The Lorentz Force as Pure Geodesic Motion in Four-Dimensional Spacetime

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Abstract: We develop a linear metric element ds in ordinary four-dimensional spacetime which, when held stationary under worldline variations, leads to the gravitational equations of geodesic motion extended to include the Lorentz force law. We see that in the presence of an electromagnetic vector potential A^μ, all that is needed to obtain this result is to follow the well-known gauge theory prescription of replacing the kinetic momentum p^μ with a canonical momentum $\pi^\mu=p^\mu+eA^\mu$ in the mass / momentum relationship $m^2=p_\sigma p^\sigma$, and then to apply variational calculus to obtain the motion of charged particles in this potential. We also show how by this same prescription, Maxwell's classical source-free field equations become embedded within the second Bianchi identity of Riemannian geometry.

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1. Introduction

In §9 of his landmark 1916 paper [1], Albert Einstein first derived the geodesic equation of motion $d^2x^\mu / ds^2 = -\Gamma^\mu{}_{\alpha\beta}\left(dx^\alpha / ds\right)\left(dx^\beta / ds\right)$ for a particle in a gravitational field based on the variation $0=\delta\int_A^B ds$ of the linear metric element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ between any two spacetime events A and B at which the worldlines of different observers meet so that their clocks and measuring rods and scales can be coordinated at the outset A and then compared at the conclusion B. Notably absent from [1], however, was a similar geodesic development of the Lorentz force law $d^2x^\mu / ds^2 = (e/m)F^\mu{}_\alpha\left(dx^\alpha / ds\right)$. Subsequent papers by Kaluza [2] and Klein [3] did succeed in explaining the Lorentz force as a type of geodesic motion and even gave a geometric explanation for the electric charge itself, but only at the cost of adding a fifth dimension to spacetime and curling that dimension into a cylinder. To date, a century later, there still does not appear to have been any fully-successful attempt to obtain the Lorentz force from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime. In this letter, we show how this is done.

2. Basis and derivation

As the basis for obtaining the Lorentz force from a geodesic variation in four dimensions, we begin with the equation $m^2 = p_\sigma p^\sigma$ that describes the relativistic relationship between any mass m and its "kinetic" energy-momentum $p^\mu = mu^\mu = m\left(dx^\mu / ds\right)$. We then promote this kinetic momentum to a "canonical" momentum π^μ via the prescription $p^\mu \to \pi^\mu = p^\mu + eA^\mu$ taught by the local gauge (really, phase) theory of Hermann Weyl developed over 1918 to 1929 in [4], [5], [6], and so obtain $m^2 = p_\sigma p^\sigma \to m^2 = \pi_\sigma \pi^\sigma$. It will be appreciated that this prescription is the momentum space equivalent of $\partial_\mu = \partial / \partial x^\mu \to D_\mu = \partial_\mu + ieA_\mu$ which is the gauge-covariant derivative specified in a configuration space for which the metric tensor of the tangent flat Minkowski space is $\mathrm{diag}\left(\eta_{\mu\nu}\right) = (+1,-1,-1,-1)$. Consequently, deconstructing into a linear equation using the Dirac matrices $\frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} = \eta^{\mu\nu}$ in flat spacetime, one can employ $m^2 = \pi_\sigma \pi^\sigma$ to obtain Dirac's equation $\left(i\gamma^\mu D_\mu - m\right)\psi = 0$ for an electron wavefunction ψ in an electromagnetic potential A_μ, which equation Dirac first derived in [7] for a free electron in a form equivalent to $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$, i.e., without yet using $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$.

So to obtain the Lorentz force from a geodesic variation in spacetime, we backtrack from $m^2 = \pi_\sigma \pi^\sigma$ to a linear metric element:

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to ds^2 = g_{\mu\nu}d\chi^\mu d\chi^\nu &= g_{\mu\nu}\left(dx^\mu + ds\,(e/m)\,A^\mu\right)\left(dx^\nu + ds\,(e/m)\,A^\nu\right) \\ &= g_{\mu\nu}dx^\mu dx^\nu + 2\,(e/m)\,A_\sigma dx^\sigma ds + (e/m)^2\, g_{\mu\nu}A^\mu A^\nu ds^2 \end{aligned}, \tag{2.1}$$

which uses a canonical gauge prescription for the spacetime coordinates themselves, namely:

$$dx^{\mu} \to d\chi^{\mu} = dx^{\mu} + ds(e/m)A^{\mu}. \tag{2.2}$$

This is just another variation of $p^{\mu} \to \pi^{\mu} = p^{\mu} + eA^{\mu}$ and $\partial_{\mu} \to D_{\mu} = \partial_{\mu} + ieA_{\mu}$. Indeed, it is easily seen that if one multiplies $ds^2 = \left(dx_{\sigma} + ds(e/m)A_{\sigma}\right)\left(dx^{\sigma} + ds(e/m)A^{\sigma}\right)$ in (2.1) through by m^2/ds^2, the result is identical to the canonical $m^2 = \pi_{\sigma}\pi^{\sigma}$. Now, all we need do is apply a variation $0=\delta\int_A^B ds$ to the linear element (2.1) and the Lorentz force naturally emerges as a geodesic equation of motion right alongside of the gravitational equation of motion.

Proceeding with this derivation which largely parallels that in the online [8], we first use (2.1) to construct the number

$$1 = \sqrt{g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}}, \tag{2.3}$$

which we then use to write the variation as:

$$0 = \delta\int_A^B ds = \delta\int_A^B ds\sqrt{g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}}. \tag{2.4}$$

Applying δ to the integrand and using (2.3) to clear the denominator, this yields:

$$0 = \delta\int_A^B ds = \frac{1}{2}\int_A^B ds\,\delta\left(g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}\right). \tag{2.5}$$

Dropping the ½ and using the product rule, while assuming that there is no variation in the charge-to-mass ratio – i.e., that $\delta(e/m)=0$ – over the path from *A* to *B*, we now distribute δ using the product rule to obtain:

$$0 = \int_A^B ds\left(\begin{array}{l}\delta g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + g_{\mu\nu}\frac{d\delta x^{\mu}}{ds}\frac{dx^{\nu}}{ds} + g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{d\delta x^{\nu}}{ds} + 2\frac{e}{m}\delta A_{\sigma}\frac{dx^{\sigma}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{d\delta x^{\sigma}}{ds}\\ +(e/m)^2\left(\delta g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\delta A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\delta A^{\nu}\right)\end{array}\right). \tag{2.6}$$

One can use the chain rule in the small variation $\delta \to \partial$ limit to show that $\delta g_{\mu\nu} = \partial_{\alpha}g_{\mu\nu}\delta x^{\alpha}$ and $\delta A_{\sigma} = \partial_{\alpha}A_{\sigma}\delta x^{\alpha}$. So the bottom line equals $\delta x^{\alpha}(e/m)^2\left(\partial_{\alpha}g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\partial_{\alpha}A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\partial_{\alpha}A^{\nu}\right)$. Likewise, we may recondense $\partial_{\alpha}\left(g_{\mu\nu}A^{\mu}A^{\nu}\right) = \partial_{\alpha}g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\partial_{\alpha}A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\partial_{\alpha}A^{\nu}$ via the product rule. Therefore, the entire integral on the bottom line contains a total derivative given by:

$$\int_A^B \delta x^\alpha \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial x^\alpha}\left(g_{\mu\nu}A^\mu A^\nu\right) ds = \delta x^\alpha \left(\frac{e}{m}\right)^2 \frac{\partial s}{\partial x^\alpha}\left(g_{\mu\nu}A^\mu A^\nu\right)\Big|_A^B = 0 \,. \tag{2.7}$$

This equals zero, because the two worldlines intersect at the boundary events A and B but have a slight variational difference between A and B otherwise, so that $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$ while $\delta x^\sigma \neq 0$ elsewhere. Consequently, the bottom line of (2.6) zeros out, leaving us with:

$$0 = \int_A^B ds \left(\delta g_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{dx^\mu}{ds}\frac{d\delta x^\nu}{ds} + 2\frac{e}{m}\delta A_\sigma \frac{dx^\sigma}{ds} + 2\frac{e}{m}A_\sigma \frac{d\delta x^\sigma}{ds} \right). \tag{2.8}$$

From here, again using $\delta g_{\mu\nu} = \partial_\alpha g_{\mu\nu}\delta x^\alpha$ and $\delta A_\sigma = \partial_\alpha A_\sigma \delta x^\alpha$, and also re-indexing and using the symmetry of $g_{\mu\nu}$ to combine the second and third terms above, we obtain:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + 2g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds} + 2\delta x^\alpha \frac{e}{m}\partial_\alpha A_\sigma \frac{dx^\sigma}{ds} + 2\frac{e}{m}A_\sigma \frac{d\delta x^\sigma}{ds} \right). \tag{2.9}$$

Next, we integrate by parts. First, we use the product rule to replace $g_{\mu\nu}\left(d\delta x^\mu / ds\right)\left(dx^\nu / ds\right) = \left(d/ds\right)\left(\delta x^\mu g_{\mu\nu}\left(dx^\nu/ds\right)\right) - \delta x^\mu \left(d/ds\right)\left(g_{\mu\nu}\left(dx^\nu/ds\right)\right)$ and likewise $A_\sigma d\delta x^\sigma / ds = \left(d/ds\right)\left(A_\sigma \delta x^\sigma\right) - \left(dA_\sigma / ds\right)\delta x^\sigma$. But the terms containing the total derivatives will vanish for the same reasons that the terms in (2.7) vanished as a result of the boundary conditions $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$. As a result, (2.9) now becomes:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu \frac{d}{ds}\left(g_{\mu\nu}\frac{dx^\nu}{ds}\right) + 2\delta x^\alpha \frac{e}{m}\partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2\delta x^\sigma \frac{e}{m}\frac{dA_\sigma}{ds} \right). \tag{2.10}$$

Applying the d/ds derivative contained in the second term above then yields:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu}\frac{d^2 x^\nu}{ds^2} - 2\delta x^\mu \frac{dg_{\mu\nu}}{ds}\frac{dx^\nu}{ds} + 2\delta x^\alpha \frac{e}{m}\partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2\delta x^\sigma \frac{e}{m}\frac{dA_\sigma}{ds} \right), \tag{2.11}$$

for the first time revealing the acceleration $d^2 x^\nu / ds^2$ in the second term above.

Next, we use the chain rules $dg_{\mu\nu}/ds = \partial_\alpha g_{\mu\nu}\left(dx^\alpha / ds\right)$ and $dA_\sigma / ds = \partial_\alpha A_\sigma \left(dx^\alpha / ds\right)$ to rewrite the third and fifth terms above, thus obtaining:

$$0 = \int_A^B ds \begin{pmatrix} \delta x^\alpha \partial_\alpha g_{\mu\nu} \dfrac{dx^\mu}{ds}\dfrac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu}\dfrac{d^2 x^\nu}{ds^2} - 2\delta x^\mu \partial_\alpha g_{\mu\nu}\dfrac{dx^\alpha}{ds}\dfrac{dx^\nu}{ds} \\ +2\delta x^\alpha \dfrac{e}{m}\partial_\alpha A_\sigma \dfrac{dx^\sigma}{ds} - 2\delta x^\sigma \dfrac{e}{m}\partial_\alpha A_\sigma \dfrac{dx^\alpha}{ds} \end{pmatrix}. \tag{2.12}$$

In the bottom line above, we may rename indexes $\alpha \leftrightarrow \sigma$ in the last term, to find that we may rewrite $\delta x^{\alpha}\partial_{\alpha}A_{\sigma}\left(dx^{\sigma}/ds\right)-\delta x^{\sigma}\partial_{\alpha}A_{\sigma}\left(dx^{\alpha}/ds\right)=\delta x^{\alpha}F_{\alpha\sigma}\left(dx^{\sigma}/ds\right)$ using the electromagnetic field strength tensor $F_{\alpha\sigma}=\partial_{\alpha}A_{\sigma}-\partial_{\sigma}A_{\alpha}$, which has now appeared as a result of the variation. So the above now simplifies to:

$$0=\int_A^B ds\left(\delta x^{\alpha}\partial_{\alpha}g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}-2\delta x^{\mu}g_{\mu\nu}\frac{d^2x^{\nu}}{ds^2}-2\delta x^{\mu}\partial_{\alpha}g_{\mu\nu}\frac{dx^{\alpha}}{ds}\frac{dx^{\nu}}{ds}+2\delta x^{\alpha}\frac{e}{m}F_{\alpha\sigma}\frac{dx^{\sigma}}{ds}\right).\tag{2.13}$$

Now we rename indexes so that the δx terms all contain the index α, that is, so all of these terms are δx^{α}. We then factor this out and interchange the first and second terms, obtaining:

$$0=\int_A^B ds\delta x^{\alpha}\left(-2g_{\alpha\nu}\frac{d^2x^{\nu}}{ds^2}+\partial_{\alpha}g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}-2\partial_{\mu}g_{\alpha\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}+2\frac{e}{m}F_{\alpha\sigma}\frac{dx^{\sigma}}{ds}\right).\tag{2.14}$$

For material worldlines, $ds\neq 0$. Likewise, while $\delta x^{\sigma}(A)=\delta x^{\sigma}(B)=0$ at the boundaries, between these boundaries where the variation occurs, $\delta x^{\sigma}\neq 0$. Thus, multiplying through by ½, for (2.14) to be true the integrand must be zero, and so we have:

$$0=-g_{\alpha\nu}\frac{d^2x^{\nu}}{ds^2}+\frac{1}{2}\partial_{\alpha}g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}-\partial_{\mu}g_{\alpha\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}+\frac{e}{m}F_{\alpha\sigma}\frac{dx^{\sigma}}{ds}.\tag{2.15}$$

Now we move the acceleration term to the left, split the term with $\partial_{\mu}g_{\alpha\nu}=\frac{1}{2}\partial_{\mu}g_{\alpha\nu}+\frac{1}{2}\partial_{\mu}g_{\alpha\nu}$ into two halves, rename some indexes while using the symmetry of $g_{\alpha\nu}$, and finally multiply through by $g^{\beta\alpha}$ and then raise indexes. This all yields:

$$\frac{d^2x^{\beta}}{ds^2}=\frac{1}{2}g^{\beta\alpha}\left(\partial_{\alpha}g_{\mu\nu}-\partial_{\mu}g_{\nu\alpha}-\partial_{\nu}g_{\alpha\mu}\right)\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}+\frac{e}{m}F^{\beta}{}_{\sigma}\frac{dx^{\sigma}}{ds}.\tag{2.16}$$

But of course, we recognize that the Christoffel symbols $-\Gamma^{\beta}{}_{\mu\nu}=\frac{1}{2}g^{\beta\alpha}\left(\partial_{\alpha}g_{\mu\nu}-\partial_{\mu}g_{\nu\alpha}-\partial_{\nu}g_{\alpha\mu}\right)$. As a consequence, the above reduces to:

$$\frac{d^2x^{\beta}}{ds^2}=-\Gamma^{\beta}{}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}+\frac{e}{m}F^{\beta}{}_{\sigma}\frac{dx^{\sigma}}{ds}.\tag{2.17}$$

In the presence of gravitational and electromagnetic fields, this contains both the equations of gravitational motion and the Lorentz force law, obtained via the geodesic variation of the canonical invariant metric length element (2.1). In the absence of gravitation, i.e., for $g_{\mu\nu}=\eta_{\mu\nu}$ over the spacetime region being considered thus $\Gamma^{\beta}{}_{\mu\nu}=0$, this reduces to the Lorentz force law.

As a result, we have proved that by using Weyl's canonical prescription in form of $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ from (2.2) to define the linear metric element by $ds^2 = g_{\mu\nu} d\chi^\mu d\chi^\nu$ as shown in (2.1), the Lorentz force law of electrodynamics may indeed be obtained from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime geometry.

3. Einstein's Equation and Maxwell's Equations

Because the metric length $ds^2 = g_{\mu\nu} d\chi^\mu d\chi^\nu$ of (2.1) under a variation $0 = \delta\int_A^B ds$ simultaneously provides a geodesic description of motion in a gravitational field and in an electromagnetic field, and because the prescription $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ is no more than a variant of Weyl's gauge prescriptions $p^\mu \rightarrow \pi^\mu = p^\mu + eA^\mu$ in momentum space and $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$ in configuration space and leads directly as well to Dirac's equation $(i\gamma^\mu D_\mu - m)\psi = 0$ for an interacting fermion, this may fairly be regarded as a classical metric-level unification of electrodynamics with gravitation, using four spacetime dimensions only. But the equations of motion in a field are only half the matter. We also need to know the equations for the fields themselves in relation to their sources. Thus we now ask, can the field equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ which specifies the gravitational field, be shown to relate in some direct fashion to Maxwell's field equations for electric and (the absence of) magnetic sources?

Because the Lorentz force (2.17) is obtained by dilating or contracting the differential coordinate elements via $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ without in any way altering the metric tensor $g_{\mu\nu}$ as is done, for example, in Kaluza-Klein theory, one might incorrectly conclude that the electromagnetic interaction does not affect spacetime curvature as represented by the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ with the field dynamics specified by $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. However, one must keep in mind that the Reimann tensor may be defined via $R^\alpha{}_{\beta\mu\nu} V_\alpha \equiv [\partial_{;\nu}, \partial_{;\mu}] V_\beta$ as a measure of the extent to which the gravitationally-covariant derivatives $\partial_{;\mu} V_\beta = \partial_\mu V_\beta - \Gamma^\alpha{}_{\mu\beta} V_\alpha$ operating on a vector V_β do not commute. Likewise, the field strength tensor $F_{\nu\mu}$ may be defined via $ieF_{\nu\mu} V_\beta \equiv [D_\nu, D_\mu] V_\beta$ as a measure of the extent to which the gauge-covariant derivatives $D_\mu V_\beta = (\partial_\mu + ieA) V_\beta$ do not commute when operating on this same vector V_β. Indeed, this latter definition results in $F_{\nu\mu} = D_{[\nu} A_{\mu]} = \partial_{[\nu} A_{\mu]} + ie[A_\nu, A_\mu]$ for a *non-abelian* gauge theory defined such that $[A_\nu, A_\mu] \neq 0$, which simplifies to $F_{\nu\mu} = \partial_{[\nu} A_{\mu]}$ for an abelian theory such as electrodynamics in which $[A_\nu, A_\mu] = 0$.

Therefore, let us now apply Weyl's canonical prescription to the gravitationally-covariant derivatives by employing:

$$\partial_{;\mu}V_\beta = \partial_\mu V_\beta - \Gamma^\alpha{}_{\mu\beta}V_\alpha \to D_{;\mu}V_\beta = \left(\partial_\mu + ieA_\mu\right)V_\beta - \Gamma^\alpha{}_{\mu\beta}V_\alpha\,, \tag{3.1}$$

for vectors V_β a.k.a. first-rank tensors, and likewise extended for second and higher-rank tensors. This is the same prescription that in the form $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ of (2.2) yielded the Lorentz force law in (2.17). If we then use these derivatives (3.1) to define a gauge-enhanced canonical Riemann tensor $\Re^\alpha{}_{\beta\mu\nu}$ as:

$$\Re^\alpha{}_{\beta\mu\nu}V_\alpha \equiv \left[D_{;\nu}, D_{;\mu}\right]V_\beta\,, \tag{3.2}$$

it can be expected as a consequence of $ieF_{\nu\mu}V_\beta \equiv \left[D_\nu, D_\mu\right]V_\beta$ that the electrodynamic fields $F_{\nu\mu}$ and potentials A_μ will appear in this Riemann tensor. Further, because $F_{\nu\mu} = D_{[\nu}A_{\mu]}$ encompasses *both abelian and non-abelian field strengths,* one would expect that the gravitational field equations using $\Re_{\beta\mu} = \Re^\alpha{}_{\beta\mu\alpha}$ and $\Re = \Re^\sigma{}_\sigma$ can be related not only to abelian electrodynamics, *but also to non-abelian such as weak and strong interactions.* So let us expressly calculate this enhanced canonical $\Re^\alpha{}_{\beta\mu\nu}$ using (3.2) and see what results.

We first calculate:

$$\begin{aligned}D_{;\nu}\left(D_{;\mu}V_\beta\right) &= \left(\partial_\nu + ieA_\nu\right)\left(\left(\partial_\mu + ieA_\mu\right)V_\beta - \Gamma^\alpha{}_{\mu\beta}V_\alpha\right)\\ &\quad - \Gamma^\tau{}_{\mu\nu}\left(\left(\partial_\tau + ieA_\tau\right)V_\beta - \Gamma^\alpha{}_{\tau\beta}V_\alpha\right) - \Gamma^\tau{}_{\nu\beta}\left(\left(\partial_\mu + ieA_\mu\right)V_\tau - \Gamma^\alpha{}_{\mu\tau}V_\alpha\right)\end{aligned} \tag{3.3}$$

as well as the like expression interchanging $\mu \leftrightarrow \nu$, then subtract the latter from the former and reduce using index renaming and the symmetries of the objects in the resulting equations. Many terms cancel, but with the vector V_α still attached as the operand on the right, what remains is:

$$\begin{aligned}&\Re^\alpha{}_{\beta\mu\nu}V_\alpha \equiv \left[D_{;\nu}, D_{;\mu}\right]V_\beta = D_{;\nu}\left(D_{;\mu}V_\beta\right) - D_{;\mu}\left(D_{;\nu}V_\beta\right)\\ &= \left(-\partial_\nu\Gamma^\alpha{}_{\mu\beta} + \partial_\mu\Gamma^\alpha{}_{\nu\beta} + \Gamma^\tau{}_{\nu\beta}\Gamma^\alpha{}_{\mu\tau} - \Gamma^\tau{}_{\mu\beta}\Gamma^\alpha{}_{\nu\tau} - ie\delta^\alpha{}_\beta F_{\mu\nu}\right)V_\alpha\end{aligned}, \tag{3.4}$$

including a *non-abelian* field strength:

$$F_{\mu\nu} = T^aF^a{}_{\mu\nu} = \partial_{[\mu}A_{\nu]} + ie\left[A_\mu, A_\nu\right] = T^a\partial_{[\mu}A^a{}_{\nu]} + ie\left[T^b, T^c\right]A^b{}_\mu A^c{}_\nu = T^a\partial_{[\mu}A^a{}_{\nu]} - ef^{abc}T^aA^b{}_\mu A^c{}_\nu\,, \tag{3.5}$$

which becomes abelian in the event $\left[A_\mu, A_\nu\right] = 0$. When we explicitly display the group structure constants f^{abc} for the non-abelian Hermitian generators T^a via $if^{abc}T^a = \left[T^b, T^c\right]$, we see that $F^a{}_{\nu\mu} = \partial_{[\nu}A^a{}_{\mu]} - ef^{abc}A^b{}_\nu A^c{}_\mu$ is real and so $ieF_{\mu\nu} = ieT^aF^a{}_{\mu\nu}$ in (3.4) is a *complex* Hermitian field owing to the T^a. With V_α removed and some index renaming and lowered to covariant form, the canonical Riemann tensor in (3.4) is then seen to be:

$$\begin{aligned}\Re_{\alpha\beta\mu\nu} &= g_{\alpha\tau}\Re^{\tau}{}_{\beta\mu\nu} = -g_{\alpha\tau}\partial_\nu\Gamma^{\tau}{}_{\beta\mu} + g_{\alpha\tau}\partial_\mu\Gamma^{\tau}{}_{\beta\nu} + g_{\alpha\tau}\Gamma^{\sigma}{}_{\beta\nu}\Gamma^{\tau}{}_{\sigma\mu} - g_{\alpha\tau}\Gamma^{\sigma}{}_{\beta\mu}\Gamma^{\tau}{}_{\sigma\nu} - ieg_{\alpha\beta}F_{\mu\nu} \\ &= R_{\alpha\beta\mu\nu} - ieg_{\alpha\beta}F_{\mu\nu}\end{aligned} \tag{3.6}$$

As expressed by $\Re_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - ieg_{\alpha\beta}F_{\mu\nu}$, the terms containing Christoffels are no different from the usual in $R_{\alpha\beta\mu\nu}$. But the new term $-ieg_{\alpha\beta}F_{\mu\nu}$ resulting from the same gauge prescription $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ that likewise brought the Lorentz force law into (2.17) changes several aspects of $\Re_{\alpha\beta\mu\nu}$ in relation to the ordinary $R_{\alpha\beta\mu\nu}$. First, while for the last two indexes $\Re_{\alpha\beta\mu\nu} = -\Re_{\alpha\beta\nu\mu}$ as with $R_{\alpha\beta\mu\nu}$, for the first two indexes $\Re_{\alpha\beta\mu\nu} \neq -\Re_{\beta\alpha\mu\nu}$ due to the presence of the symmetric $g_{\alpha\beta}$ next to the antisymmetric $F_{\mu\nu}$ in the term $g_{\alpha\beta}F_{\mu\nu}$. Thus, $\Re_{\alpha\beta\mu\nu}$ is *non-symmetric* in α, β. Second, noting that $F_{\mu\nu} = T^a F^a{}_{\mu\nu}$ is Hermitian, the term $ieg_{\alpha\beta}F_{\mu\nu}$ provides an similar *complex* aspect to $\Re_{\alpha\beta\mu\nu}$, so that overall, this enhanced $\Re_{\alpha\beta\mu\nu}$ is a *complex* object. Third, as a consequence of both these matters, the *real* part of $\Re_{\alpha\beta\mu\nu}$ has the usual symmetries of $R_{\alpha\beta\mu\nu}$, while the new *complex* part has the mixed symmetry of $g_{\alpha\beta}F_{\mu\nu}$.

It is readily seen from (3.6) after some re-indexing that the canonical Ricci tensor:

$$\Re_{\mu\nu} = \Re^{\alpha}{}_{\mu\nu\alpha} = -\partial_\alpha\Gamma^{\alpha}{}_{\mu\nu} + \partial_\nu\Gamma^{\alpha}{}_{\mu\alpha} + \Gamma^{\sigma}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\sigma\nu} - \Gamma^{\sigma}{}_{\mu\nu}\Gamma^{\alpha}{}_{\sigma\alpha} + ieF_{\mu\nu} = R_{\mu\nu} + ieF_{\mu\nu}, \tag{3.7}$$

concisely $\Re_{\mu\nu} = R_{\mu\nu} + ieF_{\mu\nu}$, is likewise non-symmetric, with the usual, real gravitational terms being symmetric and the new, complex electrodynamic term being antisymmetric in μ, ν. Finally, because $F^{\sigma}{}_{\sigma} = 0$, the canonical Ricci scalar is the usual:

$$\Re = g^{\mu\nu}\Re_{\mu\nu} = -g^{\sigma\tau}\partial_\alpha\Gamma^{\alpha}{}_{\sigma\tau} + g^{\sigma\tau}\partial_\tau\Gamma^{\alpha}{}_{\sigma\alpha} + g^{\sigma\tau}\Gamma^{\beta}{}_{\sigma\alpha}\Gamma^{\alpha}{}_{\beta\sigma} - g^{\sigma\tau}\Gamma^{\beta}{}_{\sigma\tau}\Gamma^{\alpha}{}_{\beta\alpha} = R \tag{3.8}$$

with no residual terms from electrodynamics, that is, $\Re = R$.

If we now construct $\partial_{;\sigma}\Re_{\alpha\beta\mu\nu} + \partial_{;\mu}\Re_{\alpha\beta\nu\sigma} + \partial_{;\nu}\Re_{\alpha\beta\sigma\mu}$, then because (3.6) informs us that $\Re_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - ieg_{\alpha\beta}F_{\mu\nu}$, all of the Christoffel terms will zero out as a result of the second Bianchi identity $\partial_{;\sigma}R_{\alpha\beta\mu\nu} + \partial_{;\mu}R_{\alpha\beta\nu\sigma} + \partial_{;\nu}R_{\alpha\beta\sigma\mu} = 0$, simply due to the inherent structure of the Riemannian geometry itself. All that will remain are terms containing the field strength, so that:

$$\begin{aligned}&\partial_{;\sigma}\Re_{\alpha\beta\mu\nu} + \partial_{;\mu}\Re_{\alpha\beta\nu\sigma} + \partial_{;\nu}\Re_{\alpha\beta\sigma\mu} = -ieg_{\alpha\beta}\left(\partial_{;\sigma}F_{\mu\nu} + \partial_{;\mu}F_{\nu\sigma} + \partial_{;\nu}F_{\sigma\mu}\right) \\ &= -ieg_{\alpha\beta}\left(\partial_{;\sigma}\left[A_\mu, A_\nu\right] + \partial_{;\mu}\left[A_\nu, A_\sigma\right] + \partial_{;\nu}\left[A_\sigma, A_\mu\right]\right)\end{aligned} \tag{3.9}$$

Specifically: We see here that the Hermitian part of the construct $\partial_{;\sigma}\Re_{\alpha\beta\mu\nu}+\partial_{;\mu}\Re_{\alpha\beta\nu\sigma}+\partial_{;\nu}\Re_{\alpha\beta\sigma\mu}$ contains the terms $\partial_{;\sigma}F_{\mu\nu}+\partial_{;\mu}F_{\nu\sigma}+\partial_{;\nu}F_{\sigma\mu}$ which specify magnetic charges. Because exterior calculus teaches that the differential forms $dF=ddA=0$, this set of terms must be equal to zero for any *abelian* gauge theory with $\left[A_\mu, A_\nu\right]=0$. And because this set of terms must be zero if $\left[A_\mu, A_\nu\right]=0$, this means that for an abelian interaction the canonical Riemann tensor obeys the identity $\partial_{;\sigma}\Re_{\alpha\beta\mu\nu}+\partial_{;\mu}\Re_{\alpha\beta\nu\sigma}+\partial_{;\nu}\Re_{\alpha\beta\sigma\mu}=0$ as a consequence of $dF=ddA=0$ which is Maxwell's magnetic charge equation.

Next, given the identity (3.9), we may double-contract two pairs of indexes in the customary manner to ascertain that this canonical Riemann tensor also obeys an identity:

$$\begin{aligned}&\partial_{;\sigma}\left(\Re^{\sigma\mu}-\tfrac{1}{2}g^{\mu\sigma}\Re\right)=-\tfrac{1}{2}ie\left(\partial_{;\sigma}F^{\mu\sigma}+\partial^{;\mu}F^{\sigma}{}_{\sigma}+\partial_{;\sigma}F^{\sigma\mu}\right)\\&=-\tfrac{1}{2}ie\left(\partial_{;\sigma}\left[A^\mu, A^\sigma\right]+\partial^{;\mu}\left[A^\sigma, A_\sigma\right]+\partial_{;\sigma}\left[A^\sigma, A^\mu\right]\right)=0\end{aligned} \tag{3.10}$$

that has the exact same form as the usual $\partial_{;\mu}\left(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R\right)=0$ used to ensure local energy conservation in the Einstein equation via $\partial_{;\mu}T^{\mu\nu}=\partial_{;\mu}\left(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R\right)=0$, and which in this instance is zero because $F^{\mu\nu}=-F^{\nu\mu}$ is antisymmetric. Additionally, from (3.7) we deduce that:

$$\partial_{;\mu}\Re^{\mu\nu}=\partial_{;\mu}R^{\mu\nu}+ie\partial_{;\mu}F^{\mu\nu}=\partial_{;\mu}R^{\mu\nu}+ieJ^\nu, \tag{3.11}$$

where $J^\nu=\partial_{;\mu}F^{\mu\nu}$ is recognized to be the electric charge source current.

So, combining (3.10) and (3.11) and $\Re=R$ from (3.8) and $\partial_{;\mu}\left(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R\right)=0$ we are able to deduce that:

$$0=\partial_{;\mu}\left(\Re^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}\Re\right)=\partial_{;\mu}\left(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R\right)+ieJ^\nu=0+ieJ^\nu, \tag{3.12}$$

with the net consequence that:

$$J^\nu=\partial_{;\mu}F^{\mu\nu}=\partial_{;\mu}\left(\Re^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}\Re\right)=0. \tag{3.13}$$

Here, we see that $\partial_{;\mu}\left(\Re^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}\Re\right)$ contains the electric current. But because (3.10) is equal to zero this means that this electric current that is also zero. Taken together, (3.13) and (3.9) are then seen to be Maxwell's source-free equations:

$$\begin{cases}ieJ^\nu=\partial_{;\mu}F^{\mu\nu}=\partial_{;\mu}\left(\Re^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}\Re\right)=0\\-ieg_{\alpha\beta}\left(\partial_{;\sigma}F_{\mu\nu}+\partial_{;\mu}F_{\nu\sigma}+\partial_{;\nu}F_{\sigma\mu}\right)=\partial_{;\sigma}\Re_{\alpha\beta\mu\nu}+\partial_{;\mu}\Re_{\alpha\beta\nu\sigma}+\partial_{;\nu}\Re_{\alpha\beta\sigma\mu}=0\end{cases}. \tag{3.14}$$

And naturally, it will be recognized that $\Re^{\mu\nu} - \frac{1}{2} g^{\mu\nu}\Re$ is the canonical gauge extension of $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ which is the "marble" heart of the Einstein equation and which in the form $-\kappa\partial_{;\mu} T^{\mu\nu} = \partial_{;\mu}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 0$ secures the local conservation of energy.

In sum, we find that if we use the canonical gauge extension $\partial_{;\mu} \to D_{;\mu} = \partial_{;\mu} + ieA_\mu$ to construct a canonical Riemann tensor via the definition $\Re^\alpha{}_{\beta\mu\nu} V_\alpha \equiv \left[D_{;\nu}, D_{;\mu}\right] V_\beta$ of (3.2), Maxwell's source-free field equations indeed become embedded in the second Bianchi identity. The magnetic equation appears in the fifth rank identity and the electric charge equation in its vector contraction. We shall not at this moment, examine the next logical question as to a geometric understanding of non-vanishing electric sources $J^\nu \neq 0$. Certainly, we expect that using the canonical extension $\partial_{;\mu} \to D_{;\mu} = \partial_{;\mu} + ieA_\mu$ for the derivatives in (3.9) through (3.14) may play a role in exploring this question.

4. Conclusion

It has been shown how the Lorentz force law may be obtained from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime geometry as a consequence, at bottom, of simply applying Weyl's gauge prescription $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ to dilate or contract the spacetime coordinate elements by $dx^\mu \to d\chi^\mu = dx^\mu + ds\left(e/m\right)A^\mu$. It has also been shown how this same prescription embeds Maxwell's source-free equations into an imaginary, antisymmetric aspect that is added to the gravitational field equations. Studying sources in Maxwell theory will likely require continuing to apply $\partial_{;\mu} \to D_{;\mu} = \partial_{;\mu} + ieA_{;\mu}$ to the second Bianchi identity and its contracted variant that relates to the local conservation of the energy tensor.

As a consequence of what has been shown here, it may well be possible to unify gravitation not only with electrodynamics – but because $F^{\sigma\tau}$ first obtained in (3.4) encompasses a non-abelian field strength $F^a{}_{\nu\mu} = \partial_{[\nu} A^a{}_{\mu]} - ef^{abc}\left[A^b{}_\nu, A^c{}_\mu\right]$ – with the remaining weak and strong interactions as well, all consistently with quantum mechanics because the canonical gauge prescriptions $p^\mu \to \pi^\mu = p^\mu + eA^\mu$ and $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ and now $dx^\mu \to d\chi^\mu = dx^\mu + ds\left(e/m\right)A^\mu$ remain at the root of the entire development. The main questions that would remain following such a unification, would be as to the specific non-abelian gauge groups that operate physically at any given energy ranging up to the Planck mass, and how the symmetry of those groups becomes broken at lower energies down to the phenomenological group $SU(3)_C \times SU(2)_W \times U(1)_Y \to SU(3)_C \times U(1)_{em}$ and the fermions on which these groups act. The author has previously published on these questions, and even shown how the three generations of quarks and leptons originate, and why their left-chiral projections engage in CKM mixing, at [9].

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