Proof of existence of integral solutions ($a_1, a_2, \ldots, a_n$) of the equation $a_1p_1^m + a_2p_2^m + \ldots + a_np_n^m = 0$ for any integer "m" greater than or equal to one, for sequence of prime $p_1, p_2, \ldots, p_n$

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Abstract: We prove using Bezout's identity that $a_1p_1^m + a_2p_2^m + \ldots + a_np_n^m = 0$ has integral solutions for $a_1, a_2, \ldots, a_n$, where $p_1, p_2, \ldots, p_n$ is a sequence of distinct prime and m is any integer larger than or equal to 1.

Proof:

If $p_1, p_2, p_3, \ldots, p_n$ be "n" distinct primes and "m" is an integer greater or equal to one, then there exists integers $a_1, a_2, a_3, \ldots, a_n$ (not all zero) such that ,

$a_1p_1^m + a_2p_2^m + \ldots + a_np_n^m = 0$

Since $p_1, p_2, p_3, \ldots, p_n$ are n distinct primes, therefore the terms $p_1^m, p_2^m, p_3^m, \ldots, p_n^m$ are pair wise co-prime and $\gcd(p_1^m, p_2^m, p_3^m, \ldots, p_n^m)=1$

This also implies $\gcd(p_1^m, p_2^m, p_3^m, \ldots, p_{n-1}^m)=1$

Therefore using Bezout's identity there must exist (n-1) integers $b_1, b_2, b_3, \ldots, b_{n-1}$ (not all zero) such that

$b_1p_1^m + b_2p_2^m + \ldots + (b_{n-1})(p_{n-1})^m = 1$

Multiplying both sides with $(-a_np_n^m)$ where we choose a_n is a non-zero integer,

$(-a_np_n^m)\, b_1p_1^m + (-a_np_n^m)\, b_2p_2^m + \ldots + (-a_np_n^m)\,(b_{n-1})(p_{n-1})^m = (-a_np_n^m)$

Replacing $(-a_np_n^m)\, b_1$ by a_1,
$(-a_np_n^m)\, b_2$ by a_2,
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$(-a_np_n^m)\,(b_{n-1})$ by a_{n-1}

We have
$a_1p_1^m + a_2p_2^m + \ldots + a_{n-1}p_{n-1}^m = (-a_np_n^m)$

or

$\mathbf{a_1p_1^m + a_2p_2^m + \ldots + a_{n-1}p_{n-1}^m + a_np_n^m = 0}$
where $a_1, a_2, a_3, \ldots, a_n$ are integers (not all zero).