Korovkin-type theorems for abstract modular convergence

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Abstract: We give some Korovkin-type theorems on convergence and estimates of rates of approximations for nets of functions, satisfying suitable axioms, whose particular cases are filter/ideal convergence, almost convergence and triangular A-statistical convergence, where A is a non-negative summability method. Furthermore, we give some comparison result between different types of convergence and applications to integral type and discrete operators.

Let $(W, \succeq)$ be a directed set, and let us consider an axiomatic abstract convergence on W, as follows.

Let $\mathcal{T}$ be the set of all real-valued nets $(x_w)_{w \in W}$. A *convergence* is a pair $(\mathcal{S}, \ell)$, where $\mathcal{S}$ is a linear subspace of $\mathcal{T}$ and $\ell : \mathcal{S} \to \mathbb{R}$ is a function, satisfying the following axioms:

(a) $\ell((a_1 \, x_w + a_2 \, y_w)_w) = a_1 \, \ell((x_w)_w) + a_2 \, \ell((y_w)_w)$ for every pair of nets $(x_w)_w$, $(y_w)_w \in \mathcal{S}$ and for each a_1, $a_2 \in \mathbb{R}$ (linearity).

(b) If $(x_w)_w$, $(y_w)_w \in \mathcal{S}$ and there is $w^* \in W$ with $x_w \leq y_w$ for every $w \succeq w^*$, then $\ell((x_w)_w) \leq \ell((y_w)_w)$ (monotonicity).

(c) If $(x_w)_w$ is such that there is $w_* \in W$ with $x_w = l$ whenever $w \succeq w_*$, then $(x_w)_w \in \mathcal{S}$ and $\ell((x_w)_w) = l$.

(d) If $(x_w)_w \in \mathcal{S}$, then $(|x_w|)_w \in \mathcal{S}$ and $\ell((|x_w|)_w) = |\ell((x_w)_w)|$.

(e) Let $(x_w)_w$, $(y_w)_w$, $(z_w)_w$, satisfying $(x_w)_w$, $(z_w)_w \in \mathcal{S}$, $\ell((x_w)_w) = \ell((z_w)_w)$ and suppose that there is $\overline{w} \in W$ with $x_w \leq y_w \leq z_w$ for every $w \geq \overline{w}$. Then $(y_w)_w \in \mathcal{S}$.

Note that $\mathcal{S}$ is the space of all convergent nets, ℓ will be the "limit" according to this approach, and we will denote by the symbol $\lim\limits_{w} x_w$ the quantity $\ell((x_w)_w)$.

We now give the axiomatic definition of the operators "limit superior" and "limit inferior" related with a convergence $(\mathcal{S}, \ell)$, which we denote by the symbols lim sup and lim inf

Let $\mathcal{T}$, $\mathcal{S}$ be as above. We define two functions $\overline{\ell}$, $\underline{\ell} : \mathcal{T} \to \widetilde{\mathbb{R}}$, satisfying the following axioms:

(f) If $(x_w)_w$, $(y_w)_w \in \mathcal{T}$, then $\underline{\ell}((x_w)_w) \leq \overline{\ell}((x_w)_w)$ and $\overline{\ell}((x_w)_w) = -\underline{\ell}((-x_w)_w)$.

(g) If $(x_w)_w \in \mathcal{T}$, then

(i) $\overline{\ell}((x_w + y_w)_w) \leq \overline{\ell}((x_w)_w) + \overline{\ell}((y_w)_w)$ (subadditivity);

(ii) $\underline{\ell}((x_w + y_w)_w) \geq \underline{\ell}((x_w)_w) + \underline{\ell}((y_w)_w)$ (superadditivity).

(h) If $(x_w)_w$, $(y_w)_w \in \mathcal{T}$ and $x_w \leq y_w$ definitely, then $\overline{\ell}((x_w)_w) \leq \overline{\ell}((y_w)_w)$ and $\underline{\ell}((x_w)_w) \leq \underline{\ell}((y_w)_w)$ (monotonicity).

(j) A net $(x_w)_w \in \mathcal{T}$ belongs to $\mathcal{S}$ if and only if $\overline{\ell}((x_w)_w) = \underline{\ell}((x_w)_w)$.

We will denote by the symbols $\limsup_w x_w$ and $\liminf_w x_w$ the quantities $\overline{\ell}((x_w)_w)$ and $\overline{\ell}((x_w)_w)$, respectively.

Let $(x_w)_w$, $(y_w)_w \in \mathcal{S}$ with $\lim_w x_w = \lim_w y_w = 0$ and $y_w \neq 0$ for every $w \in W$. We say that $x_w = o(y_w)$ iff $\lim_w \dfrac{|x_w|}{|y_w|} = 0$, and that $x_w = O(y_w)$ iff $\limsup_w \dfrac{|x_w|}{|y_w|} \in \mathbb{R}$.

Let $G = (G, d)$ be a metric space, $\mathcal{B}$ be the σ-algebra of all Borel subsets of G, and μ be a positive finite regular measure defined on $\mathcal{B}$. Let $L^0(G)$ be the space of all real-valued μ-measurable functions on G with identification up to sets of measure μ zero, $\mathcal{C}_b(G)$ be the space of all real-valued continuous and bounded functions on G, $\mathcal{C}_c(G)$ be the subspace of $\mathcal{C}_b(G)$ of all functions with compact support on G and $\mathrm{Lip}(G)$ be the space of all real-valued Lipschitz functions on G.

A functional $\rho : L^0(G) \to \widetilde{\mathbb{R}^+_0}$ is called a *modular* on $L^0(G)$ iff it satisfies the following conditions:

i) $\rho[f] = 0 \Longleftrightarrow f = 0$ μ-almost everywhere on G;

ii) $\rho[-f] = \rho[f]$ for every $f \in L^0(G)$;

iii) $\rho[af + bg] \leq \rho[f] + \rho[g]$ whenever f, $g \in L^0(G)$ and $a \geq 0$, $b \geq 0$ with $a + b = 1$.

A modular ρ is said to be *convex* iff it satisfies conditions i), ii) and

iii') $\rho[af + bg] \leq a\rho[f] + b\rho[g]$ for all f, $g \in L^0(G)$ and for every a, $b \geq 0$ with $a + b = 1$.

Let $Q \geq 1$ be a real constant. We say that a modular ρ is *Q-quasi semiconvex* if $\rho[a\, f] \leq Q\, a\, \rho[Q\, f]$ for all $f \in L^0(G)$, $f \geq 0$ and $0 < a \leq 1$.

A modular ρ is *monotone* if $\rho[f] \leq \rho[g]$ for all $f, g \in L^0(G)$ with $|f| \leq |g|$.

A modular ρ is said to be *finite* if χ_A (the characteristic function associated with A) belongs to $L^\rho(G)$ whenever $A \in \mathcal{B}$ with $\mu(A) < +\infty$.

A modular ρ is *strongly finite* if χ_A belongs to $E^\rho(G)$ for all $A \in \mathcal{B}$ with $\mu(A) < +\infty$.

A modular ρ is said to be *absolutely continuous* if there is a positive constant a with the property: for all $f \in L^0(G)$ with $\rho[f] < +\infty$,

i) for each $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ with $\mu(A) < +\infty$ and $\rho[af\chi_{G\setminus A}] \leq \varepsilon$,

ii) for every $\varepsilon > 0$ there is a $\delta > 0$ with $\rho[af\,\chi_B] \leq \varepsilon$ for every $B \in \mathcal{B}$ with $\mu(B) < \delta$.

The *modular space* $L^\rho(G)$ generated by ρ is

$$L^\rho(G) = \{f \in L^0(G) : \lim_{\lambda \to 0^+} \rho[\lambda f] = 0\},$$

and the *space of the finite elements of* $L^\rho(G)$ is

$$E^\rho(G) = \{f \in L^\rho(G) : \rho[\lambda f] < +\infty \text{ for all } \lambda > 0\}.$$

A net $(f_w)_w$ of functions in $L^\rho(G)$ is (ℓ)-*modularly convergent* to $f \in L^\rho(G)$ if there is a $\lambda > 0$ with

$$\lim_w \rho[\lambda(f_w - f)] = 0.$$

A net $(f_w)_w$ in $L^\rho(G)$ is (ℓ)-*strongly convergent* to $f \in L^\rho(G)$ if

$$\lim_w \rho[\lambda(f_w - f)] = 0 \text{ for every } \lambda > 0.$$

We consider some kinds of rates of approximation associated with the Korovkin theorem in the context of modular convergence. For technical reasons, we sometimes suppose that (G, d) satisfies the following property:

H*) For every $n \in \mathbb{N}$ and s, $t \in G$, with $s \neq t$, there are $n + 1$ points x_i, $i = 0, \ldots, n+1$, such that $s = x_0$, $t = x_{n+1}$ and $d(x_i, x_{i+1}) \leq \dfrac{1}{n} d(s,t)$ for each $i = 0, \ldots, n$.

Some examples of spaces satisfying condition H*) are the Euclidean multidimensional space $\mathbb{R}^N$ endowed with the usual metric and the space $\mathbb{R}^\Lambda$ equipped with the sup-norm, where Λ is any abstract nonempty set.

For every $f \in \mathcal{C}_b(G)$ and $\delta > 0$, let

$$\omega(f;\delta) := \sup\{|f(s) - f(t)| : s, t \in G, d(s,t) \leq \delta\}$$

be the usual modulus of continuity of f. Note that $\omega(f;\delta)$ is an increasing function of δ, $|f(s) - f(t)| \leq \omega(f; d(s,t))$ for each s, $t \in G$, $\omega(f;\delta) \leq 2\,M$ for every δ, where $M = \sup_{t \in G} |f(t)|$, and

$$\omega(f;\gamma\,\delta) \leq (1+\gamma)\,\omega(f;\delta) \tag{1}$$

for every γ, $\delta > 0$.

Let T be a net of linear operators $T_w : \mathcal{D} \to L^0(G)$, $w \in W$, with $\mathcal{C}_b(G) \subset \mathcal{D} \subset L^0(G)$. Here the set $\mathcal{D}$ is the domain of the operators T_w.

We say that the net T, together with the modular ρ, satisfies *property* (ρ)-$(*)$ iff there exist a subset $X_T \subset \mathcal{D} \cap L^\rho(G)$ with $\mathcal{C}_b(G) \subset X_T$ and an $E > 0$ with

$T_w f \in L^\rho(G)$ for any $f \in X_T$ and $w \in W$, and $\limsup_w \rho[\tau(T_w f)] \le E\,\rho[\tau f]$ for every $f \in X_T$ and $\tau > 0$.

Let e_r and a_r, $r = 0, \ldots, m$, be functions in $\mathcal{C}_b(G)$, and put $e_0(t) := 1$ for every $t \in G$. Let us define

$$P_s(t) := \sum_{r=0}^{m} a_r(s) e_r(t), \quad s, t \in G, \tag{2}$$

and assume that

(P1) $P_s(s) = 0$ for all $s \in G$;

(P2) there is a $C_1 > 0$ with $P_s(t) \ge C_1\, d(s,t)$ whenever s, $t \in G$.

From now on, we suppose that $e_r \in L^\rho(G)$, $r = 0, 1, \ldots, m$. Note that this assumption is fulfilled, for example, when G is an open bounded subset of $\mathbb{R}^n$ or, more generally, a space of finite measure μ.

Theorem 0.1 *Let ρ be a strongly finite, monotone and Q-quasi semiconvex modular. Assume that e_r and a_r, $r = 0, \ldots, m$, satisfy* (P1) *and* (P2). *Let T_w, $w \in W$, be a net of positive linear operators having property (ρ)-$(*)$. If $(T_w e_r)_w$ is (ℓ)-modularly convergent to e_r in $L^\rho(G)$ for each $r = 0, \ldots, m$, then $(T_w f)_w$ is (ℓ)-modularly convergent to f in $L^\rho(G)$ for every $f \in \mathcal{C}_c(G)$.*

If $(T_w e_r)_w$ is (ℓ)-strongly convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(G)$, then $(T_w f)_w$ is (ℓ)-strongly convergent to f in $L^\rho(G)$ for every $f \in \mathcal{C}_c(G)$.

Theorem 0.2 *Let ρ be a monotone, strongly finite, absolutely continuous and Q-quasi semiconvex modular on $L^0(G)$, and T_w, $w \in W$ be a net of positive linear operators satisfying (ρ)-$(*)$. If $(T_w e_r)_w$ is (ℓ)-strongly convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(G)$, then $(T_w e_r)_w$ is (ℓ)-modularly convergent to f in $L^\rho(G)$ for every $f \in L^\rho(G) \cap \mathcal{D}$ with $f - \mathcal{C}_b(G) \subset X_T$, where $\mathcal{D}$ and X_T are as above.*

Now we present some estimates on rates of approximation for abstract Korovkin-type theorems. Let Ξ be the family of all nets ξ_w, $w \in W$, with $\xi_w \neq 0$ for each $w \in W$ and $\lim_w \xi_w = 0$.

Theorem 0.3 *Let $Q \ge 1$, ρ be a monotone, strongly finite and Q-quasi semiconvex modular, T_w, $w \in W$, be a net of positive linear operators and Ξ be as above. For every $w \in W$, let $\xi_w^r \in \Xi$, $r = 0, \ldots, m$, and set $\xi_w := \max\{\xi_w^r\colon r = 0, \ldots, m\}$. If $\gamma > 0$ is such that $\rho[\gamma(T_w e_r - e_r)] = o(\xi_w^r)$ for each $r = 0, \ldots, m$, then for every $f \in \mathcal{C}_c(G) \cap Lip(G)$ there exists a positive real number τ with $\rho[\tau(T_w f - f)] = o(\xi_w)$.*

A similar result holds also when o is replaced by O.

Theorem 0.4 *Let Q, $(T_w)_w$, ρ, Ξ be as in Theorem* 0.3, (G,d) *satisfy condition* H*), ξ_w^0, $\xi_w^* \in \Xi$, *set* $\xi_w := \max\{\xi_w^0, \xi_w^*\}$, $w \in W$, *and* $\psi(s)(t) := d(s,t)$, s, $t \in G$. *For every* $f \in \mathcal{C}_c(G)$ *and* $w \in W$ *put* $\delta_w^f = \|T_w(\psi)\|$, *where* $\|\cdot\|$ *is the sup-norm and the supremum is taken with respect to the support of* f. *If* $\gamma > 0$ *satisfies the conditions*

0.4.1) $\rho[\gamma(T_w e_0 - e_0)] = o(\xi_w^0)$ *and*

0.4.2) $\rho[\gamma\, \omega(f; \delta_w^f)] = o(\xi_w^*)$,

then for each $f \in \mathcal{C}_c(G)$ *there is a positive real number* τ *with* $\rho[\tau\,(T_w f - f)] = o(\xi_w)$.

Moreover, a similar result holds when the symbol o *is replaced by* O.

We consider filter convergence, noting that this kind of convergence satisfies the given axioms.

Let $W = (W, \succeq)$ be a directed set, then for each $w \in W$, set $M_w := \{z \in W : z \succeq w\}$. A filter $\mathcal{F}$ of W is said to be *free* iff $M_w \in \mathcal{F}$ for every $w \in W$.

Some examples frequently used in the literature are $(W, \succeq) = (\mathbb{N}, \geq)$, $W \subset [a, w_0[\subset \mathbb{R}$ with the usual order, where $w_0 \in \mathbb{R} \cup \{+\infty\}$ is a limit point of W, or $(W, \succeq) = (\mathbb{N}^2, \geq) = (\mathbb{N} \times \mathbb{N}, \geq)$, where in $\mathbb{N}^2$ the symbol $\geq$ denotes the usual componentwise order.

It is not difficult to check that, if $\mathcal{F}$ is any fixed free filter of W, in the $\mathcal{F}$-convergence setting, given $(x_w)_w$, $(y_w)_w \in \Xi$, we get $x_w = o(y_w)$ if and only if $\{w \in W\colon\ x_w \leq \varepsilon\, y_w\} \in \mathcal{F}$ for every $\varepsilon > 0$ and $x_w = O(y_w)$ if and only if there is a positive real number C with $\{w \in W\colon\ x_w \leq C\, y_w\} \in \mathcal{F}$.

In the filter convergence setting, it is possible also to relax the positivity condition on the involved linear operators. For instance, let I be a bounded interval of $\mathbb{R}$, $\mathcal{C}^2(I)$ (resp. $\mathcal{C}^2_b(I)$) be the space of all functions defined on I, (resp. bounded and) continuous together with their first and second derivatives, $\mathcal{C}_+ := \{f \in \mathcal{C}^2_b(I) : f \geq 0\}$, $\mathcal{C}^2_+ := \{f \in \mathcal{C}^2_b(I) : f'' \geq 0\}$.

Let e_r, $r = 1, \ldots, m$ and a_r, $r = 0, \ldots, m$ be functions in $\mathcal{C}^2_b(I)$, $P_s(t)$, $s, t \in I$, be as in (2), and suppose that $P_s(t)$ satisfies the above conditions (P1), (P2) and

(P3) there is a positive real number C_0 with $P_s''(t) \geq C_0$ whenever s, $t \in I$, where the second derivative is taken with respect to t.

Theorem 0.5 *Let $\mathcal{F}$ be any free filter of W, ρ be as in Theorem 0.1, e_r, a_r, $r = 0, \ldots, m$ and $P_s(t)$, s, $t \in I$, satisfy properties* (P1), (P2) *and* (P3). *Assume that* T_w, $w \in W$ *is a net of linear operators which fulfil property* (ρ)-$(*)$, *and that* $\{w \in W : T_w(\mathcal{C}_+ \cap \mathcal{C}^2_+) \subset \mathcal{C}_+\} \in \mathcal{F}$. *If* $(T_w e_r)_w$ *is* (ℓ)*-modularly* $\mathcal{F}$*-convergent to* e_r, $r = 0, \ldots, m$ *in* $L^\rho(I)$, *then* $(T_w f)_w$ *is* (ℓ)*-modularly* $\mathcal{F}$*-convergent to* f *in* $L^\rho(I)$ *for each* $f \in \mathcal{C}^2_b(I)$.

If $(T_w e_r)_w$ *is* (ℓ)*-strongly* $\mathcal{F}$*-convergent to* e_r, $r = 0, \ldots, m$ *in* $L^\rho(I)$, *then* $(T_w f)_w$ *is* (ℓ)*-strongly* $\mathcal{F}$*-convergent to* f *in* $L^\rho(I)$ *for every* $f \in \mathcal{C}^2_b(I)$.

Furthermore, if ρ is absolutely continuous and $(T_w e_r)_w$ is (ℓ)-strongly $\mathcal{F}$-convergent to e_r, $r = 0, \ldots, m$ in $L^\rho(I)$, then $(T_w f)_w$ is (ℓ)-modularly $\mathcal{F}$-convergent to f in $L^\rho(I)$ for every $f \in L^\rho(I) \cap \mathcal{D}$ with $f - \mathcal{C}_b(I) \subset X_T$.

Other examples of convergences, satisfying the given axioms, are the *single convergence* and the *almost convergence.*

Let $W = \mathbb{N}$. A sequence $(x_n)_n$ is said to *singly converge* (resp. *almost converge*) to $x \in \mathbb{R}$ iff

$$\lim_n \frac{x_{m+1} + x_{m+2} + \ldots + x_{m+n}}{n} = x$$

for every $m \geq 0$ (resp. uniformly with respect to m), where the involved limit is the usual one. It is not difficult to check that single and almost convergence satisfy the given axioms. Note that, in general, almost and singly convergence are not generated by any free filter.

We now consider a kind of "triangular statistical convergence". Let $A = (a_{i,j})_{i,j}$ be a non-negative two-dimensional infinite matrix and $\Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a fixed function. We say that A is a *summability matrix* iff it satisfies the following conditions:

$(A1)$ $\displaystyle\sum_{j \in \mathbb{N}, \Psi(i,j) \geq 0} a_{i,j} \leq 1$ for each $i \in \mathbb{N}$,

$(A2)$ $\displaystyle\lim_i \sum_{j \in \mathbb{N}, \Psi(i,j) \geq 0} a_{i,j} > 0$,

$(A3)$ $\displaystyle\lim_i a_{i,j} = 0$ for every $j \in \mathbb{N}$.

For every $K \subset \mathbb{N}^2$, set $K_i := \{j \in \mathbb{N}\colon (i,j) \in K, \Psi(i,j) \geq 0\}$. The *$\Psi$-$A$-density* of K is given by

$$\delta_A^\Psi(K) := \lim_i \sum_{j \in K_i} a_{i,j}, \tag{3}$$

provided that the limit on the right hand exists in $\mathbb{R}$.

It is not difficult to see that the Ψ-A-density satisfies the following properties:

$(D1)$ $\delta_A^\Psi(\mathbb{N}^2) > 0$.

$(D2)$ If $K \subset H$, then $\delta_A^\Psi(K) \leq \delta_A^\Psi(H)$.

$(D3)$ If $\delta_A^\Psi(K) = \delta_A^\Psi(H) = 0$, then $\delta_A^\Psi(K \cup H) = 0$.

Observe that from $(D1)$-$(D3)$ it follows that the family

$$\mathcal{F}_A^\Psi := \{K \subset \mathbb{N}^2 : \delta_A^\Psi(\mathbb{N}^2 \setminus K) = 0\} \tag{4}$$

is a filter of $\mathbb{N}^2$.

Let $A = (a_{i,j})_{i,j}$ be a summability matrix. The double sequence $(x_w)_w$ is said to *Ψ-A-statistically converge* to a real number x iff $(\mathcal{F}_A^\Psi) \lim_w x_w = x$, that is iff for every $\varepsilon > 0$ we get

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \{j \in \mathbb{N}\colon\ \Psi(i,j) \geq 0,\ |x_w - x| \geq \varepsilon\}$, and we write $st_A^{\Psi}\text{-}\lim_i x_w = x$.

Let $(a_{i,j})_{i,j}$ be defined by

$$a_{i,j} := \begin{cases} \frac{1}{i^2} & \text{if } j \leq i^2, \\ \\ 0 & \text{otherwise,} \end{cases}$$

put $\Psi(i,j) = i - j$, i, $j \in \mathbb{N}$, and pick any double sequence $(x_{i,j})_{i,j}$ in $\mathbb{R}$. For every $\varepsilon > 0$ we get $K_i(\varepsilon) := \{j \in \mathbb{N}\colon\ j \leq i,\ x_{i,j} \geq \varepsilon\} \subset \{j \in \mathbb{N}\colon\ j \leq i\}$. Thus we obtain

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} \leq \lim_i \sum_{j \leq i} \frac{1}{i^2} = \lim_i \frac{1}{i} = 0, \tag{5}$$

and thus $(x_{i,j})_{i,j}$ Ψ-A-statistically converges to 0. We get $\lim_i \sum_{j \in \mathbb{N}, \Psi(i,j) \geq 0} a_{i,j} = 0$, and so condition $(A2)$ is not fulfilled. Note that, in this case, the class $\mathcal{F}_A^{\Psi}$ defined as in (4) is not a filter, because it coincides with the family of all subsets of $\mathbb{N}^2$.

Note that, in general, filter convergence in $\mathbb{N}^2$ is not equal to Ψ-A-statistical convergence, that is there exists some filter $\mathcal{F}$ of $\mathbb{N}^2$ such that, for every summability matrix A, there is a set $K \in \mathcal{F} \setminus \mathcal{F}_A^{\Psi}$.