The generalized Stokes theorem

By J.A.J. van Leunen

Last modified: 21 december 2015

Abstract

When applied to a quaternionic manifold, the generalized Stokes theorem can provide an elucidating space-progression model in which elementary objects float on top of symmetry centers that act as their living domain. The paper elucidates the origin of the electric charges and color charges of elementary particles.

1 Introduction

This paper uses the fact that separable Hilbert spaces can only cope with number systems that are division rings. We use the most elaborate version of these division rings and that is the quaternionic number system. Quaternionic number systems exist in multiple versions, that differ in the way they are ordered. Ordering influences the arithmetic properties of the number system and it appears that it influences the behavior of quaternionic functions under integration. Another important fact is that every infinite dimensional separable Hilbert system owns a companion Gelfand triple, which is a non-separable Hilbert space. We will use these Hilbert spaces as structured storage media for discrete quaternionic data and for quaternionic manifolds. We use the reverse bra-ket method in order to relate operators and their eigenspaces to pairs of functions and their parameter spaces. Subspaces act as Hilbert space domains in relation to which manifolds are defined.

2 Without discontinuities

The generalized Stokes theorem is in fact a combination of two versions. One is the using the divergence part of the exterior derivative $d\omega$. It is also known as the generalized divergence theorem. The other version uses the curl part of the exterior derivative. The theorem can be applied when everywhere in Ω the derivative $\mathrm{d}\omega$ exists and when everywhere in $\partial\Omega$ the manifold ω is continuous.

Without discontinuities in the manifold ω the generalized Stokes theorem is represented by a simple formula [1].

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega \left(= \oint_{\partial\Omega} \omega \right) \tag{1}$$

The domain Ω is encapsulated by a boundary $\partial\Omega$.

$$\Omega \subset \partial\Omega \tag{2}$$

The manifolds ω and $d\omega$ represent quaternionic fields $\mathfrak{F}$ and $\mathrm{d}\mathfrak{F}$, while inside $\partial\Omega$ the manifold ω represents the quaternionic boundary of the quaternionic field $\mathfrak{F}$.

$d\omega$ is the exterior derivative of ω. The theorem exists in the form of a divergence based version and in the form of a curl based version [2].

2.1 A special boundary between the real part and the imaginary part of the domain

In the special case that is investigated here, the generalized Stokes theorem constructs a rim $\mathfrak{F}(\boldsymbol{x},\tau)$ between the past history of the field $[\mathfrak{F}(\boldsymbol{x},t)]_{t<\tau}$ and the future $[\mathfrak{F}(\boldsymbol{x},t)]_{t>\tau}$ of that field. It means that the boundary $\mathfrak{F}(\boldsymbol{x},\tau)$ of field $[\mathfrak{F}(\boldsymbol{x},t)]_{t<\tau}$ represents a universe wide static status quo of that field.

More specifically, in the form of the divergence version of the theorem holds:

$$\int_{t=0}^{\tau}\iiint_{V} \mathrm{d}\mathfrak{F}\,(x) = \int_{t=0}^{\tau}\left(\iiint_{V} \langle\nabla,\mathfrak{F}(x)\rangle\, dx \wedge dy \wedge dz\right) \wedge d\tau = \left[\iiint_{V} \mathfrak{F}\,(\boldsymbol{x}) d\boldsymbol{x}\right]_{t=\tau} \tag{1}$$

$$x = \boldsymbol{x} + \tau \tag{2}$$

Here $[\mathfrak{F}(\boldsymbol{x},t)]_{t=\tau}$ represents the static status quo of a quaternionic field at instance τ. V represents the spatial part of the quaternionic domain of $\mathfrak{F}$, but it may represent only a restricted part of that parameter space. This last situation corresponds to the usual form of the divergence theorem.

2.1.1 Domains and parameter spaces

The quaternionic ***domain*** Ω is supposed to be defined as part of the ***domain*** $\mathfrak{R}$ of a ***reference operator*** $\mathfrak{R}$ that resides in the non-separable Hilbert space $\mathcal{H}$. The reverse bra-ket method [3] relates the eigenspace $\{q\}$ of reference operator $\mathfrak{R}$ to a flat quaternionic ***function*** $\mathfrak{R}(q)$. The target of function $\mathfrak{R}(q)$ is its own ***parameter space*** $\{q\}$. Here we explicitly use the same symbol $\mathfrak{R}$ for all directly related objects.

$$\mathfrak{R} = |q\rangle\mathfrak{R}(q)\langle q| = |q\rangle q\langle q| \tag{1}$$

The domain $\mathfrak{R}$ is spanned by the eigenvectors $\{|q\rangle\}$ of operator $\mathfrak{R}$.

The reverse bra-ket method also relates the eigenspace $\mathfrak{R}$ to an equivalent eigenspace $\mathcal{R}$ of a reference operator $\mathcal{R}$, which resides in the separable Hilbert space $\mathfrak{H}$. Both eigenspaces are related to the same version of the quaternionic number system. However, the second eigenspace $\mathcal{R}$ only uses rational quaternions q_i.

$$\mathcal{R} = |q_i\rangle\mathfrak{R}(q_i)\langle q_i| = |q_i\rangle q_i\langle q_i| \tag{2}$$

Quaternionic number systems can be ordered in several ways. Operator $\mathcal{R}$ corresponds with one of these orderings. $\mathcal{R}$ is supposed to be ***Cartesian-ordered***. $\mathcal{R}$ is a normal operator and its eigenspace is countable. It means that the set of eigenvectors of $\mathcal{R}$ can be enumerated by the separate eigenvalues of $\mathcal{R}$. The eigenspace is the Cartesian product of four partially ordered sets in which the set, which represents the real part takes a special role. The eigenspace of the Hermitian part $\mathcal{R}_0 = ½(\mathcal{R} + \mathcal{R}^\dagger)$ of normal operator $\mathcal{R}$ can be used to enumerate a division of $\mathfrak{H}$ into a countable number of disjunctive subspaces, which are spanned by eigenvectors of $\mathcal{R}$. Cartesian ordering means partial ordering of the eigenvalues of $\mathcal{R}_0$ and additional ordering of the eigenvalues of the anti-Hermitian operator $\boldsymbol{\mathfrak{R}} = ½(\mathfrak{R} - \mathfrak{R}^\dagger)$ by selecting a Cartesian coordinate system. Eight mutually independent Cartesian coordinate systems exist. $\mathcal{R}_0 = (\mathcal{R} + \mathcal{R}^\dagger)/2$ is a self-adjoint operator. The ordered eigenvalues of $\mathcal{R}_0$ can be interpreted as ***progression values***. The eigenvalues of $\boldsymbol{\mathcal{R}}$ can be interpreted as ***spatial values.***

The split that has been selected in the previous section set a category of operators apart that are all Cartesian-ordered in the same way as operator $\mathcal{R}$ is. It enables a space-progression model in which progression steps in the separable Hilbert space $\mathfrak{H}$ and flows in its non-separable companion $\mathcal{H}$. Via the reverse bra-ket method the Cartesian-ordering of $\mathcal{R}$ can be transferred to $\mathfrak{R}$.

In this way, parameter spaces as well as domains correspond to closed subspaces of the Hilbert spaces. The domain subspaces are subspaces of the domains of the corresponding reference operators. The parameter spaces are ordered by a selected coordinate system. The Ω domain is represented by a part of the eigenspace of reference operator $\mathfrak{R}$. The flat quaternionic function $\mathfrak{R}$ defines the parameter space $\mathfrak{R}$. It installs an ordering by selecting a Cartesian coordinate system for the eigenspace of its anti-Hermitian part $\boldsymbol{\mathfrak{R}} = ½(\mathfrak{R} - \mathfrak{R}^\dagger)$. Several mutually independent selections are possible. The chosen selection attaches a corresponding symmetry flavor to this parameter space. In the model, this symmetry flavor will become the reference symmetry flavor. Thus, the symmetry flavor of parameter space $\mathfrak{R}^{⓪}$ may be distinguished by its superscript ^{⓪}.

The manifold ω is also defined as the continuum eigenspace of a dedicated normal operator ω which is related to domain Ω and to parameter space $\mathfrak{R}^{⓪}$ via function $\mathfrak{F}$. Within this parameter space $\mathfrak{F}$ may have discontinuities, but these must be excluded from the Ω domain. This exclusion will be treated below.

2.1.2 Interpretation of the selected encapsulation

The boundary $\partial\Omega$ is selected between the real part and the imaginary part of domain $\mathfrak{R}$. But it also excludes part of the real part. That part is the range of the real part from τ to infinity. τ is interpreted as the current progression value.

The future $\mathfrak{R} - \Omega$ is kept on the outside of the boundary $\partial\Omega$. As a consequence, the mechanisms that generate new data, operate on the rim $\partial\Omega$ between past Ω and future $\mathfrak{R} - \Omega$.

This split of quaternionic space results in a space-progression model that is to a large extent similar to the way that physical theories describe their space time models. However, the physical theories apply a model that has a Minkowski signature. The quaternionic model is strictly Euclidean.

The paper does not claim that this quaternionic space-progression model reflects the structure and the habits of physical reality. The quaternionic space-progression model is merely promoted as a test model.

What happens is an ongoing process that embeds the subsequent static status quo’s of the separable Hilbert space into the Gelfand triple.

The controlling mechanisms act as a function of progression τ in a stochastic and step-wise fashion in the realm of the separable Hilbert space. The result of their actions are stored in eigenspaces of corresponding operators that reside in the separable Hilbert space. At the same instance this part of the separable Hilbert space is embedded into its companion Gelfand triple.

The controlling mechanisms will provide all generated data with a ***progression stamp*** τ. This progression stamp reflects the state of a model wide clock tick. The whole model, including its physical fields will proceed with these progression steps. However, in the Gelfand triple this progression can be considered to flow.

At the defined rim, any forecasting will be considered as mathematical cheating. Thus, at the rim, the uncertainty principle does not work for the progression part of the parameter spaces. Differential equations that offer advanced as well as retarded solutions must reinterpret the advanced solutions and turn them in retarded solutions, which in that case represent another kind of object. If the original object represents a particle, then the reversed particle is the anti-particle.

As a consequence of the construct, the history, which is stored-free from any uncertainty-in the already processed part of the eigenspaces of the physical operators, is no longer touched. Future is unknown or at least it is inaccessible.

3 Symmetry centers as floating parameter spaces

If we tolerate discontinuities, then these artifacts must be encapsulated by boundaries $\partial \mathrm{H}_n^x$ and in that way they are separated from the main domain Ω.

In that case the model may apply different parameter spaces, which have their own private symmetry flavor [4]. A separable quaternionic Hilbert space can cope with coexisting parameter spaces and these spaces are served by dedicated operators. The reverse bra-ket method relates the parameter space to a corresponding operator. For example [4]:

Let $\{q_i\}$ be the set of ***rational*** quaternions in a selected quaternionic number system and let $\{|q_i\rangle\}$ be the set of corresponding base vectors. They are eigenvectors of a normal operator $\mathcal{R}$.

Here we enumerate the eigenvalues and the base vectors with the same index i. This shows how the reverse bra-ket method works.

$$\mathcal{R} \equiv |q_i\rangle q_i \langle q_i| \tag{1}$$

For all bra's $\langle x|$ and ket's $|y\rangle$ hold:

$$\langle x|\mathcal{R}\, y\rangle = \sum_i \langle x|q_i\rangle q_i \langle q_i|y\rangle \tag{2}$$

$\mathcal{R}_0 = \left(\mathcal{R} + \mathcal{R}^\dagger\right)/2$ is a self-adjoint and thus Hermitian operator. Its eigenvalues can be used to arrange the order of the eigenvectors by enumerating them with the eigenvalues. The ordered eigenvalues can be interpreted as ***progression values***.

$\boldsymbol{\mathcal{R}} = \left(\mathcal{R} - \mathcal{R}^\dagger\right)/2$ is the corresponding anti-Hermitian operator.

We will use the same symbol for the operator $\mathcal{R}$, for the eigenspace $\{q_i\}$ and for the defined parameter space. $\boldsymbol{\mathcal{R}}$ is supposed to be ordered by using a selected Cartesian coordinate system. Eight mutually independent selections are possible. Together with the ordering of the real part $\mathcal{R}_0$, the Cartesian ordering of the imaginary part $\boldsymbol{\mathcal{R}}$ determines the symmetry flavor of the eigenspace of $\mathcal{R}$.

We define a category of anti-Hermitian operators $\{\mathfrak{S}_n^x\}$ that have no Hermitian part and that are distinguished by the way that their eigenspace is ordered by applying a polar coordinate system. We call them symmetry centers $\mathfrak{S}_n^x$. A polar ordering always start with a selected Cartesian ordering. The geometric center of the eigenspace of the symmetry center floats on a background parameter space of the normal reference operator $\mathcal{R}$, whose eigenspace defines a full quaternionic parameter space. The eigenspace of the symmetry center $\mathfrak{S}_n^x$ acts as a three dimensional spatial parameter space. The super script ^{x} refers to the symmetry flavor of $\mathfrak{S}_n^x$. The subscript $_n$enumerates the symmetry centers. Sometimes we omit the subscript.

$$\mathfrak{S}^x = |\mathfrak{s}_i^x\rangle \mathfrak{s}_i^x \langle \mathfrak{s}_i^x| \tag{4}$$

$$\mathfrak{S}^{x\dagger} = -\mathfrak{S}^x \tag{5}$$

In the companion Gelfand triple of an infinite dimensional separable Hilbert space the reverse bra-ket method can define continuum parameter spaces and relate them to corresponding operators. In this way the countable parameter space $\mathcal{R}$ relates to a continuum parameter space $\mathfrak{R}$.

The quaternionic field $\mathfrak{F}$ can also be represented by a dedicated operator. Here we use a parameter space $\mathfrak{R}$ that is spanned by a full quaternionic number system.

For all bra's $\langle x|$ and ket's $|y\rangle$ hold:

$$\langle x|\mathfrak{R}\, y\rangle = \int_q \langle x|q\rangle q \langle q|y\rangle \; dq \tag{6}$$

$$\langle x|\mathfrak{F}\, y\rangle = \int_q \langle x|q\rangle \mathfrak{F}(q) \langle q|y\rangle \; dq \tag{7}$$

Here, we use the symbol $\mathfrak{F}$ for the field, the function and the operator. However, another parameter space R would deliver another function F for the same field $\mathfrak{F}$. So, what determines the field $\mathfrak{F}$ is stored in the eigenspace $\mathfrak{F}$ of operator $\mathfrak{F}$ and can be coupled to different pairs of functions and parameter spaces.

4 The detailed generalized Stokes theorem

Symmetry centers represent spherically ordered parameter spaces in regions H_n^x that float on a background parameter space $\mathfrak{R}$. The boundaries $\partial \mathrm{H}_n^x$ separate the regions H_n^x from the domain Ω. The regions H_n^x are platforms for local discontinuities in basic fields [2]. These fields are continuous in domain Ω.

The symmetry centers are encapsulated and the encapsulating boundary is part of the disconnected boundary which encapsulates all continuous parts of physical fields that exist in the quaternionic model.

$$\int_{\Omega-\mathrm{H}} d\omega = \int_{\partial\Omega} \omega - \sum_n \int_{\partial \mathrm{H}_n^x} \omega \tag{1}$$

Here domain Ω corresponds to part of the reference parameter space $\mathfrak{R}^{⓪}$. As mentioned before the symmetry centers $\{\mathfrak{S}_n^x\}$ represent encapsulated regions $\{\mathrm{H}_n^x\}$ that float on parameter space $\mathfrak{R}^{⓪}$.

The geometric center of symmetry center $\mathfrak{S}_n^x$ is represented by a location on parameter space $\mathfrak{R}^{⓪}$.

$$H = \bigcup_n \mathrm{H}_n^x \tag{2}$$

The relation between the ***subspace*** S_Ω that corresponds to the domain Ω and the ***subspace*** $S_\mathfrak{R}$ that corresponds to the parameter space $\mathfrak{R}^{⓪}$ is given by:

$$\underbrace{\Omega}_{S_\Omega} \subset \underbrace{\mathfrak{R}^{⓪}}_{S_\mathfrak{R}} \tag{3}$$

Similarly:

$$\underbrace{\mathrm{H}_n^x}_{S_{\mathrm{H}_n^x}} \subset \underbrace{\mathfrak{S}_n^x}_{S_{\mathfrak{S}_n^x}} \tag{4}$$

4.1 Discrepant regions

The symmetry centers correspond to point-like discontinuities. However, also large connected regions of $\mathfrak{R}^{⓪}$ may exist that disrupt the continuity of the manifold. For example a region that is surrounded by a boundary where the deformation is so strong that information contained in ω cannot pass the boundary of this region. These regions must also be separated from domain Ω. In this way these regions will correspond to ***cavities*** in the domain Ω. ***The information contained in the manifold cannot pass the surface of the cavity.*** The cavities act as information holes. Within the cavity the manifold can be considered to be non-existent. Within that region it has no defining function.

4.2 Symmetry flavor of the symmetry center

The symmetry center $\mathfrak{S}_n^x$ is characterized by a private symmetry flavor. That symmetry flavor relates to the Cartesian ordering of this parameter space. When the orientation of the coordinate axes is fixed, then eight independent Cartesian orderings are possible [4]. We use the Cartesian ordering of $\mathfrak{R}^{⓪}$ as the reference for the orientation of the axes. $\mathfrak{R}^{⓪}$ has the same Cartesian ordering as $\mathcal{R}^{⓪}$ has.

$$\int_{\Omega-\mathrm{H}} d\omega = \int_{\partial\Omega} \omega - \sum_n \int_{\partial \mathrm{H}_n^x} \omega \tag{1}$$

In this formula the boundaries $\partial\Omega$ and $\partial\mathrm{H}_n^x$ are subtracted. This subtraction is affected by the ordering of the domains Ω and H_n^x.

This can best be comprehended when the encapsulation $\partial\mathrm{H}_n^x$ is performed by a cubic space form that is aligned along the Cartesian axes. Now the six sides of the cube contribute different to the effects of the encapsulation when the ordering differs from the Cartesian ordering of the reference parameter space $\mathfrak{R}^{⓪}$. Each discrepant axis ordering corresponds to one third of the surface of the cube. This effect is represented by the symmetry related charge and the color charge of the symmetry center [4]. It is easily related to the algorithm which is introduced for the computation of the symmetry related charge. Also the relation to the color charge will be clear.

The symmetry related charge and the color charge of symmetry center $\mathfrak{S}_n^x$ are located at the geometric center of the symmetry center. A Green's function together with these charges can represent the defining function of the contribution to the symmetry related field $\mathfrak{A}^x$ within and beyond the realm of the floating region H_n^x.

Nothing else than the discrepancy of the ordering of symmetry center $\mathfrak{S}_n^x$ with respect to the ordering of the parameter spaces $\mathcal{R}^{⓪}$and $\mathfrak{R}^{⓪}$ causes the existence of the symmetry related charge, which is related to the symmetry center. Anything that resides on this symmetry center will inherit that symmetry related charge.

4.3 Path of the symmetry center

The symmetry center $\mathfrak{S}_n^x$ that conforms to encapsulated region H_n^x, keeps its private symmetry flavor. At the passage through the boundary the symmetry flavor of the background parameter space $\mathfrak{R}^{⓪}$ flips from history to future. As a consequence the symmetry related charge of the symmetry center will flip.

However, the passage of the symmetry center through the rim may also be interpreted as the annihilation of the historic symmetry center and the creation of a new symmetry center with a reverse symmetry flavor that will extend its live in the future.

The passage of the symmetry centers through the rim goes together with annihilation and creation phenomena for the objects that reside on these platforms. Thus, this passage is related to the annihilation and creation of elementary particles.

In the quaternionic space-progression model the existence of symmetry centers is independent of progression. With other words the number of symmetry centers is a model constant. The passage through the rim does not influence this number. Only the characteristics of the combination of the symmetry center and the background parameter space are affected by the passage.

4.4 The embedding field

Apart from the symmetry related fields $\mathfrak{A}^x$ that are raised by the charges of the symmetry centers at least one other fields exists. That field is the embedding field. The embedding field is not directly affected by the symmetry related charges of the symmetry centers. However, this field is affected by the embedding of artifacts that are picked by controlling mechanisms from the private domain of a symmetry center H_n^x. and then embedded by the controlling mechanism into the embedding continuum, which is represented by the continuum eigenspace of operator $\mathfrak{C}$. Each of these mechanisms operates in a cyclic and stochastic fashion. The result is a recurrently regenerated coherent location swarm that also represent a stochastic hopping path. The swarm is generated within the symmetry center $\mathfrak{S}_n^x$ and is encapsulated by $\partial\mathrm{H}_n^x$. Since the embedding artifacts live near the geometric center of the symmetry center, the domain Ω also holds for the field $\mathfrak{C}$. The actions of the mechanisms deform the field $\mathfrak{C}$ inside the floating regions H_n^x. The deformation reaches beyond the region H_n^x.

The mechanism creates an elementary object, which is able to deform the embedding field $\mathfrak{C}$ and inherits the symmetry related charge from the symmetry center. The deformation represents the gravitation potential of the elementary object that owns the swarm. This is treated in more detail in reference [5].

4.5 At the start of progression

At progression value $\tau = 0$, the mechanisms that generate the artifacts, which cause discontinuities in the embedding manifold $\mathfrak{C}$ have not yet done any work. It means that this manifold was flat and its defining function equaled its parameter space at instance $\tau = 0$.

The model offers the possibility that the domain Ω expands as a function of τ. In that case it is possible that domain Ω covers a growing amount of symmetry centers.

5 Discussion

This paper only considered the divergence based version of the generalized Stokes theorem. The consequences for the curl based version are not investigated. From fluid dynamics it is known that artifacts that are embedded in a fluid may suffer from the vorticity of the embedding field [2].

The concept of exterior derivative is carefully crafted by skillful mathematicians, such that it becomes independent of the selection of parameter spaces. However, in a situation like this in which one parameter space floats on top of another, the selection of the parameter space does matter. The symmetry flavors of the coupled parameter spaces determine the values of the integrals that account for the contributions of the artifacts. It is represented by the symmetry related charges of these artifacts [5]. These symmetry related charges are supposed to be located at the geometric centers of the symmetry centers.

As happens so often, physical reality reveals facts (the symmetry related charges) that cannot easily be discovered by skilled mathematicians. The standard model contains a short list of electric charges that correspond to the symmetry related charges. The standard model does not give an explanation for the existence of this short list. Here it becomes clear that the electric charge and the color charge are a properties of connected parameter spaces and not a properties of the objects that use these parameter spaces. The objects inherit the charge properties from the platform on which they reside.

This model is no more and no less than a mathematical test case. The paper does not pretend that physical reality behaves like this model. But the methods used and the results obtained in this paper might learn more about how physical reality can be structured and how it can behave.

References

[1] https://en.wikipedia.org/wiki/Stokes%27_theorem#General_formulation.

[2] Justin Shaw, Invariant Vector Calculus.
http://www.math.uwaterloo.ca/~j9shaw/Invariant%20Vector%20Calculus.pdf

[3] The reverse bra-ket method. http://vixra.org/abs/1511.0266

[4] Fermion Symmetry Flavors. http://vixra.org/abs/1512.0225

[5] Foundation of a Mathematical Model of Physical Reality. http://vixra.org/abs/1511.0074