Sedeonic Equations of Ideal Fluid

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In the present paper we develop the description of ideal liquid on the basis of space-time algebra of sixteen-component sedeons. We demonstrate that the dynamics of isentropic fluid is described by the first-order sedeonic wave equation. The second-order relations for the potentials analogues to the Pointing theorem in electrodynamics are derived. The plane wave solution of sedeonic equation for sound in liquid is disused.

1. Introduction

In recent years, there were many publications devoted to the reformulation of ideal liquid (*IL*) equations in a form similar to the equations of electrodynamics [1-4]. The authors show that the system of *IL* equations can be represented in the form of second-order wave equation similar to the equation for the electromagnetic field. Based on this analogy, the authors discuss the possibility of introducing the vector fields analogous to electric and magnetic fields, which satisfy the equations similar to the Maxwell's equations. However, from our point of view, the procedure to increase the order of wave equation used by the authors of these works leads to an incorrect description of *IL*. In our opinion, the first-order wave equation is more appropriate for *IL* description.

However, the Gibbs-Heaviside vector algebra, which is usually used for the formulation of *IL* equations, can not be used for analysis of first-order wave equations. Some time ago we proposed the alternative space-time algebra of sedeons, which is the Clifford algebra realizing the scalar-vector representation of Poincare group [5,6]. The sedeonic algebra enables the consideration of non-scalar first-order wave equations and analyse the spin properties of wave fields [6,7]. In the present paper we develop the description of *IL* on the basis of sedeonic potentials and space-time operators.

2. Equations of isentropic liquid in symmetric form

As is well known [8], the dynamics of an ideal fluid is described by a system of equations in traditional vector algebra, which includes the Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho}\nabla p = 0 , \tag{2.1}$$

continuity equation

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho + \rho(\nabla \cdot \boldsymbol{v}) = 0 , \tag{2.2}$$

and adiabatic equation

$$\frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \nabla)s = 0 . \tag{2.3}$$

Here $\boldsymbol{v}$ is a local liquid velocity, ρ is a density, p is a pressure, s is entropy per unit mass, ∇ is the Hamilton nabla-operator. In the case of isentropic fluid these equations can be reduced to a system of two symmetric equations [1]. Let us use the well-known thermodynamic relation:

$$dw = Tds + Vdp , \tag{2.4}$$

where w is a thermal function (enthalpy) of a unit mass of liquid, T is a temperature, V is the volume of unit mass of liquid ($V = 1/\rho$).For the case $s = const$ we obtain the following relation for differentials

$$dw = \frac{1}{\rho}dp = \frac{c^2}{\rho}d\rho , \tag{2.5}$$

where c is the speed of sound ($c^2 = (\partial p/\partial\rho)_s$). It follows that

$$\frac{1}{\rho}\nabla p = \nabla w\,, \tag{2.6}$$

$$\frac{\partial\rho}{\partial t} = \frac{\rho}{c^2}\frac{\partial w}{\partial t}\,, \tag{2.7}$$

$$\nabla\rho = \frac{\rho}{c^2}\nabla w\,. \tag{2.8}$$

Then the system of equations (2.1) - (2.3) is reduced to the following form:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla w = 0\,, \tag{2.9}$$

$$\frac{\partial w}{\partial t} + (\boldsymbol{v}\cdot\nabla)w + c^2(\nabla\cdot\boldsymbol{v}) = 0\,. \tag{2.10}$$

3. Space-time sedeons

The sedeonic algebra [5,6] encloses four groups of values, which are differed with respect to spatial and time inversion.

- Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
- Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
- Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
- Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Here indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{V}}$, which is defined by the following expression:

$$\tilde{\mathbf{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}}\,. \tag{3.1}$$

Let us introduce a scalar-vector basis $\mathbf{a}_0$, $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, where the element $\mathbf{a}_0$ is an absolute scalar unit ($\mathbf{a}_0 \equiv 1$), and the values $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ are absolute unit vectors generating the right Cartesian basis. Further we will indicate the absolute unit vectors by symbols without arrows as $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. We also introduce the four space-time units $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, where $\mathbf{e}_0$ is an absolute scalar unit ($\mathbf{e}_0 \equiv 1$); $\mathbf{e}_1$ is a time scalar unit ($\mathbf{e}_1 \equiv \mathbf{e}_{\mathbf{t}}$); $\mathbf{e}_2$ is a space scalar unit ($\mathbf{e}_2 \equiv \mathbf{e}_{\mathbf{r}}$); $\mathbf{e}_3$ is a space-time scalar unit ($\mathbf{e}_3 \equiv \mathbf{e}_{\mathbf{tr}}$). Using space-time basis $\mathbf{e}_\alpha$ and scalar-vector basis $\mathbf{a}_\beta$ (Greek indexes $\alpha, \beta = 0,1,2,3$), we can introduce unified sedeonic components $V_{\alpha\beta}$ in accordance with following relations:

$$\begin{aligned}
V &= \mathbf{e}_0 V_{00}\mathbf{a}_0\,,\\
\vec{V} &= \mathbf{e}_0\left(V_{01}\mathbf{a}_1 + V_{02}\mathbf{a}_2 + V_{03}\mathbf{a}_3\right),\\
V_{\mathbf{t}} &= \mathbf{e}_1 V_{10}\mathbf{a}_0\,,\\
\vec{V}_{\mathbf{t}} &= \mathbf{e}_1\left(V_{11}\mathbf{a}_1 + V_{12}\mathbf{a}_2 + V_{13}\mathbf{a}_3\right),\\
V_{\mathbf{r}} &= \mathbf{e}_2 V_{20}\mathbf{a}_0\,,\\
\vec{V}_{\mathbf{r}} &= \mathbf{e}_2\left(V_{21}\mathbf{a}_1 + V_{22}\mathbf{a}_2 + V_{23}\mathbf{a}_3\right),\\
V_{\mathbf{tr}} &= \mathbf{e}_3 V_{30}\mathbf{a}_0\,,\\
\vec{V}_{\mathbf{tr}} &= \mathbf{e}_3\left(V_{31}\mathbf{a}_1 + V_{32}\mathbf{a}_2 + V_{33}\mathbf{a}_3\right).
\end{aligned} \tag{3.2}$$

Then sedeon (3.1) can be written in the following expanded form:

$$\tilde{\mathbf{V}} = \mathbf{e}_0\left(V_{00}\mathbf{a}_0 + V_{01}\mathbf{a}_1 + V_{02}\mathbf{a}_2 + V_{03}\mathbf{a}_3\right)$$

$$+\mathbf{e}_1\left(V_{10}\mathbf{a}_0+V_{11}\mathbf{a}_1+V_{12}\mathbf{a}_2+V_{13}\mathbf{a}_3\right)$$
$$+\mathbf{e}_2\left(V_{20}\mathbf{a}_0+V_{21}\mathbf{a}_1+V_{22}\mathbf{a}_2+V_{23}\mathbf{a}_3\right) \qquad (3.3)$$
$$+\mathbf{e}_3\left(V_{30}\mathbf{a}_0+V_{31}\mathbf{a}_1+V_{32}\mathbf{a}_2+V_{33}\mathbf{a}_3\right).$$

The sedeonic components $V_{\alpha\beta}$ are numbers (complex in general). Further we will omit units $\mathbf{a}_0$ and $\mathbf{e}_0$ for the simplicity. The important property of sedeons is that the equality of two sedeons means the equality of all sixteen components $V_{\alpha\beta}$.

Let us consider the multiplication rules for the basis elements $\mathbf{a}_\mathbf{n}$ and $\mathbf{e}_\mathbf{k}$ (Latin indexes **n**, **k** = 1, 2, 3). The unit vectors $\mathbf{a}_\mathbf{n}$ have the following multiplication and commutation rules:

$$\mathbf{a}_\mathbf{n}\mathbf{a}_\mathbf{n}=\mathbf{a}_\mathbf{n}^2=1, \qquad (3.4)$$

$$\mathbf{a}_\mathbf{n}\mathbf{a}_\mathbf{k}=-\mathbf{a}_\mathbf{k}\mathbf{a}_\mathbf{n} \text{ (for } \mathbf{n}\neq\mathbf{k}\text{)}, \qquad (3.5)$$

$$\mathbf{a}_1\mathbf{a}_2=i\mathbf{a}_3,\quad \mathbf{a}_2\mathbf{a}_3=i\mathbf{a}_1,\quad \mathbf{a}_3\mathbf{a}_1=i\mathbf{a}_2, \qquad (3.6)$$

while the space-time units $\mathbf{e}_\mathbf{k}$ satisfy the following rules:

$$\mathbf{e}_\mathbf{k}\mathbf{e}_\mathbf{k}=\mathbf{e}_\mathbf{k}^2=1, \qquad (3.7)$$

$$\mathbf{e}_\mathbf{n}\mathbf{e}_\mathbf{k}=-\mathbf{e}_\mathbf{k}\mathbf{e}_\mathbf{n} \text{ (for } \mathbf{n}\neq\mathbf{k}\text{)}, \qquad (3.8)$$

$$\mathbf{e}_1\mathbf{e}_2=i\mathbf{e}_3,\quad \mathbf{e}_2\mathbf{e}_3=i\mathbf{e}_1,\quad \mathbf{e}_3\mathbf{e}_1=i\mathbf{e}_2. \qquad (3.9)$$

Here and further the value i is imaginary unit ($i^2=-1$). The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a}_\mathbf{n}$ and space-time units $\mathbf{e}_\mathbf{k}$ can be presented for obviousness as the tables 1 and 2.

Table 1. Multiplication rules for absolute unit vectors $\mathbf{a}_\mathbf{n}$.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	1	$i\mathbf{a}_3$	$-i\mathbf{a}_2$
$\mathbf{a}_2$	$-i\mathbf{a}_3$	1	$i\mathbf{a}_1$
$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	1

Table 2. Multiplication rules for space-time units $\mathbf{e}_\mathbf{k}$.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	1	$i\mathbf{e}_3$	$-i\mathbf{e}_2$
$\mathbf{e}_2$	$-i\mathbf{e}_3$	1	$i\mathbf{e}_1$
$\mathbf{e}_3$	$i\mathbf{e}_2$	$-i\mathbf{e}_1$	1

Note that units $\mathbf{e}_\mathbf{k}$ commute with vectors $\mathbf{a}_\mathbf{n}$:

$$\mathbf{a}_\mathbf{n}\mathbf{e}_\mathbf{k}=\mathbf{e}_\mathbf{k}\mathbf{a}_\mathbf{n} \qquad (3.10)$$

for any $\mathbf{n}$ and $\mathbf{k}$.

In sedeonic algebra we assume the Clifford multiplication of vectors. The sedeonic product of two vectors $\vec{A}$ and $\vec{B}$ can be presented in the following form:

$$\vec{A}\vec{B}=\left(\vec{A}\cdot\vec{B}\right)+\left[\vec{A}\times\vec{B}\right]. \qquad (3.11)$$

Here we denote the sedeonic scalar multiplication of two vectors (internal product) by symbol "$\cdot$" and round brackets

$$\left(\vec{A}\cdot\vec{B}\right) = A_1B_1 + A_2B_2 + A_3B_3\,, \tag{3.12}$$

and sedeonic vector multiplication (external product) by symbol “$\times$” and square brackets

$$\left[\vec{A}\times\vec{B}\right] = i\left(A_2B_3 - A_3B_2\right) + i\left(A_3B_1 - A_1B_3\right) + i\left(A_1B_2 - A_2B_1\right). \tag{3.13}$$

Note that in sedeonic algebra the expression for the vector product differs from analogous expression in Gibbs vector algebra.

4. Generalized sedeonic equation of ideal liquid

Let us assume a constant speed of sound ($c = const$). Then, if we introduce the vector potential $\vec{A}$

$$\vec{A} = c\vec{v}\,, \tag{4.1}$$

the equations (2.9) and (2.10) can be represented in the following symmetric form:

$$\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)\vec{A} + \vec{\nabla}w = 0\,, \tag{4.2}$$

$$\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)w + (\vec{\nabla}\cdot\vec{A}) = 0\,. \tag{4.3}$$

In sedeon algebra, the system of equations (4.2) - (4.3) can be written as a single first-order wave equation:

$$\left\{i\mathbf{e_t}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right) + \mathbf{e_r}\vec{\nabla}\right\}\tilde{\mathbf{W}} = 0\,, \tag{4.4}$$

where the sedeonic wave function $\tilde{\mathbf{W}}$ is

$$\tilde{\mathbf{W}} = \left(w + \mathbf{e_{tr}}\vec{A}\right), \tag{4.5}$$

so, in expanded form the equation (4.4) can be written as

$$\left\{i\mathbf{e_t}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right) + \mathbf{e_r}\vec{\nabla}\right\}\left(w + \mathbf{e_{tr}}\vec{A}\right) = 0\,. \tag{4.6}$$

Indeed, after the act of operator on the wave function in (4.6) we get

$$\begin{aligned}&\mathbf{e_r}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)\vec{A} + i\mathbf{e_t}\left(\vec{\nabla}\cdot\vec{A}\right) + i\mathbf{e_t}\left[\vec{\nabla}\times\vec{A}\right]\\ &+ i\mathbf{e_t}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)w + \mathbf{e_r}\vec{\nabla}w = 0.\end{aligned} \tag{4.7}$$

Separating the values with various space-time properties we obtain the system of equations:

$$\begin{aligned}&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)\vec{A} + \vec{\nabla}w = 0,\\ &\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)w + \left(\vec{\nabla}\cdot\vec{A}\right) = 0,\\ &\left[\vec{\nabla}\times\vec{A}\right] = 0.\end{aligned} \tag{4.8}$$

The first two equations in (4.8) coincide with equations (4.2) и (4.3). The last equation shows that in the case of constant sound speed the fluid motion is potential.

5. Maxwell equations for ideal liquid

Of course, the potentials w and $\vec{A}$ satisfy also the second order wave equation. By analogy with electrodynamics we may also introduce the appropriate field strengths, which satisfy the equations similar to Maxwell's equations.

Let us introduce the operator

$$D = \frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right). \tag{5.1}$$

Then the sedeonic first-order wave equation (4.6) is rewritten as

$$\left(i\mathbf{e}_t D + \mathbf{e}_r \vec{\nabla}\right)\left(w + \mathbf{e}_{tr}\vec{A}\right) = 0. \tag{5.2}$$

Appling the operator $\left(i\mathbf{e}_t D + \mathbf{e}_r \vec{\nabla}\right)$ to the equation (5.2), we obtain second-order wave equation

$$\left(i\mathbf{e}_t D + \mathbf{e}_r \vec{\nabla}\right)\left(i\mathbf{e}_t D + \mathbf{e}_r \vec{\nabla}\right)\left(w + \mathbf{e}_{tr}\vec{A}\right) = 0. \tag{5.3}$$

Multiplying operators in the left-hand side of (5.3), we obtain the wave equation in the following form:

$$\left(-D^2 + \Delta\right)\left(w + \mathbf{e}_{tr}\vec{A}\right) = 0. \tag{5.4}$$

On the other hand, we can introduce the scalar-vector field strengths in accordance with the following definitions:

$$\begin{aligned} \vec{E} &= -D\vec{A} - \vec{\nabla}w, \\ \vec{H} &= -i\left[\nabla\times\vec{A}\right], \\ \varepsilon &= Dw + \left(\vec{\nabla}\cdot\vec{A}\right). \end{aligned} \tag{5.5}$$

Then the second-order wave equation (5.3) can be represented as

$$\left(i\mathbf{e}_t D + \mathbf{e}_r \vec{\nabla}\right)\left(i\mathbf{e}_t \varepsilon - \mathbf{e}_r \vec{E} - \mathbf{e}_t \vec{H}\right) = 0. \tag{5.6}$$

The field strengths ε , $\vec{E}$ and $\vec{H}$ satisfy the system of equations, similar to the Maxwell equations in electrodynamics [6,7]. Indeed, performing the action of the operators in the equation (5.6) and separating the variables with different spatial and temporal properties, we get

$$\begin{aligned} &\left(\vec{\nabla}\cdot\vec{E}\right) + D\varepsilon = 0, \\ &\left(\vec{\nabla}\cdot\vec{H}\right) = 0, \\ &D\vec{E} + i\left[\vec{\nabla}\times\vec{H}\right] + \vec{\nabla}\varepsilon = 0, \\ &D\vec{H} - i\left[\vec{\nabla}\times\vec{E}\right] = 0. \end{aligned} \tag{5.7}$$

If we require the calibrating condition similar to the Lorentz gauge in electrodynamics

$$Dw + \left(\vec{\nabla}\cdot\vec{A}\right) = 0, \tag{5.8}$$

then the scalar filed ε can be eliminated ($\varepsilon = 0$). In this case the wave equation (5.6) is rewritten as

$$\left(i\mathbf{e}_t D + \mathbf{e}_r \vec{\nabla}\right)\left(\mathbf{e}_r \vec{E} + \mathbf{e}_t \vec{H}\right) = 0, \tag{5.9}$$

and the system (5.7) takes the following form:

$$\begin{aligned} &\left(\vec{\nabla}\cdot\vec{E}\right) = 0, \\ &\left(\vec{\nabla}\cdot\vec{H}\right) = 0, \\ &D\vec{E} + i\left[\vec{\nabla}\times\vec{H}\right] = 0, \\ &D\vec{H} - i\left[\vec{\nabla}\times\vec{E}\right] = 0. \end{aligned} \tag{5.10}$$

The analogy with electrodynamics is obvious. However the second-order wave equation (5.3) has the superfluous solutions, since the initial wave function satisfies the first order equation (4.6).

If we consider the fluid motion with speeds $|\vec{v}| << c$, then the convective derivative $(\vec{v}\cdot\nabla)\vec{v}$ can be neglected [8] and operator D is

$$D = \frac{1}{c}\frac{\partial}{\partial t}. \tag{5.11}$$

Then we get the linearized wave equation (5.3) in the following form:

$$\left(i\mathbf{e}_{\mathrm{t}}\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{e}_{\mathrm{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathrm{t}}\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{e}_{\mathrm{r}}\vec{\nabla}\right)\left(w+\mathbf{e}_{\mathrm{tr}}\vec{A}\right)=0\,. \tag{5.12}$$

The form of this equation coincides with the wave equation for the electromagnetic field [7].

6. Sound waves

Let us consider the oscillatory motion of a fluid with speeds $|\vec{v}| << c$. In this case, the convective derivative $(\vec{v}\cdot\nabla)\vec{v}$ can be neglected [8] and sedeonic wave equation (4.4) takes the form:

$$\left\{i\mathbf{e}_{\mathrm{t}}\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{e}_{\mathrm{r}}\vec{\nabla}\right\}\tilde{\mathbf{W}}=0\,. \tag{6.1}$$

The first-order wave equation (7.1) has the solution in the form of plane wave:

$$\tilde{\mathbf{W}}=\tilde{\mathbf{U}}\exp\left\{-i\omega t+i\left(\vec{k}\cdot\vec{r}\right)\right\}. \tag{6.2}$$

Here ω is a frequency, $\vec{k}$ is a wave vector and the wave amplitude $\tilde{\mathbf{U}}_v$ does not depend on coordinates and time. In this case the dependence of the frequency on the wave vector has two branches:

$$\omega_{\pm}=\pm ck\,, \tag{6.3}$$

where k is the modulus of wave vector ($k=\left|\vec{k}\right|$). In general, the solution of equation (6.1) can be written as a plane wave of the following form:

$$\tilde{\mathbf{W}}=\left(\mathbf{e}_1\frac{\omega_{\pm}}{c}-i\mathbf{e}_2\vec{k}\right)\tilde{\mathbf{M}}\exp\left\{-i\omega_{\pm}t+i\left(\vec{k}\cdot\vec{r}\right)\right\}, \tag{6.4}$$

where $\tilde{\mathbf{M}}$ is arbitrary sedeon with constant components, which do not depend on coordinates and time. Indeed the expression

$$\left(\mathbf{e}_1\frac{\omega_{\pm}}{c}-i\mathbf{e}_2\vec{k}\right) \tag{6.5}$$

is so-called zero divisor since

$$\left(\mathbf{e}_1\frac{\omega_{\pm}}{c}-i\mathbf{e}_2\vec{k}\right)\left(\mathbf{e}_1\frac{\omega_{\pm}}{c}-i\mathbf{e}_2\vec{k}\right)\equiv 0\,. \tag{6.6}$$

Let us analyze the structure of the plane wave solution (6.4) in detail. Note that the internal structure of this wave is changed under space and time conjugation [7,9]. Further we suppose that wave vector $\vec{k}$ is directed along the Z axis. Then the first-order equation (6.1) can be rewritten in the following equivalent form:

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\mathbf{e}_{\mathrm{tr}}\mathbf{a}_3\frac{\partial}{\partial z}\right)\tilde{\mathbf{W}}'=0\,, \tag{6.7}$$

where $\tilde{\mathbf{W}}'=i\mathbf{e}_{\mathrm{t}}\tilde{\mathbf{W}}$. The solution of (6.7) can be presented in form of two waves:

$$\tilde{\mathbf{W}}'_{+}=-\left(1+\mathbf{e}_{\mathrm{tr}}\mathbf{a}_3\right)k\tilde{\mathbf{M}}\exp\left\{-i\omega_{+}t+ikz\right\}, \tag{6.8}$$

$$\tilde{\mathbf{W}}'_{-}=\left(1-\mathbf{e}_{\mathrm{tr}}\mathbf{a}_3\right)k\tilde{\mathbf{M}}\exp\left\{-i\omega_{-}t+ikz\right\}. \tag{6.9}$$

Note that the wave function $\tilde{\mathbf{W}}'_{+}$ corresponds to the positive branch of dispersion law (6.3), while $\tilde{\mathbf{W}}'_{-}$ corresponds to the negative branch of dispersion law (6.3). The wave functions (6.8) and (6.9) can be interpreted as the eigenfunctions of spin operator

$$\hat{S}_z=\frac{1}{2}\mathbf{e}_{\mathrm{tr}}\mathbf{a}_3\,. \tag{6.10}$$

Indeed, it is simple to check that $\tilde{\mathbf{W}}'$ satisfies the following equation:

$$\hat{S}_z\tilde{\mathbf{W}}'=S_z\tilde{\mathbf{W}}'\,, \tag{6.11}$$

where eigenvalue $S_z = \pm 1/2$. Thus, $\tilde{\mathbf{W}}'_+$ describes the wave with spirality $S_z = +1/2$, while $\tilde{\mathbf{W}}'_-$ describes the wave with spirality $S_z = -1/2$ [7,9].

7. The second-order relations for sound potentials

Multiplying the equation (6.1) on the potential ($w - \mathbf{e}_{\mathbf{tr}}\vec{A}$) from the left, we have the following sedeonic equation:

$$\left(w - \mathbf{e}_{\mathbf{tr}}\vec{A}\right)\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(w + \mathbf{e}_{\mathbf{tr}}\vec{A}\right) = 0 . \tag{7.1}$$

Performing the sedeonic multiplication and separating the quantities with different spatial and temporal properties, we get the relations for the field potentials:

$$\frac{1}{2c}\frac{\partial}{\partial t}\left(w^2 + \vec{A}^2\right) + \left(\vec{A}\cdot\vec{\nabla}w\right) + w\left(\vec{\nabla}\cdot\vec{A}\right) = 0 , \tag{7.2}$$

$$\frac{1}{2}\vec{\nabla}w^2 + w\frac{1}{c}\frac{\partial\vec{A}}{\partial t} + \vec{A}\frac{1}{c}\frac{\partial w}{\partial t} + \vec{A}\left(\vec{\nabla}\cdot\vec{A}\right) + \left[\vec{A}\times\left[\vec{\nabla}\times\vec{A}\right]\right] = 0 , \tag{7.3}$$

$$w\left[\vec{\nabla}\times\vec{A}\right] + \left[\vec{A}\times\vec{\nabla}w\right] + \frac{1}{c}\left[\vec{A}\times\frac{\partial\vec{A}}{\partial t}\right] = 0 , \tag{7.4}$$

$$\left(\vec{A}\cdot\left[\vec{\nabla}\times\vec{A}\right]\right) = 0 . \tag{7.5}$$

The expressions (7.2) and (7.3) are an analog of the Poynting's theorem for the field described by the first-order wave equation [7].

8. Entropy accounting

From the point of view of relativistic thermodynamics, enthalpy and entropy are not changed in the transition to the other inertial system [10], i.e. they are invariants of Lorentz transformations. To account for the effects associated with the entropy of the fluid the equation (4.6) can be modified as follows:

$$\left\{i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right) + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right\}\left(w + \mathbf{e}_{\mathbf{tr}}\vec{A} + \mathbf{e}_{\mathbf{tr}}s\right) = 0 . \tag{8.1}$$

Performing sedeonic multiplication in (8.1) we get:

$$\begin{aligned}
&\left\{i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right) + \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right\}\left(w + \mathbf{e}_{\mathbf{tr}}\vec{A} + \mathbf{e}_{\mathbf{tr}}s\right) = \\
&\mathbf{e}_{\mathbf{r}}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)\vec{A} + i\mathbf{e}_{\mathbf{t}}\left(\vec{\nabla}\cdot\vec{A}\right) + i\mathbf{e}_{\mathbf{t}}\left[\vec{\nabla}\times\vec{A}\right] \\
&+ i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)h + \mathbf{e}_{\mathbf{r}}\vec{\nabla}h + \\
&+ \mathbf{e}_{\mathbf{r}}\frac{1}{c}\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right)s + i\mathbf{e}_{\mathbf{t}}\vec{\nabla}s.
\end{aligned} \tag{8.2}$$

Now, separating the values with different space-time properties, we get the following system of equations:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)\vec{A}+\vec{\nabla}h=0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)h+\left(\vec{\nabla}\cdot\vec{A}\right)=0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right)s=0,\\
&\left[\vec{\nabla}\times\vec{A}\right]+\vec{\nabla}s=0.
\end{aligned} \tag{8.3}$$

9. Summary

Thus we have shown that the equations describing the dynamics of isentropic fluid can be rewritten in the compact form of single nonlinear first-order sedeonic wave equation. We have shown that in linear approximation the equation for sound waves is similar to the wave equation of neutrino field. The plane wave solution of sedeonic equation for sound in liquid was considered. The second-order relations for the sound potentials analogues to the Pointing theorem in electrodynamics have been derived. Additionally, we have proposed the sedeonic equation describing the entropy effects.

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