The Lorentz Force as Pure Geodesic Motion in Four-Dimensional Spacetime

Jay R. Yablon
910 Northumberland Drive
Schenectady, New York 12309-2814
yablon@alum.mit.edu

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Abstract: We develop a linear element ds in ordinary four-dimensional spacetime which, when held stationary under worldline variations, leads to the gravitational equations of geodesic motion extended to include the Lorentz force law. We see that in the presence of an electromagnetic vector potential A^μ, all that is needed to obtain this result is to follow the well-known gauge theory prescription of replacing the kinetic momentum p^μ with a canonical momentum $\pi^\mu=p^\mu+eA^\mu$ in the mass / momentum relationship $m^2=p_\sigma p^\sigma$, and then to apply variational calculus to obtain the motion of charged particles in this potential.

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1. Introduction

In §9 of his landmark 1916 paper [1], Albert Einstein first derived the geodesic equation of motion $d^2x^\mu / ds^2 = -\Gamma^\mu{}_{\alpha\beta}\left(dx^\alpha / ds\right)\left(dx^\beta / ds\right)$ for a particle in a gravitational field based on the variation $0=\delta\int_A^B ds$ of the linear metric element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ between any two spacetime events *A* and *B* at which the worldliness of different observers meet so that their clocks and measuring rods can be coordinated at the outset *A* and then compared at the conclusion *B*. Notably absent from [1], however, was a similar geodesic development of the Lorentz force law $d^2x^\mu / ds^2 = \left(e/m\right)F^\mu{}_\alpha\left(dx^\alpha / ds\right)$. Subsequent papers by Kaluza [2] and Klein [3] did succeed in explaining the Lorentz force as a type of geodesic motion and even gave a geometric explanation for the electric charge itself, but only at the cost of adding a fifth dimension to spacetime and curling that dimension into a cylinder. To date, a century later, there still does not appear to have been any fully-successful attempt to obtain the Lorentz force from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime. In this letter, we show how this is done.

2. Basis and derivation

As the basis for obtaining the Lorentz force from a geodesic variation in four dimensions, we begin with the equation $m^2 = p_\sigma p^\sigma$ that describes the relativistic relationship between any mass *m* and its "kinetic" energy-momentum $p^\mu = mu^\mu = m\left(dx^\mu / ds\right)$. We then promote this kinetic momentum to a "canonical" momentum π^μ via the prescription $p^\mu \to \pi^\mu = p^\mu + eA^\mu$ taught by the local gauge (really, phase) theory of Hermann Weyl developed over 1918 to 1929 in [4], [5], [6], and so obtain $m^2 = p_\sigma p^\sigma \to m^2 = \pi_\sigma \pi^\sigma$. It will be appreciated that this prescription is the momentum space equivalent of $\partial_\mu = \partial / \partial x^\mu \to D_\mu = \partial_\mu + ieA_\mu$ which is the gauge-covariant derivative specified in a configuration space for which the metric tensor of the tangent flat Minkowski space is $\mathrm{diag}\left(\eta_{\mu\nu}\right) = \left(+1,-1,-1,-1\right)$. Consequently, deconstructing into a linear equation using the Dirac matrices $\frac{1}{2}\left\{\gamma^\mu,\gamma^\nu\right\} = \eta^{\mu\nu}$ in flat spacetime, one can employ $m^2 = \pi_\sigma \pi^\sigma$ to obtain Dirac's equation $\left(i\gamma^\mu D_\mu - m\right)\psi = 0$ for an electron wavefunction ψ in an electromagnetic potential A_μ, which equation Dirac first derived in [7] for a free electron in a form equivalent to $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$, i.e., without yet using $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$.

So to obtain the Lorentz force from a geodesic variation in spacetime, we backtrack from $m^2 = \pi_\sigma \pi^\sigma$ to the linear metric element:

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to ds^2 &= g_{\mu\nu}d\chi^\mu d\chi^\nu = g_{\mu\nu}\left(dx^\mu + ds\left(e/m\right)A^\mu\right)\left(dx^\nu + ds\left(e/m\right)A^\nu\right) \\ &= g_{\mu\nu}dx^\mu dx^\nu + 2\left(e/m\right)A_\sigma dx^\sigma ds + \left(e/m\right)^2 g_{\mu\nu}A^\mu A^\nu ds^2 \end{aligned}, \qquad (2.1)$$

which uses a canonical gauge prescription for the spacetime coordinates themselves, namely:

$$dx^{\mu} \to d\chi^{\mu} = dx^{\mu} + ds\left(e/m\right)A^{\mu} . \tag{2.2}$$

This is just another variation of $p^{\mu} \to \pi^{\mu} = p^{\mu} + eA^{\mu}$ and $\partial_{\mu} \to D_{\mu} = \partial_{\mu} + ieA_{\mu}$. Indeed, it is easily seen that if one multiplies $ds^2 = \left(dx_{\sigma} + ds\left(e/m\right)A_{\sigma}\right)\left(dx^{\sigma} + ds\left(e/m\right)A^{\sigma}\right)$ in (2.1) through by m^2/ds, the result is identical to the canonical $m^2 = \pi_{\sigma}\pi^{\sigma}$. Now, all we need do is apply a variation $0=\delta\int_A^B ds$ to the linear element (2.1) and the Lorentz force naturally emerges as a geodesic equation of motion right alongside of the gravitational equation of motion.

Proceeding with this derivation which largely parallels that in [8] to which it may be helpful to refer, we first use (2.1) to construct the number

$$1 = \sqrt{g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}} , \tag{2.3}$$

which we then use to write the variation as:

$$0 = \delta\int_A^B ds = \delta\int_A^B ds\sqrt{g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}} . \tag{2.4}$$

Applying δ to the integrand and using (2.3) to clear the denominator, this yields:

$$0 = \delta\int_A^B ds = \frac{1}{2}\int_A^B ds\delta\left(g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}\right). \tag{2.5}$$

Dropping the ½ and using the product rule, while assuming that there is no variation in the charge-to-mass ratio – i.e., that $\delta(e/m)=0$ – over the path from *A* to *B*, we now distribute δ using the product rule to obtain:

$$0 = \int_A^B ds\left(\begin{array}{l}\delta g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + g_{\mu\nu}\frac{d\delta x^{\mu}}{ds}\frac{dx^{\nu}}{ds} + g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{d\delta x^{\nu}}{ds} + 2\frac{e}{m}\delta A_{\sigma}\frac{dx^{\sigma}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{d\delta x^{\sigma}}{ds} \\ +\left(e/m\right)^2\left(\delta g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\delta A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\delta A^{\nu}\right)\end{array}\right). \tag{2.6}$$

One can use the chain rule in the small variation $\delta \to \partial$ limit to show that $\delta g_{\mu\nu} = \partial_{\alpha}g_{\mu\nu}\delta x^{\alpha}$ and $\delta A_{\sigma} = \partial_{\alpha}A_{\sigma}\delta x^{\alpha}$. Thus the bottom line equals $\delta x^{\alpha}\left(e/m\right)^2\left(\partial_{\alpha}g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\partial_{\alpha}A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\partial_{\alpha}A^{\nu}\right)$. Likewise, we may recondense $\partial_{\alpha}\left(g_{\mu\nu}A^{\mu}A^{\nu}\right) = \partial_{\alpha}g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\partial_{\alpha}A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\partial_{\alpha}A^{\nu}$ via the product rule. As a result, the entire integral on the bottom line contains a total derivative given by:

$$\int_A^B \delta x^\alpha \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial x^\alpha}\left(g_{\mu\nu}A^\mu A^\nu\right) ds = \delta x^\alpha \left(\frac{e}{m}\right)^2 \frac{\partial s}{\partial x^\alpha}\left(g_{\mu\nu}A^\mu A^\nu\right)\Big|_A^B = 0 . \tag{2.7}$$

This equals zero, because the two worldlines intersect at events A and B but have a slight variational difference between A and B otherwise, so that $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$ while $\delta x^\sigma \neq 0$ elsewhere. Consequently, the bottom line of (2.6) zeros out, leaving us with:

$$0 = \int_A^B ds \left(\delta g_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{dx^\mu}{ds}\frac{d\delta x^\nu}{ds} + 2\frac{e}{m}\delta A_\sigma \frac{dx^\sigma}{ds} + 2\frac{e}{m}A_\sigma \frac{d\delta x^\sigma}{ds}\right). \tag{2.8}$$

From here, we again use $\delta g_{\mu\nu} = \partial_\alpha g_{\mu\nu}\delta x^\alpha$ and $\delta A_\sigma = \partial_\alpha A_\sigma \delta x^\alpha$, also with a renaming of indexes and using the symmetry of $g_{\mu\nu}$ to combine the second and third terms above, to obtain:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + 2g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds} + 2\delta x^\alpha \frac{e}{m}\partial_\alpha A_\sigma \frac{dx^\sigma}{ds} + 2\frac{e}{m}A_\sigma \frac{d\delta x^\sigma}{ds}\right). \tag{2.9}$$

Next, we integrate by parts. First, we use the product rule to replace $g_{\mu\nu}\left(d\delta x^\mu / ds\right)\left(dx^\nu / ds\right) = \left(d/ds\right)\left(\delta x^\mu g_{\mu\nu}\left(dx^\nu / ds\right)\right) - \delta x^\mu \left(d/ds\right)\left(g_{\mu\nu}\left(dx^\nu / ds\right)\right)$ and likewise $\left(dA_\sigma / ds\right)\delta x^\sigma = \left(d/ds\right)\left(A_\sigma \delta x^\sigma\right) - A_\sigma d\delta x^\sigma / ds$. But the terms containing the total derivatives will vanish for the same reasons that the terms in (2.7) vanished as a result of the boundary conditions $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$. As a result, (2.9) now becomes:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu \frac{d}{ds}\left(g_{\mu\nu}\frac{dx^\nu}{ds}\right) + 2\delta x^\alpha \frac{e}{m}\partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2\delta x^\sigma \frac{e}{m}\frac{dA_\sigma}{ds}\right). \tag{2.10}$$

Applying the d/ds derivative contained in the second term above then yields:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu}\frac{d^2 x^\nu}{ds^2} - 2\delta x^\mu \frac{dg_{\mu\nu}}{ds}\frac{dx^\nu}{ds} + 2\delta x^\alpha \frac{e}{m}\partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2\delta x^\sigma \frac{e}{m}\frac{dA_\sigma}{ds}\right), \tag{2.11}$$

for the first time revealing the acceleration $d^2 x^\nu / ds^2$ in the second term above.

Nest, we use the chain rules $dg_{\mu\nu}/ds = \partial_\alpha g_{\mu\nu}\left(dx^\alpha / ds\right)$ and $dA_\sigma / ds = \partial_\alpha A_\sigma \left(dx^\alpha / ds\right)$ to rewrite the third and fifth terms above, thus obtaining:

$$0 = \int_A^B ds \begin{pmatrix} \delta x^\alpha \partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu}\frac{d^2 x^\nu}{ds^2} - 2\delta x^\mu \partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{ds}\frac{dx^\nu}{ds} \\ +2\delta x^\alpha \frac{e}{m}\partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2\delta x^\sigma \frac{e}{m}\partial_\alpha A_\sigma \frac{dx^\alpha}{ds} \end{pmatrix}. \tag{2.12}$$

In the bottom line above, we may rename indexes $\alpha \leftrightarrow \sigma$ in the last term, to find that we may rewrite $\delta x^\alpha \partial_\alpha A_\sigma \left(dx^\sigma / ds \right) - \delta x^\sigma \partial_\alpha A_\sigma \left(dx^\alpha / ds \right) = \delta x^\alpha F_{\alpha\sigma} \left(dx^\sigma / ds \right)$ using the electromagnetic field strength tensor $F_{\alpha\sigma} = \partial_\alpha A_\sigma - \partial_\sigma A_\alpha$, which has now appeared as a result of the variation. So the above now simplifies to:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu} \frac{d^2 x^\nu}{ds^2} - 2\delta x^\mu \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} + 2\delta x^\alpha \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds} \right). \tag{2.13}$$

Now we rename indexes so that the δx terms all contain the index α, that is, so all of these terms are δx^α. We then factor this out and interchange the first and second terms, obtaining:

$$0 = \int_A^B ds \delta x^\alpha \left(-2 g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} + \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\partial_\mu g_{\alpha\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds} \right). \tag{2.14}$$

For material worldlines, $ds \neq 0$. Likewise, while $\delta x^\sigma \left(A \right) = \delta x^\sigma \left(B \right) = 0$ at the boundaries, between these boundaries where the variation occurs, $\delta x^\sigma \neq 0$. Thus, multiplying through by ½, for (2.14) to be true we must have:

$$0 = -g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} + \frac{1}{2} \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \partial_\mu g_{\alpha\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds}. \tag{2.15}$$

Now we move the acceleration term to the left, split the term with $\partial_\mu g_{\alpha\nu} = \frac{1}{2} \partial_\mu g_{\alpha\nu} + \frac{1}{2} \partial_\mu g_{\alpha\nu}$ into two halves, rename some indexes while using the symmetry of $g_{\alpha\nu}$, and finally multiply through by $g^{\beta\alpha}$ and then raise indexes. This all yields:

$$\frac{d^2 x^\beta}{ds^2} = \frac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F^\beta{}_\sigma \frac{dx^\sigma}{ds}. \tag{2.16}$$

But of course, we recognize that the Christoffel symbols $-\Gamma^\beta{}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right)$. As a consequence, the above reduces to:

$$\frac{d^2 x^\beta}{ds^2} = -\Gamma^\beta{}_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F^\beta{}_\sigma \frac{dx^\sigma}{ds}. \tag{2.17}$$

In the presence of gravitational and electromagnetic fields, this contains both the equations of gravitational motion and the Lorentz force law, obtained via the geodesic variation of the canonical invariant metric length element (2.1). In the absence of gravitation, i.e., for $g_{\mu\nu} = \eta_{\mu\nu}$ thus $\Gamma^\beta{}_{\mu\nu} = 0$, this reduces to the Lorentz force law. As a result, we have proved that the Lorentz force law of electrodynamics may indeed be obtained from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime geometry.

3. Conclusion and further lines for development

The result (2.17) proves that charged particles moving according to the Lorentz force law are in fact simply following geodesic paths in spacetime, so long as we use Weyl's canonical prescription in form of $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ from (2.2) to define the linear metric element by $ds^2 = g_{\mu\nu}d\chi^\mu d\chi^\nu$ as shown in (2.1).

Because the metric length $ds^2 = g_{\mu\nu}d\chi^\mu d\chi^\nu$ of (2.1) under a variation $0=\delta\int_A^B ds$ simultaneously provides a geodesic description of motion in a gravitational field and in an electromagnetic field, and because the prescription $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ is no more than a variant of Weyl's gauge prescriptions $p^\mu \to \pi^\mu = p^\mu + eA^\mu$ in momentum space and $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ in configuration space and leads directly as well to Dirac's equation $(i\gamma^\mu D_\mu - m)\psi = 0$ for an interacting fermion, this may fairly be regarded as a classical metric-level unification of electrodynamics with gravitation, using four spacetime dimensions only. But what about the field equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ which specifies the gravitational dynamics?

Because the Lorentz force (2.17) is obtained by dilating the differential coordinate elements via $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ without in any way altering the metric tensor $g_{\mu\nu}$ as is done, for example, in Kaluza-Klein theory, one may incorrectly conclude that the electromagnetic interaction does not affect spacetime curvature as represented by the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ with the field dynamics specified by $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. However, one must keep in mind that the Reimann tensor may be defined via $R^\alpha{}_{\beta\mu\nu}V_\alpha \equiv \left[\partial_{;\nu}, \partial_{;\mu}\right]V_\beta$ as a measure of the extent to which the gravitationally-covariant derivatives $\partial_{;\mu}V_\beta = \partial_\mu V_\beta - \Gamma^\alpha{}_{\mu\beta}V_\alpha$ operating on a vector V_β do not commute. Likewise, the field strength tensor $F_{\nu\mu}$ may be defined via $ieF_{\nu\mu}V_\beta \equiv \left[D_\nu, D_\mu\right]V_\beta$ as a measure of the extent to which the gauge-covariant derivatives $D_\mu V_\beta = (\partial_\mu + ieA)V_\beta$ do not commute when operating on this same vector V_β. Indeed, this latter definition results in $F_{\nu\mu} = D_{[\nu}A_{\mu]} = \partial_{[\nu}A_{\mu]} + ie\left[A_\nu, A_\mu\right]$ for a *non-abelian* gauge theory where $\left[A_\nu, A_\mu\right] \neq 0$, which simplifies to $F_{\nu\mu} = \partial_{[\nu}A_{\mu]}$ for an abelian theory such a electrodynamics in which $\left[A_\nu, A_\mu\right] = 0$.

Therefore, if we apply Weyl's canonical prescription to the gravitationally-covariant derivatives by employing:

$$\partial_{;\mu}V_\beta = \partial_\mu V_\beta - \Gamma^\alpha{}_{\mu\beta}V_\alpha \to D_{;\mu}V_\beta = (\partial_\mu + ieA_\mu)V_\beta - \Gamma^\alpha{}_{\mu\beta}V_\alpha, \tag{3.1}$$

which is the same prescription that in the form $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ of (2.2) yielded the Lorentz force law in (2.17), and if we then use *these derivatives (3.1)* to canonically *define* the Riemann tensor as:

$$R^\alpha{}_{\beta\mu\nu}V_\alpha \equiv \left[D_{;\nu}, D_{;\mu}\right]V_\beta , \tag{3.2}$$

it can be expected as a consequence of $ieF_{\nu\mu}V_\beta \equiv \left[D_\nu, D_\mu\right]V_\beta$ that the electrodynamic fields $F_{\nu\mu}$ and possibly potentials A_μ will appear in the Riemann tensor and therefore in the field equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. Further, because $ieF_{\nu\mu}V_\beta \equiv \left[D_\nu, D_\mu\right]V_\beta$ encompasses both *abelian and non-abelian field strengths*, the gravitational field equations using $R_{\beta\mu} = R^\alpha{}_{\beta\mu\alpha}$ and $R = R^\sigma{}_\sigma$ can be made not only to govern abelian electrodynamics, *but also the non-abelian weak and strong interactions*.

Via such a path for further development, it may well be possible to unify gravitation not only with electrodynamics, but with the remaining weak and strong interactions, all while maintaining full consistency with quantum mechanics because the canonical gauge prescriptions $p^\mu \to \pi^\mu = p^\mu + eA^\mu$ and $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ and now $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ remain at the root of the entire development. The main questions that would remain following such a unification, would be as to the specific non-abelian gauge groups that operate at any given energy ranging up to the Planck mass, and how the symmetry of those groups becomes broken at lower energies leading to the phenomenological group $SU(3)_C \times SU(2)_W \times U(1)_Y \to SU(3)_C \times U(1)_{em}$ and the fermions on which these groups act. The author has previously published on these questions, and even shown how the three generations of quarks and leptons originate, at [9].

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