LCC: QA1-939
UDC: 517.9

MODULATED CHAOS AS A SOURCE OF IMAGES FOR NUMBERS

J.C. Tiago de Oliveira

Department of Mathematics, University of Évora
Institute of Scientific Research Bento da Rocha Cabral **(Portugal)**
E-mail: **zct@uevora.pt**

ABSTRACT

Mario Markus, a Chilean scientist and artist from Dortmund Max Planck Institute, has exposed a large set of images of Lyapunoff exponents for the logistic equation modulated through rhythmic oscillation of parameters. The pictures display features like foreground/background contrast, visualizing superstability, structural instability and, above all, multistability, in a way visually analogous to three-dimensional representation.

See, for instance, http://www.mariomarkus.com/hp4.html.

The present papers aims to classify, through codification of numbers in the unit interval, the ensemble of images thus generated. The above is intended as a part of a still unfulfilled work in progress, the classification of style in visual fractal images-a common endeavour to Art and Science.

Key words: chaos, fractals, pictures of Lyapounoff

1. INTRODUCTIONS

The pictures of Lyapounoff exponents, published by Markus et al (1), corresponding to symbolic sequences which control a quadratic – or topologically conjugate – iteration, is, through the alternative nature of parameters A and B, equivalent to a succession of zeros and ones. For such a symbolic sequence, the natural topology is the one of a Cantos set.

Now, it is also true that the infinite ensemble generated by sequencing zero and one is the transcription of the unit interval in basis 2; to what conditions is such a sequence bound to obey, in order to designate a generic real number, instead of a point in a meagre, zero – measure set such as a Cantor set?

Are such conditions compatible with the hypothesis basilar to the works of Markus?

If so it was, this would be another instantiation of a new paradigm – constructing fractals as number images.

A procedure developed in divergent ways by a few authors:

- **Jenny Harrison** (2), through a construction inspired by Denjoy and leading into continued fractals
- **Michel Mendes-France** (3), when he considered number-theoretic properties of sequences of coefficients for trigonometric sums.
- The early deported **Mário Sarreira and José Sousa Ramos** (4) in their codification of a cubic Mandelbrot set.

2. DEVELOPMENT

(1) 0,111111... = 1.000... is the constitutive equation of the binary number system. For a radix b > 2, it will not happen. $0.111\ldots = \frac{1}{b} + \frac{1}{b^2} + \ldots < 1.000\ldots$; therefore, the free semigroup generated by the two symbols 0 and 1 will define a Cantos set.

The inverse inequality (0.1111... > 1.00...) is accomplished for b < 2, were a multiplicity of developments, with a magnitude dependent on the location, is present for each point of the unit segment, thus structuring it as a multifractal.

Equation (1) fills the same role as the “just-touching collage” in Barnsley (5); similar conditions may be found in Williams, Dubuc, Hata (6, 7, 8), along with iterated function systems, whose sum is the unit matrix, giving rise to connected sets.

The apparent tridimensionality of the pictures in (1) is a consequence of the difference between the basins of attraction, the cardinality of which is conjectured to be equal to the periodicity in the control signal (the A B sequence).

Such a conjecture would validate the following corollaries:

a)The existence of a shift symmetry in the control sequence, with the same period as it
b)The irrelevance of the first N terms of the control sequence, for whatever N, in periodic A B sequences

c) Modulo the postulate of equivalence between images, generated by points with a multiple definition of basins of attraction (depending on the initial terms of control sequence), then the shift symmetry of period P reported in a) becomes,, by generalization, a unit-shift symmetry.

From these points we are bound to deduce, for every 01 sequence:

d) Every multiple or sub multiple by a power of 2 is equivalent and

e) Every two sequences diverging in a finite number of transient terms are equivalent

Thus: e) => 0.001001... ≡ 0.010010... ≡ 0.1001001...≡

f) => 0.111001001... = 0.001101001...

This means that for the set of numbers, in the unit interval, expanded in binary system in a periodic way (dyadic rationals), a quotient set is generated through these equivalences.

Are these quotient classes the real numbers, or something alike in a Cantos set?

An answer is suggested by the geometric significance of equation (1): we are dealing with sequences ABBB... and BAAA..., which as the same as (through the exclusion of the first symbol, a transient) every constant sequence (of period one).

Their identity, after (1), simply means that both display the same structure (the "haircomb" generated by the cartesian product of the chaotic windows in the parameters of a Feigenbaum bifurcation × the unit interval); the interchange between A/B being displayed as different orientation (horizontal versus vertical).

This is only a part of a more general isomorphism, the one of duality.

It can be expressed in 3 ways:

a) Symbolically, by the replacement of all A's by B's and vice-versa (reminiscent of the ambivalence of symbolizing, once quoted by Wihgsenstein: it is necessary that P and and P mas designate the same utterance)

b) Geometrically, by transposition: reflecting along the diagonal of the unit square

c) Arithmetically, as the symmetry between X and 1-X and the unit interval (dual its) replacing the unit interval by its half

Illustrating c) by an exemple, we have:

AAABBABAB		000110101
=	means	≡
BBBAABABA		111011010
	Which sm is	111111111... =1

3. THE TRIVIAL CASES: PERIODS 1, 2, 3

<u>Period 1</u>: AAA... or BBB..., 000... ≡ 111...as above (Feigenbaun ∩ chaotic window X [01])

<u>Period 2</u>: of the 4 possibilities (AA, AB, BA e BB). Two reinstate the period 1 are to by excluded: 00 and 11. The two others, 01 and 10, are identical both by shift and by duality.

Period 3: 000 ≡ 111 are excluded (period 1)

All the other 6 are equivalent:

001 → shift → 010 → 100
↓ ↓ ↓
Duality
110 → shift → 101 → 011

Therefore, until now, only one sequence modular above equivalences exists in each period 1, 2, 3. We might name then after their numeric values in binary – 1, 2, 4 corresponding to 1, 10, 100.

From now on everything will became more complex.

4. PERIODS 4 TO 7

Period 4: 16 sequences

- 0000 ≡ 1111 are excluded (period 1)
- 1010 → shift → 0101
  ↓
  Duality
  ↓
  0101

Are identifiably excluded as their repeat the sequence of period 2

The excluded sequences correspond to numbers 0,19, 9,10.

As for the other i 2 sequences, we shall do arithmetic in decimal, instead of binary.

(duality = (2^4 - 1) - X = = 15 – X

	1 →	2 →	4 →	8
	↓	↓	↓	↓
	14	13	11	7

(shift = product by 2)

As well as

3 → 6 → 12
↓ ↓ ↓
12 9 3

As a convention, we shall designate them as the 8 and 12 sequences, in order th enhance the leading digit 1

8 : 1000
12 : 1100

Period 5:

- Exclusion of 00000 and 11111
- No repeating of sub multiple sub sequences as 5 is a prime number
- Generation of their respective sequences

Shift = 2x, Duality = 31-x

1 → 2 → 4 → 8 → 16
↓ ↓ ↓ ↓ ↓
30 29 27 23 15

Also, there is

3 → 6 → 12 → 24
↓ ↓ ↓ ↓
28 25 19 7

Which merges with

14 → 28
↓ ↓
17 3

In fact, the numbers subsuming them (24,17) represent the same pattern, ajust from a shift:

24 : 11000
17 : 10001

The same phenomenon occurs between

5 → 10 → 20
↓ ↓ ↓
26 21 11

and

9 → 18
↓ ↓
22 13

20 : 10100 being equivalent by 2 Shift to 18 : 10010

Therefore, we shall select, as tokens for periods sequences:

16 : 10000
17 : 10001
18 : 10010

Period 6: 64 sequences; elimination of 0 and 1, as usual

6 being a multiple of 2 and 3, we are bound to eliminate repetitions of segments 10 and 100, live
101010 : 42
100100 : 36

But it proves better to start by their duals

010101 : 21
011011 : 27

```
21 → 42                          27 → 54    45
↓      ↓                         ↓      ↓     ↑
42    21                         36     9 → 18 → 36
```

54 numbers remain, to be distributed.

```
1 → 2 → 4 → 8 → 16 → 32
↓    ↓    ↓    ↓     ↓      ↓
62  61   59   55    47     31

 3 → 6 → 12 → 24 → 48     33
↓     ↓      ↓      ↓      ↓       ↑
60   57    51     39     15 → 31 → 60

 5 → 10 → 20 → 40     17 → 34
↓      ↓        ↓      ↓       ↑        ↓
58    53     43     23 → 46     29 → 58

 7 → 14 → 28
↓      ↓       ↓
56    49     35
11 → 22 → 44     25 → 50      37
↓      ↓       ↓        ↑        ↓        ↑
52    41     19 → 38     13 → 26 → 52
```

Therefore the sequences will be codified as:

```
32  :  100000
33  :  100001
34  :  100010          (or 56 = 111000)
35  :  100011
36  :  100101
```

Period 7: excluding 0 and 1, and there being no submultiples, we need to explore 126 numbers

```
  1 → 2 → 4 → 8 → 16 → 32 → 64
 ↓     ↓     ↓     ↓      ↓        ↓       ↓
126  125  123  119  111     95     63 → 126

  3 → 6 → 12 → 24 → 48 → 96    65
 ↓     ↓      ↓       ↓      ↓       ↓       ↑
124  121   115    103    79     31 → 62 → 124

  5 → 10 → 20 → 40 → 80     33 → 66
 ↓      ↓        ↓      ↓         ↓      ↑        ↓
122   117    107    77      47 → 94      61 → 122

  7 → 14 → 28 → 56 → 112    97     67
 ↓      ↓        ↓       ↓         ↓       ↑       ↑
120   113     99     71       15 → 30 → 60

  9 → 18 → 36 → 72     17 → 34 → 68
 ↓      ↓        ↓       ↓        ↑       ↓       ↓
118   109     91     55 → 110    93     59

 11 → 22 → 44 → 88 → 49 → 98     69
 ↓      ↓         ↓        ↓        ↑       ↓       ↑
116   105     83      39      78    29  → 58

 13 → 26 → 52 → 104    81     35 → 70
 ↓       ↓       ↓         ↓       ↑        ↑       ↓
114    101    65      23 → 46 → 92     57
```

15 and 17 are not suitable generators, as previously they were found. We might remark that these sequences always have 14 elements; therefore, we need 2 more (7 + 2 = 9, 9 x 14 = 126).

The last elements in the 1st line grow according to arithmetic progression (647,70).
71 is present in one previous sequence, as is 72.
So, 73 and 74 should be good generators.

73 19 → 38 → 76 25 → 50 → 100
↓ ↑ ↓ ↓ ↑ ↓ ↓
54 → 108 89 51 → 102 77 27
74 21 → 42 → 84 41 → 82 37
↓ ↑ ↓ ↓ ↑ ↓ ↑
53 → 106 85 43 → 86 45 → 90

Until now, we have the followings symbolic sequences

Periods

1, 2, 3	**4**	**5**	**6**	**7**
1 (= 1	8 (=1000	16-10000	32	64
2 (= 10	9 (= 1001	17-10001	33	65
4 (= 100		18-10010	34	66
			35	67
			37	68
				69
				70
				73
				74

5. GENERALIZING

If ^{p} is a prime number, this having no sub-multiples, the only excluded sequences are 0 and $2^p - 1$ (0000... and 1111...).

Therefore, if every single set of isomorphic numbers (in terms of duality + shift) has the same quantity of terms, as they did until now, their quality must depend on the factorization of $2^p - 2 = 2(2^p - 1)$.

Let us try it for the next prime numbers: 11, 13, 17.

11: $2^{11} = 2048$; $2^{11} - 2 = 2046 = 2 \times 1023 = 6 \times 341 = 6 \times 31 \times 11 = 11 \times 186 =$
$= 22 \times 93$

Therefore, there should be 93 ensembles of 22 equivalent numbers.

13: $2^{13} - 2 = 8190 = 10 \times 819 = 13 \times 10 \times 63 = 26 \times 315$
315 equivalent segments

17: $(2^{(17-1)}-1)$ being divisible by 17, this quotient should be the member of different sequences.

For composite numbers, the problem is my difficult, we should eliminate the instantiation of sub-multiples. For instance, in the period 8 ($2^8 = 256$) we eliminate:

- 0 and 255
- The translates of period 2 (10101010 and 01010101)
- The duals and translate of 8 and 9

8 → 16 → 32 → 64 → 128 1 → 2 → 4
↓ ↓ ↓ ↓ ↓ ↑ ↓ ↓
247 239 223 191 127 → 254 253 251

9, along with fifteen others.
Therefore, 256 – 2 – 2 – 16 – 16 = 220 will remain, to be distributed through 16 – long ensembles.

6. CONCLUSIONS

The number of image-generations sequences grows with N in a was majorated by $(2^{(p-1)} - 1) / p$, for prime p; which means, exponentially in N.

Therefore, it is not a denumerable set. Is it a continuation, or a Cantor set, or something in between (a “fat” Cantor set) ?

As a tentative answer, let us be inspired by the following:

- A number such as 0.897897897... in decimal radix expansion is = 897.0.001001001... = 897 / 999

Therefore, in fractal number system, the corresponding numbers are to be
1 / 1, 2 / 3, 4 / 7, 8 / 19, 9 / 15 = 3 / 5
16 / 31, 17 / 31, 18 / 31, 32 / 63, 33 / 63 = 11 / 21
34 / 63, 35 / 63, 37 / 63, 64 / 127, 65 / 127, 66 / 127, 67 / 127, 68 / 127, 69 / 127, 70 / 127, 73 / 127, 74 / 127

How for did such a sequence fil the [$\frac{1}{2}, 1$] interval ? Is it mifuncly deuse ? Fat or meagre ? What about its Hausdorff dimension ?

Sequences, essentially because shift symmetric fails (it was a consequence of periodicity); therefore applet tridimensionality (multimodality) is lost.

However, duality, as well as irrelevance of transients, are maintained.

We might understand for the above that, restrictions of the a periodic, being less severe, on stating hypothesis – Markus pictures as images for numbers – which was not refuted in the realm of dyadic nrationals, maintains the pertinence of a conjecture.

REFERENCES

1. **Rossler Kiwi Hess Markus** (1989) Modulated Non-linear Processes and a Novel Mechanism to Induce Chaos Phiss. Rev. A, 39, 11
2. **Harrison** (1989) In R.L. Devanels ed, Chaos and Fractals, AMS
3. **Mendes France** (1991) La Constante de Plack d'une Comba Fractal Geometry and analysis
4. **Sarreira, Sousa Ramos** (1991) Symbolic Dynamics of Hercition of Cubic Complex Maps
5. **European Conference on Heration Theory**
6. **Barnsley** (1989) Fractals Everywhere
7. **Williams** (1992 The composition of Constructive Transposation Boletim Sociedade Matemática-Brasil
8. **Dahac** (1987) In Bull. Soc. Math. Quebec
9. **Hata** (1991) In Fractal Geometry and Analysis