Interpreting the summation notation when the lower limit is greater than the upper limit *

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Abstract

In interpreting the sigma notation for finite summation, it is generally assumed that the lower limit of summation is less than or equal to the upper limit. This presumption has led to certain misconceptions, especially concerning what constitutes an empty sum. This paper addresses how to construe the sigma notation when the lower limit is greater than the upper limit.

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1 Introduction

When $b \geq a$, with $a, b \in \mathbb{Z}$, the notation $\sum_{i=a}^{b} f(i)$ is understood to mean $f(a) + f(a+1) + \ldots + f(b)$. In particular

$$\sum_{i=a}^{a} f(i) \equiv f(a)\,. \tag{1.1}$$

As long as the conditions $c \geq a$ and $b \geq c+1$ hold, the following splitting of summation is intuitive and straightforward:

$$\sum_{i=a}^{c} f(i) + \sum_{i=c+1}^{b} f(i) = \sum_{i=a}^{b} f(i)\,. \tag{1.2}$$

The aim of this paper is to extend the interpretation of the sigma notation so that $\sum_{i=a}^{b} f(i)$ becomes meaningful for all $a, b \in \mathbb{Z}$ for which the summand $f(i)$ is defined for every integer i in the interval $[a, b]$ if $a \leq b$ (or interval (b, a) if $a > b$).

2 Empty sum

To achieve the purpose stated in the last paragraph of the Introduction section we first define an *empty sum* by setting $c = b$ in (1.2) to obtain

$$\sum_{i=a}^{b} f(i) + \sum_{i=b+1}^{b} f(i) = \sum_{i=a}^{b} f(i)\,,$$

which makes sense only if we adopt the following familiar interpretation:

$$\boxed{\sum_{i=b+1}^{b} f(i) = 0, \quad b \in \mathbb{Z}}\,. \tag{2.1}$$

It appears that it is in view of this result that many authors adopt the misconceived interpretation that whenever the upper limit of a summation is less than its lower limit, the sum evaluates to zero. This interpretation, which is inconsistent with the theory of summation, is found in scientific literature (see for example [1]) and software (as implemented in PARI-GP and GNU Emacs Calc, for example), as well as in various informal writings and posts on the internet [2, 3, 4]. In the next section we give an interpretation that is consistent with summation theory.

3 Summation with lower limit greater than upper limit

Setting $b = a - 1$ in (1.2) we obtain

$$\sum_{i=a}^{c} f(i) + \sum_{i=c+1}^{a-1} f(i) = \sum_{i=a-1}^{a} f(i)\,,$$

which on account of (2.1) gives

$$\sum_{i=a}^{c} f(i) + \sum_{i=c+1}^{a-1} f(i) = 0\,. \tag{3.1}$$

Since $a > c$ whenever $a - 1 > c + 1$, (3.1) allows the interpretation of the summation notation whenever the lower limit is greater than the upper limit of summation, and we have

$$\boxed{\sum_{i=a}^{c} f(i) \equiv -\sum_{i=c+1}^{a-1} f(i), \quad a, c \in \mathbb{Z} \text{ and } a > c}\,,$$

provided $f(i)$ is defined for every integer i in the interval (c, a).

Finally, setting $c = a$ in (3.1) and using (1.1) we obtain

$$\boxed{\sum_{i=a+1}^{a-1} f(i) \equiv -f(a)}\,.$$

4 Conclusion

We have extended the interpretation of the sigma summation notation to allow the evaluation of a sum in which the lower limit is greater than the upper limit of summation. The scheme is

$$\boxed{\sum_{i=a}^{c} f(i) \equiv -\sum_{i=c+1}^{a-1} f(i), \quad a, c \in \mathbb{Z} \text{ and } a > c}\,.$$

In particular

$$\sum_{i=b+1}^{b} f(i) = 0, \quad b \in \mathbb{Z}\,,$$

and

$$\sum_{i=a+1}^{a-1} f(i) \equiv -f(a)\,.$$

References

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