $\mathbf{\Omega}^{*\mathcal{D}}$ may also be differentiated by the same operator $D_{\mathbf{n}}\times$ to produce a DCGA GOPNS anti-symmetric directional derivative entity $\partial_{\mathbf{n}}\mathbf{\Omega}^{*\mathcal{D}}$.

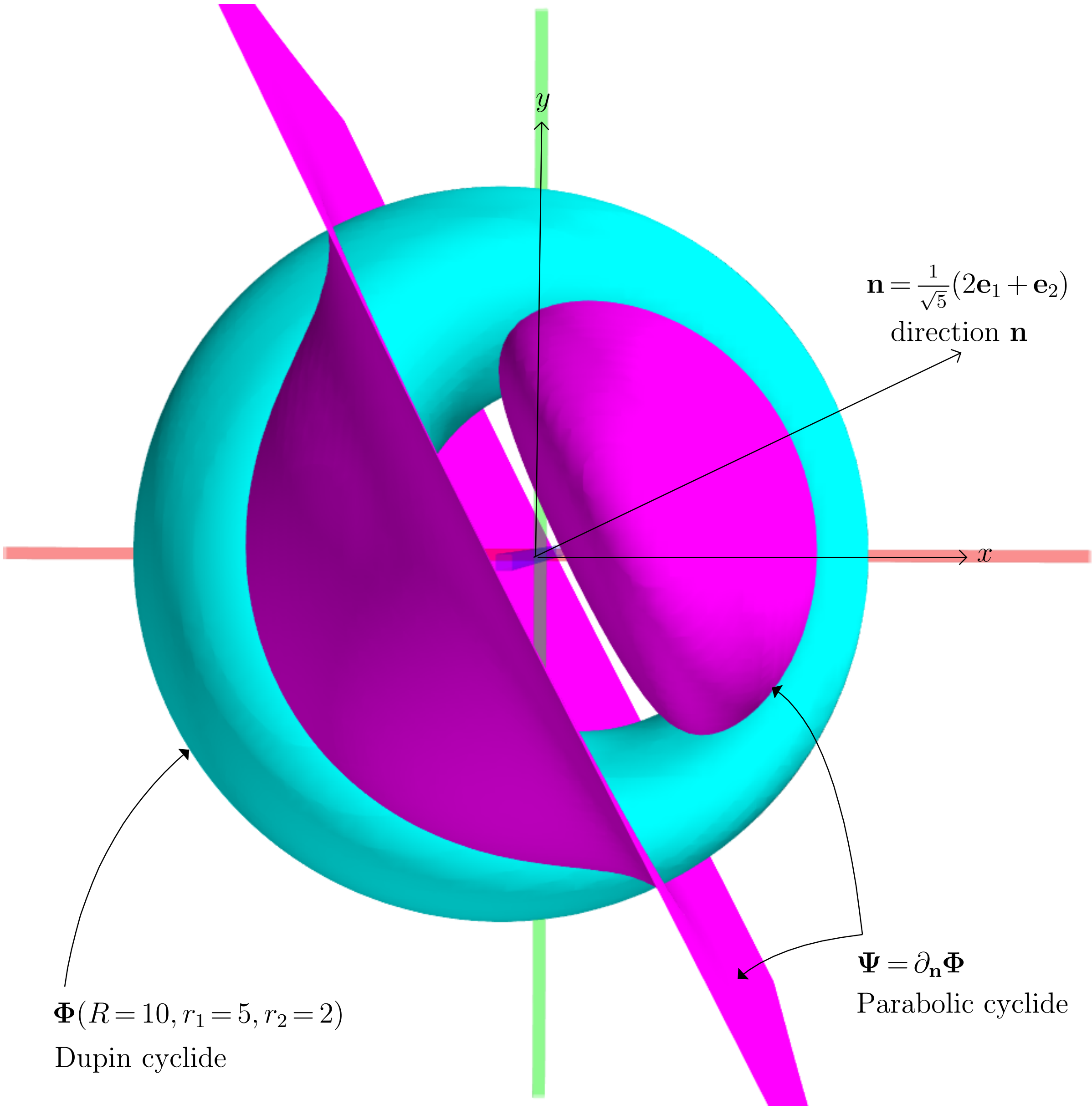

Figure 1. Derivative $\mathbf{\Psi} = \partial_{\mathbf{n}}\mathbf{\Phi}$ of Dupin cyclide $\mathbf{\Phi}$ in direction $\mathbf{n}$

Figure 1 shows the $\mathbf{n}$-directional derivative $\boldsymbol{\Psi} = \partial_{\mathbf{n}}\boldsymbol{\Phi}$ of a Dupin cyclide $\boldsymbol{\Phi}$ with major radius $R = 10$ and minor radii $r_1 = 5$ and $r_2 = 2$. See reference [7] for full definitions of Dupin cyclide $\boldsymbol{\Phi}$ and parabolic cyclide $\boldsymbol{\Psi}$. The Dupin cyclide $\boldsymbol{\Phi}$ represents an implicit surface function of polynomial degree 4. The derivative entity $\boldsymbol{\Psi} = \partial_{\mathbf{n}}\boldsymbol{\Phi}$ represents a parabolic cyclide implicit surface function of polynomial degree 3. The intersection of the surface $\boldsymbol{\Phi}$ and its derivative surface $\partial_{\mathbf{n}}\boldsymbol{\Phi}$ is the set of points $\{\ \mathbf{T}\ :\ \mathbf{T}\cdot\partial_{\mathbf{n}}\boldsymbol{\Phi} = 0, \mathbf{T}\cdot\boldsymbol{\Phi} = 0\ \}$ where surface tangents on $\boldsymbol{\Phi}$ are parallel to $\mathbf{n}$. This figure was produced using *Mayavi* [14], which is available in the *Anaconda* and *SciPy* python distributions. Geometric Algebra calculations in DCGA were performed using the *Geometric Algebra Module for Sympy* [1].

3.6 DCGA symmetric differential operators

The DCGA *symmetric differential operators* can be defined as

$$\begin{aligned}
\bar{\partial}_x = D_x = D_x\bar{\times} &= (2T_x/T_{x^2})\bar{\times} = (2T_xT_{x^2}^{-1})\bar{\times}\\
\bar{\partial}_y = D_y = D_y\bar{\times} &= (2T_y/T_{y^2})\bar{\times} = (2T_yT_{y^2}^{-1})\bar{\times}\\
\bar{\partial}_z = D_z = D_z\bar{\times} &= (2T_z/T_{z^2})\bar{\times} = (2T_zT_{z^2}^{-1})\bar{\times}.
\end{aligned}$$

The DCGA symmetric differential operators are known to be valid on any DCGA GIPNS 2-vector *quadric surface* entity $B = \mathbf{Q}$, and also on any DCGA GIPNS 4-vector *conic section* entity $B = \mathbf{Q}\wedge\boldsymbol{\Pi}$ cut from a quadric surface $\mathbf{Q}$ by a standard DCGA plane $\boldsymbol{\Pi}$. The possible meanings of products that are produced by the DCGA symmetric differential operators applied to other DCGA entities could be researched in future work.

For a DCGA GIPNS 2-vector *quadric surface* entity $B = \mathbf{Q}$ or DCGA GIPNS 4-vector *conic section* entity $B = \mathbf{Q}\wedge\boldsymbol{\Pi}$ representing an implicit surface function $F(x, y, z)$, the symmetric ∞*-derivative surface entity* $\bar{\partial}_{\mathbf{n}}B$ can represent the surface or curve of points $\mathbf{T}$ on which the anti-symmetric *0-derivative surface entity* gives $(\mathbf{T}\times D_{\mathbf{n}})\cdot B = \infty$, or where $\partial_{\mathbf{n}}F = \infty$ in standard calculus.

3.7 DCGA symmetric directional derivative operator

The DCGA *symmetric directional derivative operator* $D_{\mathbf{n}}\bar{\times}$ in the direction of a unit vector $\mathbf{n} = n_x\mathbf{e}_1 + n_y\mathbf{e}_2 + n_z\mathbf{e}_3$ can be defined as

$$\bar{\partial}_{\mathbf{n}} = D_{\mathbf{n}}\bar{\times} = ((\mathbf{n}\cdot\mathbf{e}_1)D_x + (\mathbf{n}\cdot\mathbf{e}_2)D_y + (\mathbf{n}\cdot\mathbf{e}_3)D_z)\bar{\times}$$

The DCGA symmetric directional derivative operator is known to be valid on any DCGA GIPNS 2-vector *quadric surface* entity $B = \mathbf{Q}$, and also on any DCGA GIPNS 4-vector *conic section* entity $B = \mathbf{Q}\wedge\boldsymbol{\Pi}$ cut from a quadric surface $\mathbf{Q}$ by a standard DCGA plane $\boldsymbol{\Pi}$. The possible meanings of products that are produced by the DCGA symmetric directional derivative operator applied to other DCGA entities could be researched in future work.

For any DCGA GIPNS entity $B = \mathbf{Q}$ or $B = \mathbf{Q}\wedge\boldsymbol{\Pi}$ of types just described representing an implicit surface function $F(x,\ y,\ z)$, the DCGA GIPNS symmetric *directional* ∞*-derivative surface entity* $D_{\mathbf{n}}\bar{\times}B$ can represent the surface or curve, if it exists, on which $\partial_{\mathbf{n}}F = \infty$ in standard calculus. The zero points on the surface of $\bar{\partial}_{\mathbf{n}}B$ that intersect the surface B are where $\mathbf{n}$ is perpendicular to the surface B.

The dual DCGA GOPNS entity $B^{*\mathcal{D}}$ of types B just described may also be differentiated by the same operator $D_{\mathbf{n}}\bar{\times}$ to produce a DCGA GOPNS symmetric directional ∞-*derivative surface entity* $\bar{\partial}_{\mathbf{n}}B^{*\mathcal{D}}$.

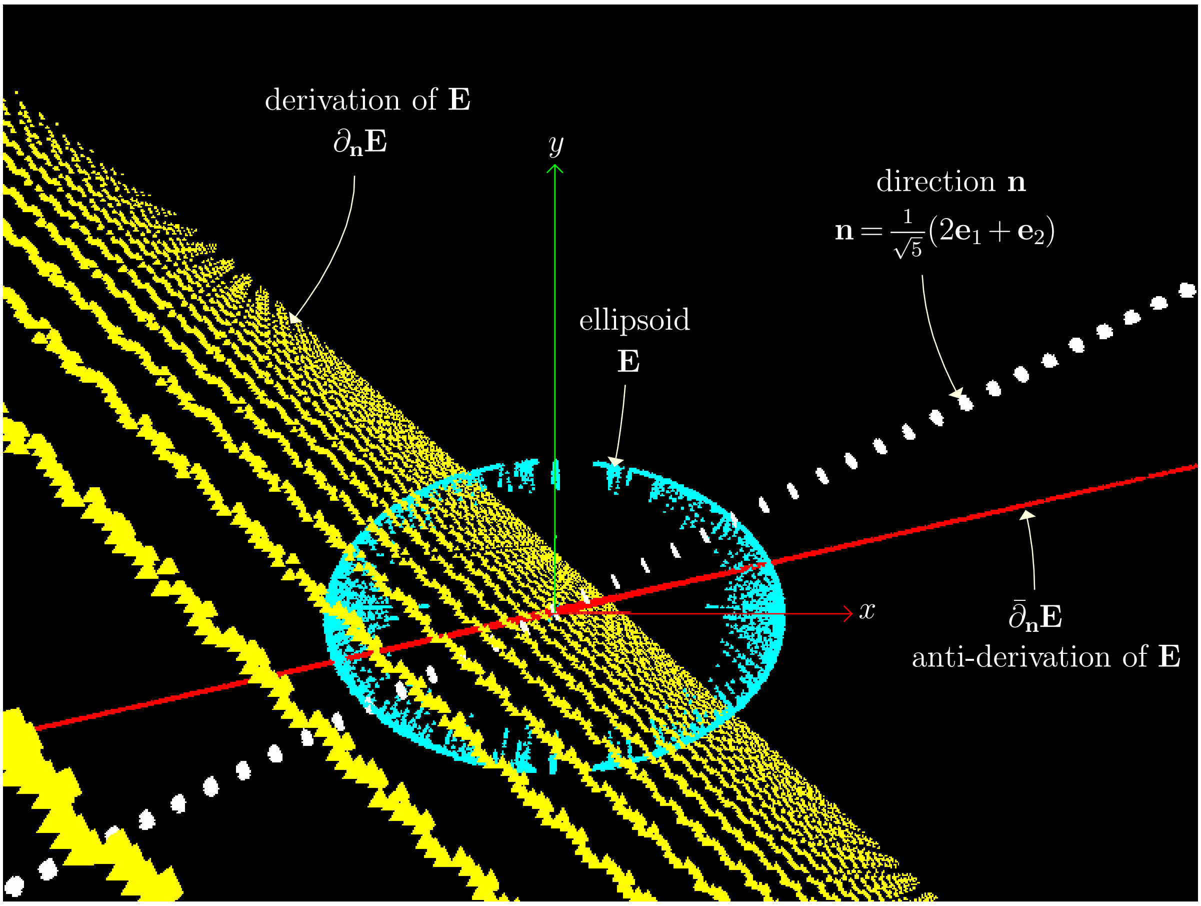

Figure 2. Derivation $\partial_{\mathbf{n}}\mathbf{E}$ and anti-derivation $\bar{\partial}_{\mathbf{n}}\mathbf{E}$ of ellipsoid $\mathbf{E}$

Figure 2 shows the $\mathbf{n}$-directional derivation $\partial_{\mathbf{n}}\mathbf{E}$ and anti-derivation $\bar{\partial}_{\mathbf{n}}\mathbf{E}$ of an ellipsoid $\mathbf{E}$. The ellipsoid $\mathbf{E}$ has center point $\mathbf{e}_o$ and radii $r_x = 3$, $r_y = 2$, and $r_z = 1$. The derivation $\partial_{\mathbf{n}}\mathbf{E}$ represents a plane that intersects the ellipsoid $\mathbf{E}$ in an elliptical set of points $\{ \mathbf{T} : \mathbf{T} \cdot \partial_{\mathbf{n}}\mathbf{E} = 0, \mathbf{T} \cdot \mathbf{E} = 0 \}$ where tangents to $\mathbf{E}$ are parallel to $\mathbf{n}$. The anti-derivation $\bar{\partial}_{\mathbf{n}}\mathbf{E}$ represents a line that intersects the ellipsoid $\mathbf{E}$ in two points $\{ \mathbf{T} : \mathbf{T} \cdot \bar{\partial}_{\mathbf{n}}\mathbf{E} = 0, \mathbf{T} \cdot \mathbf{E} = 0 \}$ where tangents to $\mathbf{E}$ are perpendicular to $\mathbf{n}$. This figure was produced using *Gaalop* (http://www.gaalop.de), which is introduced in [11].

3.8 DCGA differential point

The DCGA null 2-vector *differential point* $-\partial_{\mathbf{n}}\mathbf{T}$ of a DCGA null 2-vector *point* $\mathbf{T} = \mathcal{D}(\mathbf{t})$ is defined as

$$-\partial_{\mathbf{n}}\mathbf{T} = -D_{\mathbf{n}} \times \mathbf{T} = \mathbf{T} \times D_{\mathbf{n}}.$$

For any DCGA GIPNS 2-vector entity B, it can be verified that

$$(\mathbf{T} \times D_{\mathbf{n}}) \cdot B = \mathbf{T} \cdot (D_{\mathbf{n}} \times B).$$

For any dual DCGA GOPNS 8-vector entity $B^{*\mathcal{D}}$, it can be verified that

$$(\mathbf{T} \times D_{\mathbf{n}}) \wedge B^{*\mathcal{D}} = \mathbf{T} \wedge (D_{\mathbf{n}} \times B^{*\mathcal{D}}).$$

In the expression $(\mathbf{T} \times D_{\mathbf{n}}) \cdot B$, the entity B extracts differentiated values $\partial_{\mathbf{n}} s$ from the differential point $-\partial_{\mathbf{n}}\mathbf{T}$ to represent an implicit surface $\partial_{\mathbf{n}} F(x,\ y,\ z) = 0$ that is differentiated in a direction $\mathbf{n}$.

In the expression $\mathbf{T} \cdot (D_{\mathbf{n}} \times B)$, the differential entity $\partial_{\mathbf{n}} B$ extracts differentiated values $\partial_{\mathbf{n}} s$ from the point $\mathbf{T}$ to represent an implicit surface $\partial_{\mathbf{n}} F(x, y, z) = 0$ that is differentiated in a direction $\mathbf{n}$.

3.9 DCGA tangent point

The DCGA null 4-vector *tangent point* $\bar{\partial}_{\mathbf{n}}\mathbf{T}$ of a DCGA null 2-vector *point* $\mathbf{T} = \mathcal{D}(\mathbf{t})$ is defined as

$$\bar{\partial}_{\mathbf{n}}\mathbf{T} \;=\; \mathbf{T} \bar{\times} D_{\mathbf{n}} = D_{\mathbf{n}} \bar{\times} \mathbf{T}$$

The point $\mathbf{T}$ is a tangent point of the DCGA GIPNS 2-vector *quadric surface* $\mathbf{Q}$ in the direction $\mathbf{n}$ if the test

$$(\mathbf{T} \bar{\times} D_{\mathbf{n}}) \cdot \mathbf{Q} \;=\; 0$$

holds good. The set of all such points $\mathbf{T}$ on the surface of $\mathbf{Q}$ is the *curve of tangency* where the quadric surface $\mathbf{Q}$ and its derivative surface $D_{\mathbf{n}} \times \mathbf{Q}$ intersect.

The *tangent point* test on surfaces other than quadrics is left undefined in this paper, but could be the subject of further research.

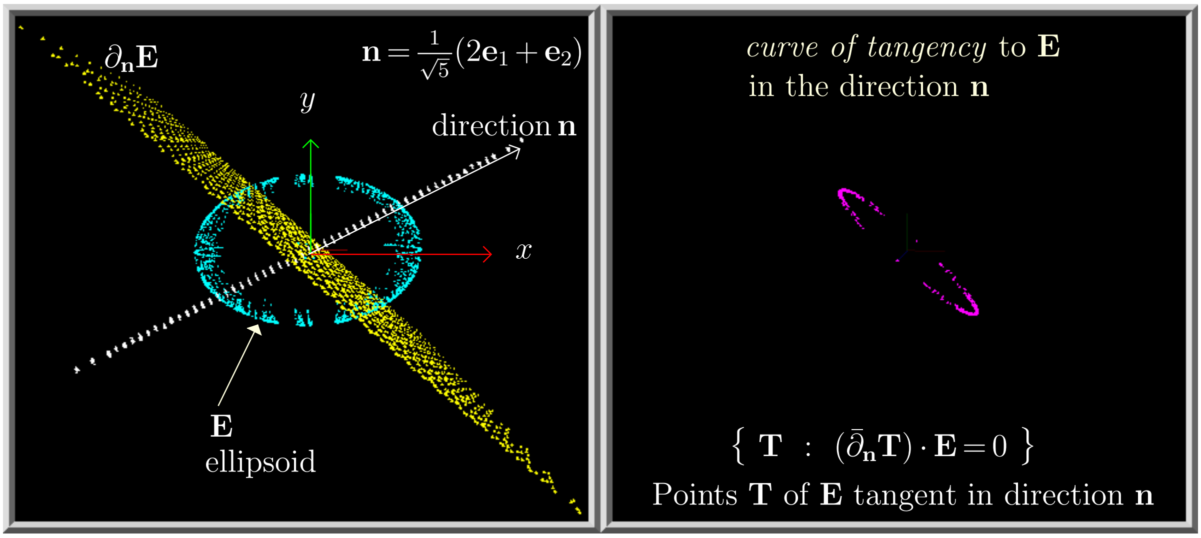

Figure 3. The $\mathbf{n}$-directional tangent points $\bar{\partial}_{\mathbf{n}}\mathbf{T}$ of ellipsoid $\mathbf{E}$

Figure 3 shows the $\mathbf{n}$-directional tangent points $\bar{\partial}_{\mathbf{n}}\mathbf{T}$ of ellipsoid $\mathbf{E}$, which are the same points

$$\{\ \mathbf{T}\ :\ (\bar{\partial}_{\mathbf{n}}\mathbf{T}) \cdot \mathbf{E} = 0\ \} \;=\; \{\ \mathbf{T}\ :\ \mathbf{T} \cdot \mathbf{E} = 0, \mathbf{T} \cdot \partial_{\mathbf{n}}\mathbf{E} = 0\ \}$$

as those on the intersection of the ellipsoid $\mathbf{E}$ and its derivation $\partial_{\mathbf{n}}\mathbf{E}$. The ellipsoid $\mathbf{E}$ is centered at the origin $\mathbf{e}_o$ with radii $r_x = 3$, $r_y = 2$, and $r_z = 1$. The derviation $\partial_{\mathbf{n}}\mathbf{E}$ represents a plane. This figure was produced using *Gaalop* (http://www.gaalop.de), which is introduced in [11].

3.10 DCGA pseudo-integral operators

The operators that are presented in this subsection are being called here, for lack of any other known terminology, *pseudo-integral* or *pseudo anti-derivative* operators. They do *not* produce the exact anti-derivatives, but they do produce results or entities that have a relation to the correct anti-derivatives.

Each extraction element T_s has a pseudo-inverse T_s^+ such that

$$T_s \cdot T_s^+ \;=\; 1.$$

Some of these pseudo-inverses T_s^+ are

$$\begin{aligned} T_1^+ &= -\frac{1}{4}T_{\mathbf{t}^4} \\ T_x^+ &= T_{x\mathbf{t}^2} \\ T_y^+ &= T_{y\mathbf{t}^2} \\ T_z^+ &= T_{z\mathbf{t}^2}. \end{aligned}$$

Using these pseudo-inverses, the following ratios can be defined as *pseudo-integral elements* for a pseudo-integration with respect to x, y, and z

$$\begin{aligned} I_x^+ &= T_x \times T_1^+ = \frac{1}{2}T_{x^2} \times T_x^+ = \frac{1}{2}T_{x^2}T_x^+ = \frac{1}{2}T_{x\mathbf{t}^2}/T_{x^2} \\ I_y^+ &= T_y \times T_1^+ = \frac{1}{2}T_{y^2} \times T_y^+ = \frac{1}{2}T_{y^2}T_y^+ = \frac{1}{2}T_{y\mathbf{t}^2}/T_{y^2} \\ I_z^+ &= T_z \times T_1^+ = \frac{1}{2}T_{z^2} \times T_z^+ = \frac{1}{2}T_{z^2}T_z^+ = \frac{1}{2}T_{z\mathbf{t}^2}/T_{z^2}. \end{aligned}$$

The *pseudo-integral operators* $\int_x^+$, $\int_y^+$, $\int_z^+$, and $\int_{\mathbf{n}}^+$ in the x, y, z, and unit vector $\mathbf{n} = n_x\mathbf{e}_1 + n_y\mathbf{e}_2 + n_z\mathbf{e}_3$ directions can be defined as

$$\begin{aligned} \int_x^+ = \partial_x^+ = I_x^+ \times &= \left(\frac{1}{2}T_{x^2}T_x^+\right) \times = \left(\frac{1}{2}T_{x\mathbf{t}^2}/T_{x^2}\right) \times \\ \int_y^+ = \partial_y^+ = I_y^+ \times &= \left(\frac{1}{2}T_{y^2}T_y^+\right) \times = \left(\frac{1}{2}T_{y\mathbf{t}^2}/T_{y^2}\right) \times \\ \int_z^+ = \partial_z^+ = I_z^+ \times &= \left(\frac{1}{2}T_{z^2}T_z^+\right) \times = \left(\frac{1}{2}T_{z\mathbf{t}^2}/T_{z^2}\right) \times \\ \int_{\mathbf{n}}^+ = \partial_{\mathbf{n}}^+ = I_{\mathbf{n}}^+ \times &= (n_x I_x^+ + n_y I_y^+ + n_z I_z^+) \times \end{aligned}$$

This notation is only suggestive as compared to standard calculus. A raised or exponentiated + indicates a pseudo-inverse element, pseudo-inverse operator, or the pseudo-integral operator, none of which are exact inverses or operators. A pseudo element or operator is an approximation for an element or operator that does not exist in an exact form. The pseudo-integral operators should be used with caution as experimental operators unless specific results are obtained using them.

These operators can be experimented on any DCGA GIPNS 2-vector entity $\mathbf{\Omega}$, which may generally represent a Darboux cyclide. For example, an entity $\mathbf{\Omega}$ may be differentiated into its derivative entity $\partial_{\mathbf{n}}\mathbf{\Omega} = D_{\mathbf{n}} \times \mathbf{\Omega}$ and then operated on to produce a third surface entity called its *pseudo-integral entity* in the $\mathbf{n}$ direction

$$\int_{\mathbf{n}}^+ \partial_{\mathbf{n}}\mathbf{\Omega} \;=\; I_{\mathbf{n}}^+ \times (D_{\mathbf{n}} \times \mathbf{\Omega}) - CT_1.$$

A constant of integration C is subtracted as the extraction element $-CT_1$ so that the result corresponds to an implicit surface function $F(x,y,z)-C=0$.

$\times$	T_1	T_x	T_y	T_z	T_{x^2}	T_{y^2}	T_{z^2}	T_{xy}	T_{yz}	T_{zx}	$T_{\mathbf{t}^2}$	$T_{x\mathbf{t}^2}$	$T_{y\mathbf{t}^2}$	$T_{z\mathbf{t}^2}$	$T_{\mathbf{t}^4}$
I_x^+	T_x	$\frac{2}{4}T_{x^2}+\frac{1}{4}T_{\mathbf{t}^2}$	$\frac{1}{2}T_{xy}$	$\frac{1}{2}T_{zx}$	$\frac{1}{2}T_{x\mathbf{t}^2}$	0	0	$\frac{1}{4}T_{y\mathbf{t}^2}$	0	$\frac{1}{4}T_{z\mathbf{t}^2}$	$\frac{1}{2}T_{x\mathbf{t}^2}$	$\frac{1}{4}T_{\mathbf{t}^4}$	0	0	0
I_y^+	T_y	$\frac{1}{2}T_{xy}$	$\frac{2}{4}T_{y^2}+\frac{1}{4}T_{\mathbf{t}^2}$	$\frac{1}{2}T_{yz}$	0	$\frac{1}{2}T_{y\mathbf{t}^2}$	0	$\frac{1}{4}T_{x\mathbf{t}^2}$	$\frac{1}{4}T_{z\mathbf{t}^2}$	0	$\frac{1}{2}T_{y\mathbf{t}^2}$	0	$\frac{1}{4}T_{\mathbf{t}^4}$	0	0
I_z^+	T_z	$\frac{1}{2}T_{zx}$	$\frac{1}{2}T_{yz}$	$\frac{2}{4}T_{z^2}+\frac{1}{4}T_{\mathbf{t}^2}$	0	0	$\frac{1}{2}T_{z\mathbf{t}^2}$	0	$\frac{1}{4}T_{y\mathbf{t}^2}$	$\frac{1}{4}T_{x\mathbf{t}^2}$	$\frac{1}{2}T_{z\mathbf{t}^2}$	0	0	$\frac{1}{4}T_{\mathbf{t}^4}$	0

Table 2. Pseudo-integral operations $I_{\mathbf{n}}^+ \times$ on extraction elements T_s

Table 2 shows the results of pseudo-integral operations on extraction entities. These results are not the correct anti-derivatives. In most cases, the pseudo-integral operation transforms an extraction element into an extraction element of a higher degree that has some relation to the correct anti-derivative. The pseudo-integral operations and the differential operations may find uses in entity analysis for the manipulation of entities and the extraction of geometric surface parameters.

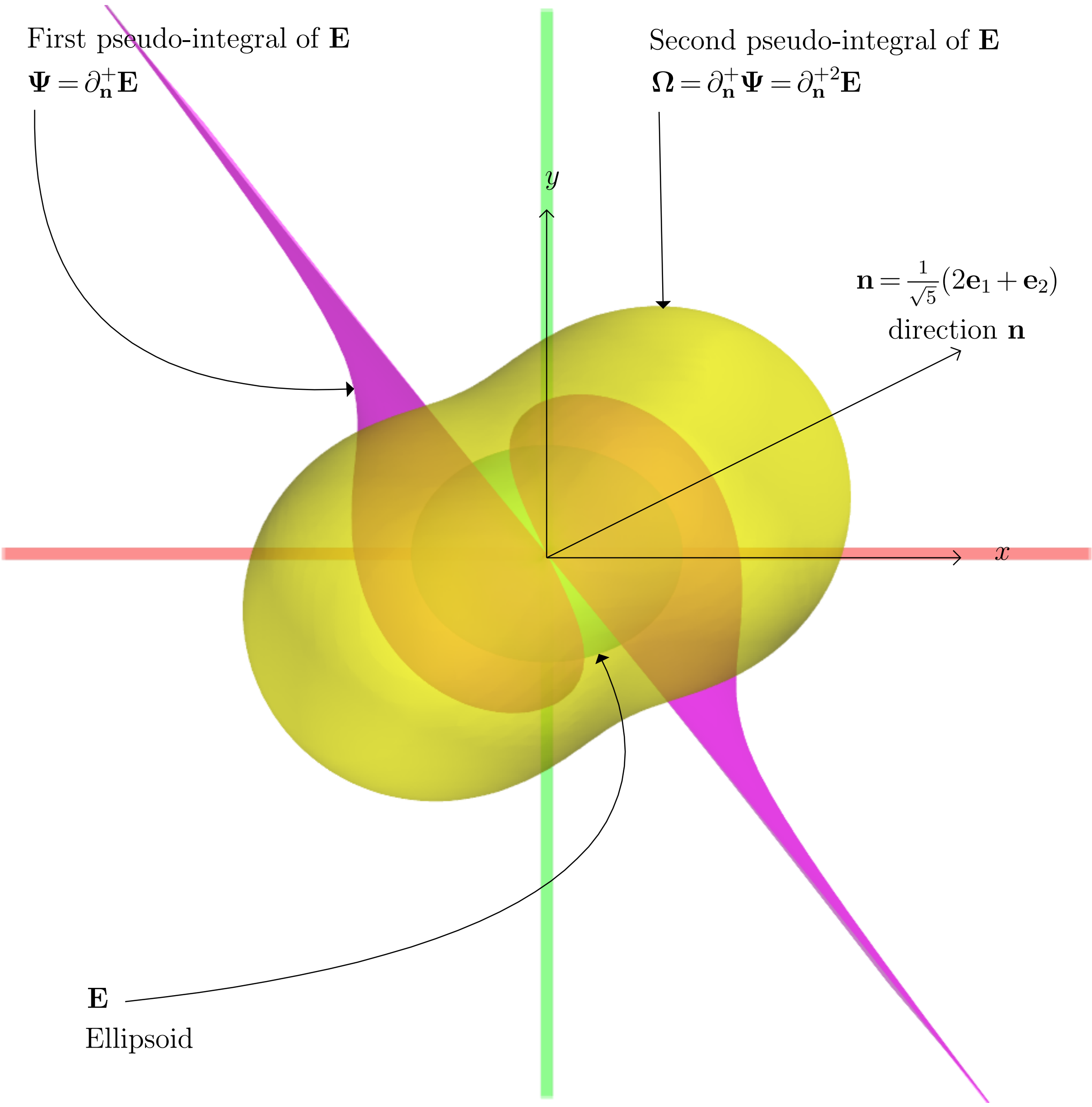

Figure 4. Pseudo-integrals $\partial_{\mathbf{n}}^{+}\mathbf{E}$ and $\partial_{\mathbf{n}}^{+2}\mathbf{E}$

Figure 4 shows the pseudo-integral operation $I_{\mathbf{n}}^{+}\times$ applied to ellipsoid $\mathbf{E}$. The first pseudo-integration $\mathbf{\Psi}=\partial_{\mathbf{n}}^{+}\mathbf{E}=I_{\mathbf{n}}^{+}\times\mathbf{E}$ produces a parabolic cyclide $\mathbf{\Psi}$, which is a cubic surface. The second pseudo-integration $\mathbf{\Omega}=\partial_{\mathbf{n}}^{+2}\mathbf{E}=\partial_{\mathbf{n}}^{+}\mathbf{\Psi}=I_{\mathbf{n}}^{+}\times(I_{\mathbf{n}}^{+}\times\mathbf{E})$ produces a Darboux cyclide $\mathbf{\Omega}$, which is a quartic surface. The surfaces that are produced appear to have an alignment with the direction of pseudo-integration $\mathbf{n}$. This figure was produced using *Mayavi* [14].

4 Examples

4.1 Vector calculus

The vector calculus, also called vector analysis, is standard engineering mathematics. In this subsection, it is shown how vector calculus concepts can be applied within the limitations of DCGA.

4.1.1 Dot product

In the subsections that follow, vectors of the form

$$\mathbf{F} = F\mathbf{e}_1+G\mathbf{e}_2+H\mathbf{e}_3$$

are used, and the coefficients F, G, and H are always to be taken symbolically as scalars, even where they are bivector expressions or differential operators.

The *dot product* $\mathbf{F}\cdot\mathbf{G}$ of $\mathbf{F}$ with another vector $\mathbf{G}=I\mathbf{e}_1+J\mathbf{e}_2+K\mathbf{e}_3$ is symbolically defined as

$$\mathbf{F}\cdot\mathbf{G} = FI+GJ+HK.$$

The products are non-commutative. For example, FI is not the same as IF unless they are actually scalars or scalar-valued expressions.

4.1.2 Cross product

The *cross product* $\mathbf{F}\times\mathbf{G}$ of two vectors

$$\begin{aligned}\mathbf{F} &= F\mathbf{e}_1+G\mathbf{e}_2+H\mathbf{e}_3\\ \mathbf{G} &= I\mathbf{e}_1+J\mathbf{e}_2+K\mathbf{e}_3\end{aligned}$$

is symbolically defined as the determinant

$$\begin{aligned}\mathbf{F}\times\mathbf{G} &= \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ F & G & H\\ I & J & K\end{vmatrix}\\ &= (GK-HJ)\mathbf{e}_1+(HI-FK)\mathbf{e}_2+(FJ-GI)\mathbf{e}_3\\ &= -\mathbf{G}\times\mathbf{F}\end{aligned}$$

The products are non-commutative. For example, FJ is not the same as JF unless they are actually scalars or scalar-valued expressions.

4.1.3 Scalar-valued function or field

Within the limitations of DCGA, a *scalar field* F of the form

$$\begin{aligned} F(x,y,z) &= A\mathbf{t}^4 + B\mathbf{t}^2 + \\ & \quad Cx\mathbf{t}^2 + Dy\mathbf{t}^2 + Ez\mathbf{t}^2 + \\ & \quad Fx^2 + Gy^2 + Hz^2 + \\ & \quad Ixy + Jyz + Kzx + \\ & \quad Lx + My + Nz + O \end{aligned}$$

can be represented by a DCGA GIPNS 2-vector *Darboux cyclide* surface entity $\mathbf{\Omega}_F$. The function F is evaluated at a point $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ as

$$F(x,y,z) = F(\mathbf{p}) = \mathbf{P} \cdot \mathbf{\Omega}_F$$

where $\mathbf{P} = \mathcal{D}(\mathbf{p})$ is the DCGA *point* embedding of $\mathbf{p}$.

It will be convenient to identify a *scalar function* F with its entity as

$$F = \mathbf{\Omega}_F$$

and to define the *scalar function evaluation operation* $F(\mathbf{p})$ as

$$F(\mathbf{p}) = \mathcal{D}(\mathbf{p}) \cdot \mathbf{\Omega}_F = \mathbf{P} \cdot \mathbf{\Omega}_F$$

which evaluates the scalar function F at the point $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$.

4.1.4 Vector-valued function or field

Within the limitations of DCGA, a *vector field*

$$\mathbf{F}(x,y,z) = F(x,y,z)\mathbf{e}_1 + G(x,y,z)\mathbf{e}_2 + H(x,y,z)\mathbf{e}_3$$

can be represented by functions F, G, and H, which are scalar functions as defined in Section 4.1.3. Three entities $\mathbf{\Omega}_F$, $\mathbf{\Omega}_G$, and $\mathbf{\Omega}_H$ can be identified with F, G, and H, respectively. The vector function $\mathbf{F}$ is evaluated at a point $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ as

$$\mathbf{F}(x,y,z) = \mathbf{F}(\mathbf{p}) = (\mathbf{P} \cdot \mathbf{\Omega}_F)\mathbf{e}_1 + (\mathbf{P} \cdot \mathbf{\Omega}_G)\mathbf{e}_2 + (\mathbf{P} \cdot \mathbf{\Omega}_H)\mathbf{e}_3$$

where $\mathbf{P} = \mathcal{D}(\mathbf{p})$ is the DCGA *point* embedding of $\mathbf{p}$.

It will be convenient to identify a *vector function* $\mathbf{F}$ with its *symbolic vector*

$$\mathbf{F} = \mathbf{\Omega}_F\mathbf{e}_1 + \mathbf{\Omega}_G\mathbf{e}_2 + \mathbf{\Omega}_H\mathbf{e}_3$$

and to define the *vector function evaluation operation* $\mathbf{F}(\mathbf{p})$ as

$$\mathbf{F}(\mathbf{p}) = (\mathbf{P} \cdot \mathbf{\Omega}_F)\mathbf{e}_1 + (\mathbf{P} \cdot \mathbf{\Omega}_G)\mathbf{e}_2 + (\mathbf{P} \cdot \mathbf{\Omega}_H)\mathbf{e}_3$$

which evaluates the vector function $\mathbf{F}$ at the point $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$.

4.1.5 The gradient operator

The *gradient operator* ∇ is defined as

$$\begin{aligned} \nabla &= \frac{\partial}{\partial x}\mathbf{e}_1 + \frac{\partial}{\partial y}\mathbf{e}_2 + \frac{\partial}{\partial z}\mathbf{e}_3 \\ &= \partial_x\mathbf{e}_1 + \partial_y\mathbf{e}_2 + \partial_z\mathbf{e}_3 \\ &= (D_x \times)\mathbf{e}_1 + (D_y \times)\mathbf{e}_2 + (D_z \times)\mathbf{e}_3. \end{aligned}$$

Del ∇ is a symbolic vector-valued operator. The coefficients on $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are bivector-valued differential operators, but they are algebraically handled as scalars. The differential operators $D_{\mathbf{n}}\times$ have the anti-commutative property $D_{\mathbf{n}}\times B=-B\times D_{\mathbf{n}}$ and our definition of ∇ expects a right-hand side operand B, such as ∇B, $\nabla\cdot B$, or $\nabla\times B$.

The del operator ∇ can symbolically operate on a scalar function F as ∇F, or on a vector function $\mathbf{F}$ as $\nabla\cdot\mathbf{F}$ or $\nabla\times\mathbf{F}$. In the next subsections, these symbolic operations are defined as results that are known in vector calculus.

4.1.6 Gradient vector of a scalar field

The vector-valued *gradient* function ∇F of a scalar field F (per Section 4.1.3) can be defined as

$$\begin{aligned}\nabla F &= \nabla\mathbf{\Omega}_F\\ &= (D_x\times\mathbf{\Omega}_F)\mathbf{e}_1+(D_y\times\mathbf{\Omega}_F)\mathbf{e}_2+(D_z\times\mathbf{\Omega}_F)\mathbf{e}_3.\end{aligned}$$

The product ∇F is symbolic and the bivector-valued coefficients on $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are handled as scalars. The symbolic vector-valued gradient function has bivector-valued coefficients until it is evaluated at a point into a vector with scalar-valued coefficients.

The vector function evaluation of the vector-valued gradient function ∇F at a point $\mathbf{p}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$ is

$$\nabla F(\mathbf{p}) = (\mathbf{P}\cdot(D_x\times\mathbf{\Omega}_F))\mathbf{e}_1+(\mathbf{P}\cdot(D_y\times\mathbf{\Omega}_F))\mathbf{e}_2+(\mathbf{P}\cdot(D_z\times\mathbf{\Omega}_F))\mathbf{e}_3$$

where $\mathbf{P}=\mathcal{D}(\mathbf{p})$ is the DCGA *point* embedding of $\mathbf{p}$. The evaluation transforms the coefficients on $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ into scalar expressions, transforming the symbolic gradient vector into an algebraic gradient vector result.

4.1.7 Directional derivative of a scalar field

The scalar-valued *directional derivative* function $\partial_{\mathbf{n}}F$ of a scalar field $F=\mathbf{\Omega}_F$ in the direction of a *unit directional vector* $\mathbf{n}=n_x\mathbf{e}_1+n_y\mathbf{e}_2+n_z\mathbf{e}_3$ can be written

$$\begin{aligned}\partial_{\mathbf{n}}F &= (\nabla F)\cdot\mathbf{n}=(\nabla\mathbf{\Omega}_F)\cdot\mathbf{n}\\ &= ((D_x\times\mathbf{\Omega}_F)\mathbf{e}_1+(D_y\times\mathbf{\Omega}_F)\mathbf{e}_2+(D_z\times\mathbf{\Omega}_F)\mathbf{e}_3)\cdot\mathbf{n}\\ &= (n_xD_x\times\mathbf{\Omega}_F)+(n_yD_y\times\mathbf{\Omega}_F)+(n_zD_z\times\mathbf{\Omega}_F)\\ &= (\nabla\cdot\mathbf{n})F=(\nabla\cdot\mathbf{n})\mathbf{\Omega}_F\\ &= (((D_x\times)\mathbf{e}_1+(D_y\times)\mathbf{e}_2+(D_z\times)\mathbf{e}_3)\cdot\mathbf{n})\mathbf{\Omega}_F\\ &= (n_xD_x\times\mathbf{\Omega}_F)+(n_yD_y\times\mathbf{\Omega}_F)+(n_zD_z\times\mathbf{\Omega}_F).\end{aligned}$$

The product $\partial_{\mathbf{n}}F$ is symbolic and the bivector-valued coefficients are handled as scalars. The symbolic scalar-valued directional derivative function is bivector-valued until it is evaluated at a point into a scalar-valued function.

The scalar function evaluation of the scalar-valued directional derivative function $\partial_n F$ at a point $\mathbf{p}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$ is

$$\begin{aligned}\partial_{\mathbf{n}}F(\mathbf{p}) &= \mathcal{D}(\mathbf{p})\cdot\partial_{\mathbf{n}}F=\mathbf{P}\cdot\partial_{\mathbf{n}}F\\ &= \mathbf{P}\cdot((n_xD_x\times\mathbf{\Omega}_F)+(n_yD_y\times\mathbf{\Omega}_F)+(n_zD_z\times\mathbf{\Omega}_F))\end{aligned}$$

where $\mathbf{P}=\mathcal{D}(\mathbf{p})$ is the DCGA *point* embedding of $\mathbf{p}$.

4.1.8 Divergence of a vector field

The scalar-valued divergence function $\nabla\cdot\mathbf{F}$ of a vector field $\mathbf{F}$ (per Section 4.1.4) can be written

$$\begin{aligned}\nabla\cdot\mathbf{F} &= ((D_x\times)\mathbf{e}_1+(D_y\times)\mathbf{e}_2+(D_z\times)\mathbf{e}_3)\cdot(\mathbf{\Omega}_F\mathbf{e}_1+\mathbf{\Omega}_G\mathbf{e}_2+\mathbf{\Omega}_H\mathbf{e}_3)\\ &= (D_x\times\mathbf{\Omega}_F)+(D_y\times\mathbf{\Omega}_G)+(D_z\times\mathbf{\Omega}_H).\end{aligned}$$

The dot product $\nabla\cdot\mathbf{F}$ is symbolic and the bivector-valued coefficients are handled as scalars. The symbolic scalar-valued divergence function is bivector-valued until it is evaluated at a point into a scalar-valued function.

The scalar function evaluation of the scalar-valued divergence function $\nabla\cdot\mathbf{F}$ at a point $\mathbf{p}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$ is

$$\nabla\cdot\mathbf{F}(\mathbf{p}) = \mathbf{P}\cdot((D_x\times\mathbf{\Omega}_F)+(D_y\times\mathbf{\Omega}_G)+(D_z\times\mathbf{\Omega}_H))$$

where $\mathbf{P}=\mathcal{D}(\mathbf{p})$ is the DCGA *point* embedding of $\mathbf{p}$.

4.1.9 Circulation of a vector field

The vector-valued circulation function $\nabla\times\mathbf{F}$ of a vector field $\mathbf{F}=\mathbf{\Omega}_F\mathbf{e}_1+\mathbf{\Omega}_G\mathbf{e}_2+\mathbf{\Omega}_H\mathbf{e}_3$ can be written

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ D_x\times & D_y\times & D_z\times\\ \mathbf{\Omega}_F & \mathbf{\Omega}_G & \mathbf{\Omega}_H\end{vmatrix}\\ &= (D_y\times\mathbf{\Omega}_H-D_z\times\mathbf{\Omega}_G)\mathbf{e}_1+\\ &\quad (D_z\times\mathbf{\Omega}_F-D_x\times\mathbf{\Omega}_H)\mathbf{e}_2+\\ &\quad (D_x\times\mathbf{\Omega}_G-D_y\times\mathbf{\Omega}_F)\mathbf{e}_3.\end{aligned}$$

The cross product $\nabla\times\mathbf{F}$ is symbolic and the bivector-valued coefficients are handled as scalars. The symbolic vector-valued circulation function has bivector-valued coefficients until it is evaluated at a point into scalar-valued coefficients.

The vector function evaluation of the vector-valued circulation function $\nabla\times\mathbf{F}$ at a point $\mathbf{p}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$ is

$$\begin{aligned}\nabla\times\mathbf{F}(\mathbf{p}) &= (\mathbf{P}\cdot(D_y\times\mathbf{\Omega}_H-D_z\times\mathbf{\Omega}_G))\mathbf{e}_1+\\ &\quad (\mathbf{P}\cdot(D_z\times\mathbf{\Omega}_F-D_x\times\mathbf{\Omega}_H))\mathbf{e}_2+\\ &\quad (\mathbf{P}\cdot(D_x\times\mathbf{\Omega}_G-D_y\times\mathbf{\Omega}_F))\mathbf{e}_3\end{aligned}$$

where $\mathbf{P}=\mathcal{D}(\mathbf{p})$ is the DCGA *point* embedding of $\mathbf{p}$.

4.2 Differential equations

Using the differential elements $\partial_{\mathbf{n}}T_s=D_{\mathbf{n}}\times T_s$ given in Table 1, it is possible to write a differential equation of a limited form and represent it as a scalar-valued function F that is identified with an entity $\mathbf{\Omega}_F$. The entity $\mathbf{\Omega}_F$ can be a linear combination of i differential elements

$$\mathbf{\Omega}_F = \sum_i c_i D_{\mathbf{n}_i}\times T_{s_i}$$

each having a scalar c_i. The geometric surface represented by $\boldsymbol{\Omega}_F$ is the solution set $\{\ \mathbf{p}\ \mid\ \mathcal{D}(\mathbf{p}) \cdot \boldsymbol{\Omega}_F = 0\ \}$ of all points $\mathbf{p}$ for the differential equation $F(\mathbf{p}) = 0$. The entity $\boldsymbol{\Omega}_F$ could also be a linear combination of i other differential entities

$$\boldsymbol{\Omega}_F \;=\; \sum_i c_i \partial_{\mathbf{n}_i} \boldsymbol{\Omega}_{F_i} = \sum_i c_i D_{\mathbf{n}_i} \times \boldsymbol{\Omega}_{F_i}$$

each having a scalar c_i.

4.3 Entity analysis

In the $\mathcal{G}_{4,1}$ Conformal Geometric Algebra (CGA), all CGA geometric entities can be formed as blades and the *analysis of blades* includes how to extract the parameters describing an entity [13]. In DCGA, most of the geometric entities cannot be formed as blades by wedging surface points. Therefore, a different analysis approach is required.

The DCGA differential operators provide the standard operations for analyzing any DCGA geometric entity. The following subsection shows how to extract parameters from an ellipsoid entity.

4.3.1 DCGA GIPNS 2-vector ellipsoid entity analysis

The implicit surface equation of an ellipsoid is

$$F(x, y, z) = \frac{(x - p_x)^2}{r_x^2} + \frac{(y - p_y)^2}{r_y^2} + \frac{(z - p_z)^2}{r_z^2} - 1 \;=\; 0$$

with center point (p_x, p_y, p_z) and radii r_x, r_y, and r_z. The LHS is the implicit surface function $F(x, y, z)$. Expanding this function and using the DCGA extraction elements, the ellipsoid entity can be defined as follows.

The DCGA GIPNS 2-vector ellipsoid entity is defined as

$$\mathbf{E} \;=\; \frac{T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} + \frac{T_{z^2}}{r_z^2} - \frac{2p_x T_x}{r_x^2} - \frac{2p_y T_y}{r_y^2} - \frac{2p_z T_z}{r_z^2} + \frac{p_x^2 T_1}{r_x^2} + \frac{p_y^2 T_1}{r_y^2} + \frac{p_z^2 T_1}{r_z^2} - T_1.$$

A DCGA point $\mathbf{T}$ is on the ellipsoid surface $\mathbf{E}$ if the Geometric Inner Product Null Space (GIPNS) condition $\mathbf{T} \cdot \mathbf{E} = 0$ holds good.

In standard differential calculus, the parameter r_x can be extracted as

$$r_x \;=\; \sqrt{\left(\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)^{-1}} = \sqrt{\left(\frac{1}{2}\frac{2}{r_x^2}\right)^{-1}}$$

and similarly for r_y and r_z. Also in standard differential calculus, and using r_x, the center point coordinate p_x can be extracted as

$$p_x \;=\; -\frac{r_x^2}{2}\frac{\partial F(0,0,0)}{\partial x} = -\frac{r_x^2}{2}\frac{2(0 - p_x)}{r_x^2}$$

and similarly for p_y and p_z. These standard differential operations can be translated into DCGA differential operations as

$$r_x \;=\; \sqrt{\left(\frac{1}{2}\mathbf{e}_o \cdot (D_x \times (D_x \times \mathbf{E}))\right)^{-1}}$$

$$p_x \;=\; -\frac{r_x^2}{2}\mathbf{e}_o \cdot (D_x \times \mathbf{E}).$$

The evaluation at zero $F(0,0,0)$ is translated into a GIPNS evaluation at the DCGA origin point $\mathbf{e}_o$. The extraction formulas for the parameters of other quadric surfaces are similar.

If the ellipsoid or other quadric surface is rotated, then the analysis may be more complicated, but should still be possible by using differential operations to differentiate with respect to one axis and then another axis to obtain the coefficients of cross terms and determine the rotated principal axes. It should also be possible to perform inverse rotations to transform a rotated quadric into a principal axes-aligned quadric that is not rotated, where the analysis is simple.

5 Conclusion

The DCGA geometric differential operators produce surface entities that represent the derivative of an implicit surface function that may have the general form of a Darboux cyclide. For quadric and conic section entities, the symmetric anti-commutator differential operator can produce a surface entity that represents where the derivative is infinite. Plotting these surfaces using software such as Gaalop, which is introduced in [11], shows interesting results that could be researched further.

The differential operators may have many applications not covered in this paper. For example, the differential and pseudo-integral operators may provide the ability to manipulate and analyze an entity for purposes such as determining the surface type of an entity and extracting the implicit surface function parameters. This paper looked at one example of entity analysis, for the extraction of parameters of a DCGA ellipsoid entity, but much more could be worked out as a subject of additional research.

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