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General two-sided Clifford Fourier transform, convolution and Mustard convolution

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Abstract. In this paper we use the general steerable two-sided Clifford Fourier transform (CFT), and relate the classical convolution of Clifford algebra-valued signals over $\mathbb{R}^{p,q}$ with the (equally steerable) Mustard convolution. A Mustard convolution can be expressed in the spectral domain as the point wise product of the CFTs of the factor functions. In full generality do we express the classical convolution of Clifford algebra signals in terms of finite linear combinations of Mustard convolutions, and vice versa the Mustard convolution of Clifford algebra signals in terms of finite linear combinations of classical convolutions.

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1. Introduction

The steerable two-sided Clifford Fourier transformation (CFT) was introduced in [23]. It generalizes related transforms, like the two-sided quaternion Fourier transform [12], and the space-time Fourier transform [25] to higher dimensions. The classical complex Fourier transform needs only one kernel factor, due to the commutativity of complex numbers. To have different kernel factors under the transform integral on both sides of the signal function is meaningful due to the inherent non-commutativity in Clifford algebras.

A key strength of the classical complex Fourier transform is its easy and fast application to filtering problems. The convolution of a signal with its filter function becomes in the spectral domain a point wise product of the respective Fourier transformations. This is not the case for the convolution

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of two Clifford algebra-valued signals (Clifford signals) over $\mathbb{R}^{p,q}$, due to Clifford algebra non-commutativity. Yet it is possible to define from the point wise product of the CFTs of two Clifford signals a new type of steerable convolution, called Mustard convolution [29]. This Mustard convolution can be expressed in terms of sums of classical convolutions and vice versa. For the left-sided QFT this has recently been carried out in [7], for two-sided QFT in [24] and for the space-time Fourier transform in [25]. Here we extend this approach in full generality to the steerable two-sided CFT for signal functions which map non-degenerate quadratic form vector spaces to Clifford algebras in all dimensions.

This paper is organized as follows. Section 2 introduces Clifford algebra, multivector signal functions, and the continuous manifolds of multivector square roots of -1. Next, Section 3 briefly reviews an important decomposition (split) of multivectors with respect to a pair of multivector square roots of -1. Then, Section 4 gives some background on the steerable two-sided CFT and newly defines two related (steerable) exponential-sine Clifford Fourier transforms. Finally, Section 5 defines the classical convolution of two Clifford signal functions, as well as two types of steerable Mustard convolutions. The rest of the section is devoted to representing the classical convolution in terms of finite sums of Mustard convolutions. First is the general case in terms of the two types of Mustard convolutions in Theorem 5.4. Second, Corollary 5.6 expresses for a commuting pair of square roots of -1 in the CFT, the convolution in terms of the standard Mustard convolution. Third, Theorem 5.7 generally expresses the classical convolution in terms of the standard Mustard convolution. Fourth, for a pair of anticommuting square roots of -1 in the CFT, Theorem 5.8 gives the classical convolution in terms of standard Mustard convolutions. At the end, Theorem 5.9 states the Mustard convolution in terms of classical convolutions.

2. Clifford's geometric algebra

Definition 2.1 (Clifford's geometric algebra [9, 27, 11, 18]**).** Let $\{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n\}$, with $n = p+q$, $e_k^2 = Q(e_k)1 = \varepsilon_k$, $\varepsilon_k = +1$ for $k = 1, \ldots, p$, $\varepsilon_k = -1$ for $k = p+1, \ldots, n$, be an *orthonormal base* of the non-degenerate inner product vector space $(\mathbb{R}^{p,q}, Q)$, Q the non-degenerate quadratic form, with a geometric product according to the multiplication rules

$$e_k e_l + e_l e_k = 2\varepsilon_k \delta_{k,l}, \qquad k, l = 1, \ldots n, \tag{2.1}$$

where $\delta_{k,l}$ is the Kronecker symbol with $\delta_{k,l} = 1$ for $k = l$, and $\delta_{k,l} = 0$ for $k \neq l$. This non-commutative product and the additional axiom of *associativity* generate the 2^n-dimensional Clifford geometric algebra $Cl(p,q) = Cl(\mathbb{R}^{p,q}) = Cl_{p,q} = \mathcal{G}_{p,q} = \mathbb{R}_{p,q}$ over $\mathbb{R}$. For Euclidean vector spaces ($n = p$) we use $\mathbb{R}^n = \mathbb{R}^{n,0}$ and $Cl(n) = Cl(n,0)$. The set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$, $e_\emptyset = 1$, the unity in the Clifford algebra, forms a graded (blade) basis of $Cl(p,q)$. The grades k range from 0 for scalars, 1 for vectors, 2 for bivectors, s for s-vectors, up to n for pseudoscalars.

The quadratic space $(\mathbb{R}^{p,q}, Q)$ is embedded into $C\ell_{p,q}$ as a subspace, which is identified with the subspace of 1-vectors. All linear combinations of basis elements of grade k, $0 \leq k \leq n$, form the subspace $C\ell^k_{p,q} \subset C\ell_{p,q}$ of k-vectors. The general elements of $Cl(p,q)$ are real linear combinations of basis blades e_A, called Clifford numbers, multivectors or hypercomplex numbers.

In general $\langle A \rangle_k$ denotes the grade k part of $A \in Cl(p,q)$. Following [11], the parts of grade 0 and $k+s$, respectively, of the geometric product of a k-vector $A_k \in Cl(p,q)$ with an s-vector $B_s \in Cl(p,q)$

$$A_k * B_s := \langle A_k B_s \rangle_0, \qquad A_k \wedge B_s := \langle A_k B_s \rangle_{k+s}, \tag{2.2}$$

are called *scalar product* and *outer product*, respectively. They are bilinear products mapping a pair of multivectors to a resulting product multivector in the same algebra. The outer product is also associative, the scalar product not.

Every k-vector B that can be written as the outer product $B = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots \wedge \boldsymbol{b}_k$ of k vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in \mathbb{R}^{p,q}$ is called a *simple* k-vector or *blade*.

Multivectors $M \in Cl(p,q)$ have k-vector parts $(0 \leq k \leq n)$: scalar part $Sc(M) = \langle M \rangle = \langle M \rangle_0 = M_0 \in \mathbb{R}$, vector part $\langle M \rangle_1 \in \mathbb{R}^{p,q}$, bi-vector part $\langle M \rangle_2 \in \bigwedge^2 \mathbb{R}^{p,q}$, ..., and pseudoscalar part $\langle M \rangle_n \in \bigwedge^n \mathbb{R}^{p,q}$

$$M = \sum_A M_A \boldsymbol{e}_A = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n \,. \tag{2.3}$$

The *principal reverse* of $M \in Cl(p,q)$ defined as

$$\widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle \overline{M} \rangle_k, \tag{2.4}$$

often replaces complex conjugation and quaternion conjugation. Taking the *reverse* is equivalent to reversing the order of products of basis vectors in the basis blades e_A. The operation $\overline{M}$ means to change in the basis decomposition of M the sign of every vector of negative square $\overline{e_A} = \varepsilon_{h_1} e_{h_1} \varepsilon_{h_2} e_{h_2} \ldots \varepsilon_{h_k} e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$. Reversion, $\overline{M}$, and principal reversion are all involutions. In $Cl(n)$ the principal reverse and the reverse are identical.

For $M, N \in Cl(p,q)$ we get $M * \widetilde{N} = \sum_A M_A N_A$. Two multivectors $M, N \in Cl(p,q)$ are *orthogonal* if and only if $M * \widetilde{N} = 0$. The modulus $|M|$ of a multivector $M \in Cl(p,q)$ is defined as

$$|M|^2 = M * \widetilde{M} = \sum_A M_A^2. \tag{2.5}$$

2.1. Multivector signal functions

A multivector valued function $h : \mathbb{R}^{p,q} \to Cl(p',q')$, has $2^{n'}$ blade components, $n' = p' + q'$ $(h_A : \mathbb{R}^{p,q} \to \mathbb{R})$

$$h(\boldsymbol{x}) = \sum_A h_A(\boldsymbol{x}) \boldsymbol{e}_A. \tag{2.6}$$

We define the *inner product* of two functions $h, m : \mathbb{R}^{p,q} \to Cl(p',q')$ by

$$(h,m) = \int_{\mathbb{R}^{p,q}} h(\boldsymbol{x})\widetilde{m(\boldsymbol{x})}\, d^n\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A\widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^{p,q}} h_A(\boldsymbol{x}) m_B(\boldsymbol{x})\, d^n\boldsymbol{x}, \tag{2.7}$$

with the *symmetric scalar part*

$$\langle h,m\rangle = \int_{\mathbb{R}^{p,q}} h(\boldsymbol{x}) * \widetilde{m(\boldsymbol{x})}\, d^n\boldsymbol{x} = \sum_{A} \int_{\mathbb{R}^{p,q}} h_A(\boldsymbol{x}) m_A(\boldsymbol{x})\, d^n\boldsymbol{x}, \tag{2.8}$$

and the $L^2(\mathbb{R}^{p,q}; Cl(p',q'))$-*norm*

$$\|h\|^2 = \langle (h,h)\rangle = \int_{\mathbb{R}^{p,q}} |h(\boldsymbol{x})|^2 d^n\boldsymbol{x} = \sum_{A} \int_{\mathbb{R}^{p,q}} h_A^2(\boldsymbol{x})\, d^n\boldsymbol{x}, \tag{2.9}$$

$$L^2(\mathbb{R}^{p,q}; Cl(p',q')) = \{h : \mathbb{R}^{p,q} \to Cl(p',q') \mid \|h\| < \infty\}. \tag{2.10}$$

Notation 2.2 (Argument reflection). *For a function $h : \mathbb{R}^{p,q} \to Cl(p',q')$ and a multi-index $\phi = (\phi_1, \phi_2)$ with $\phi_1, \phi_2 \in \{0,1\}$ we set*

$$h^{\phi} = h^{(\phi_1,\phi_2)}(\boldsymbol{x}) := h((-1)^{\phi_1}\boldsymbol{x}_k, (-1)^{\phi_2}\boldsymbol{x}_{(n-k)}), \tag{2.11}$$

where $\boldsymbol{x}_k = x_1 e_1 + \ldots + x_k e_k$, $\boldsymbol{x}_{(n-k)} = \boldsymbol{x} - \boldsymbol{x}_k$, $1 \leq k \leq n$, for arbitrary but fixed k.

2.2. Square roots of -1 in Clifford algebras

Every Clifford algebra $Cl(p,q)$, $s_8 = (p-q) \bmod 8$, is isomorphic to one of the following (square) matrix algebras[1] $\mathcal{M}(2d,\mathbb{R})$, $\mathcal{M}(d,\mathbb{H})$, $\mathcal{M}(2d,\mathbb{R}^2)$, $\mathcal{M}(d,\mathbb{H}^2)$ or $\mathcal{M}(2d,\mathbb{C})$. The first argument of $\mathcal{M}$ is the dimension, the second the associated ring[2] $\mathbb{R}$ for $s_8 = 0,2$, $\mathbb{R}^2$ for $s_8 = 1$, $\mathbb{C}$ for $s_8 = 3,7$, $\mathbb{H}$ for $s_8 = 4,6$, and $\mathbb{H}^2$ for $s_8 = 5$. For even n: $d = 2^{(n-2)/2}$, for odd n: $d = 2^{(n-3)/2}$.

It has been shown [17, 22] that $Sc(f) = 0$ for every square root of $-\mathbf{1}$ in every matrix algebra $\mathcal{A}$ isomorphic to $Cl(p,q)$. One can distinguish *ordinary* square roots of $-\mathbf{1}$, and *exceptional* ones. All square roots of $-\mathbf{1}$ in $Cl(p,q)$ can be computed using the package CLIFFORD for Maple [2, 3, 19, 28].

In all cases the *ordinary* square roots f of $-\mathbf{1}$ constitute a *unique conjugacy class* of dimension $\dim(\mathcal{A})/2$, which has *as many connected components as the group* $\mathrm{G}(\mathcal{A})$ of invertible elements in $\mathcal{A}$. Furthermore, for ordinary square roots of $-\mathbf{1}$ we always have $\mathrm{Spec}(f) = 0$ (zero pseudoscalar part) if the associated ring is $\mathbb{R}^2$, $\mathbb{H}^2$, or $\mathbb{C}$. The exceptional square roots of $-\mathbf{1}$ *only* exist if $\mathcal{A} \cong \mathcal{M}(2d,\mathbb{C})$.

For $\mathcal{A} = \mathcal{M}(2d,\mathbb{R})$, the centralizer (set of all elements in $Cl(p,q)$ commuting with f) and the conjugacy class of a square root f of $-\mathbf{1}$ both have $\mathbb{R}$-dimension $2d^2$ with *two connected components*. For the simplest case $d = 1$ we have the algebra $Cl(2,0)$ isomorphic to $\mathcal{M}(2,\mathbb{R})$, see the left side of Fig. 1.

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[1] Compare chapter 16 on *matrix representations and periodicity of 8*, as well as Table 1 on p. 217 of [27].

[2] Associated ring means, that the matrix elements are from the respective ring $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{H}^2$.

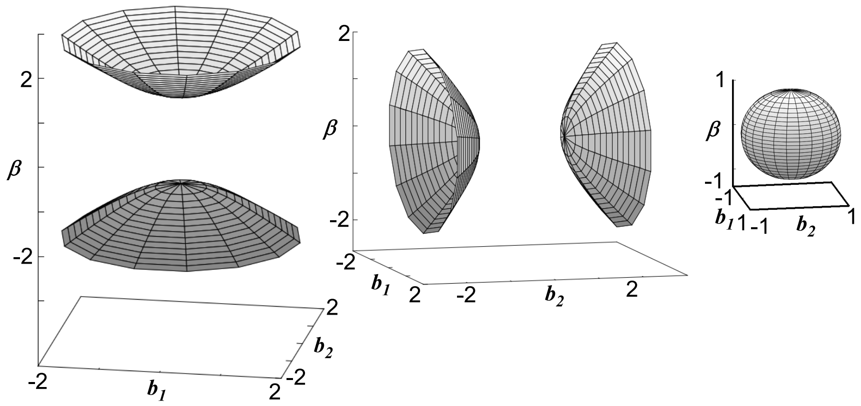

FIGURE 1. Manifolds of square roots f of -1 in $Cl(2,0)$ (left), $Cl(1,1)$ (center), and $Cl(0,2) \cong \mathbb{H}$ (right). The square roots are $f = \alpha + b_1 e_1 + b_2 e_2 + \beta e_{12}$, with $\alpha, b_1, b_2, \beta \in \mathbb{R}$, $\alpha = 0$, and $\beta^2 = b_1^2 e_2^2 + b_2^2 e_1^2 + e_1^2 e_2^2$.

For $\mathcal{A} = \mathcal{M}(2d, \mathbb{R}^2) = \mathcal{M}(2d, \mathbb{R}) \times \mathcal{M}(2d, \mathbb{R})$, the square roots of $(-\mathbf{1}, -\mathbf{1})$ are pairs of two square roots of $-\mathbf{1}$ in $\mathcal{M}(2d, \mathbb{R})$. They constitute a unique conjugacy class with *four connected components*, each of dimension $4d^2$. Regarding the four connected components, the group of inner automorphisms $\text{Inn}(\mathcal{A})$ induces the permutations of the Klein group, whereas the quotient group $\text{Aut}(\mathcal{A})/\text{Inn}(\mathcal{A})$ is isomorphic to the group of isometries of a Euclidean square in 2D. The simplest example with $d = 1$ is $Cl(2,1)$ isomorphic to $M(2, \mathbb{R}^2) = \mathcal{M}(2, \mathbb{R}) \times \mathcal{M}(2, \mathbb{R})$.

For $\mathcal{A} = \mathcal{M}(d, \mathbb{H})$, the submanifold of the square roots f of $-\mathbf{1}$ is a *single connected conjugacy class* of $\mathbb{R}$-dimension $2d^2$ equal to the $\mathbb{R}$-dimension of the centralizer of every f. The easiest example for $d = 1$ is $\mathbb{H}$, isomorphic to $Cl(0,2)$, see the right side of Fig. 1.

For $\mathcal{A} = \mathcal{M}(d, \mathbb{H}^2) = \mathcal{M}(d, \mathbb{H}) \times \mathcal{M}(d, \mathbb{H})$, the square roots of $(-\mathbf{1}, -\mathbf{1})$ are pairs of two square roots (f, f') of $-\mathbf{1}$ in $\mathcal{M}(d, \mathbb{H})$ and constitute a *unique connected conjugacy class* of $\mathbb{R}$-dimension $4d^2$. The group $\text{Aut}(\mathcal{A})$ has two connected components: the neutral component $\text{Inn}(\mathcal{A})$ connected to the identity and the second component containing the swap automorphism $(f, f') \mapsto (f', f)$. The simplest case for $d = 1$ is $\mathbb{H}^2$ isomorphic to $Cl(0,3)$.

For $\mathcal{A} = \mathcal{M}(2d, \mathbb{C})$, the square roots of $-\mathbf{1}$ are in *bijection to the idempotents* [1]. First, the *ordinary* square roots of $-\mathbf{1}$ (with $k = 0$) constitute a conjugacy class of $\mathbb{R}$-dimension $4d^2$ of a *single connected component* which is invariant under $\text{Aut}(\mathcal{A})$. Second, there are $2d$ *conjugacy classes* of *exceptional* square roots of $-\mathbf{1}$, each composed of a *single connected component*, characterized by the equality $\text{Spec}(f) = k/d$ (the pseudoscalar coefficient) with $\pm k \in \{1, 2, \ldots, d\}$, and their $\mathbb{R}$-dimensions are $4(d^2 - k^2)$. The group $\text{Aut}(\mathcal{A})$ includes conjugation of the pseudoscalar $\omega \mapsto -\omega$ which maps the conjugacy class associated with k to the class associated with $-k$. The simplest case for $d = 1$ is the Pauli matrix algebra isomorphic to the geometric algebra $Cl(3,0)$ of 3D Euclidean space $\mathbb{R}^3$, and to complex biquaternions [30].

3. The $\pm$ split with respect to two square roots of -1

With respect to any square root $f \in Cl(p,q)$ of -1, $f^2 = -1$, every multivector $A \in Cl(p,q)$ can be split into *commuting* and *anticommuting* parts [22].

Lemma 3.1. *Every multivector $A \in Cl(p,q)$ has, with respect to a square root $f \in Cl(p,q)$ of -1, i.e., $f^{-1} = -f$, the unique decomposition*

$$A_{+f} = \frac{1}{2}(A + f^{-1}Af), \qquad A_{-f} = \frac{1}{2}(A - f^{-1}Af)$$
$$A = A_{+f} + A_{-f}, \qquad A_{+f}\, f = f A_{+f}, \qquad A_{-f}\, f = -f A_{-f}, \tag{3.1}$$

$A_{+f} \in$ centralizer$(f, Cl_{p,q})$.

Furthermore, for $f, g \in Cl(p,q)$ an arbitrary pair of square roots of -1, $f^2 = g^2 = -1$, the map $f(\,)g$ is an involution, because $f^2xg^2 = (-1)^2x = x, \forall\, x \in Cl(p,q)$. In [12] a split of quaternions by means of the pure unit quaternion basis elements $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{H}$ was introduced, and generalized to a general pair of pure unit quaternions in [15, 21]. This can be *generalized to a split* of $Cl(p,q)$, see [23].

Definition 3.2 ($\pm$ split with respect to two square roots of -1 [23]). Let $f, g \in Cl(p,q)$ be an arbitrary pair of square roots of -1, $f^2 = g^2 = -1$, including the cases $f = \pm g$. The general $\pm$ split is then defined with respect to the two square roots f, g of -1 as

$$x_{\pm} = \frac{1}{2}(x \pm fxg), \qquad \forall\, x \in Cl(p,q). \tag{3.2}$$

Note that the split of Lemma 3.1 is a special case of Definition 3.2 with $g = -f$.

We observe from (3.2), that $fxg = x_+ - x_-$, i.e. under the map $f(\,)g$ the x_+ part is invariant, but the x_- part changes sign

$$fx_{\pm}g = \frac{1}{2}(fxg \pm f^2xg^2) = \frac{1}{2}(fxg \pm x) = \pm\frac{1}{2}(x \pm fxg) = \pm x_{\pm}. \tag{3.3}$$

The two parts $x_{\pm}$ can be represented with Lemma 3.1 as linear combinations of x_{+f} and x_{-f}, or of x_{+g} and x_{-g}, see [23],

$$x_{\pm} = x_{+f}\,\frac{1 \pm fg}{2} + x_{-f}\,\frac{1 \mp fg}{2} = \frac{1 \pm fg}{2}\,x_{+g} + \frac{1 \mp fg}{2}\,x_{-g}. \tag{3.4}$$

There is the following important *general identity* [23],

$$e^{\alpha f}x_{\pm}e^{\beta g} = x_{\pm}e^{(\beta \mp \alpha)g} = e^{(\alpha \mp \beta)f}x_{\pm}. \tag{3.5}$$

For $Cl(p,q) \cong \mathcal{M}(2d,\mathbb{C})$ or $\mathcal{M}(d,\mathbb{H})$ or $\mathcal{M}(d,\mathbb{H}^2)$, or for both f, g being blades in $Cl(p,q) \cong \mathcal{M}(2d,\mathbb{R})$ or $\mathcal{M}(2d,\mathbb{R}^2)$, we have $\widetilde{f} = -f$, $\widetilde{g} = -g$. We therefore obtain the following lemma.

Lemma 3.3 (Orthogonality of two ± split parts [23]). *Assume in $Cl(p,q)$ two square roots f, g of -1 with $\widetilde{f} = -f$, $\widetilde{g} = -g$. Given any two multivectors $x, y \in Cl(p,q)$ and applying the ± split (3.2) with respect to f, g we get zero for the scalar part of the mixed products*

$$Sc(x_+\widetilde{y_-}) = 0, \qquad Sc(x_-\widetilde{y_+}) = 0. \tag{3.6}$$

4. General steerable two-sided Clifford Fourier transforms

The *general steerable two-sided Clifford Fourier transform* (CFT) [23], can both be understood as a generalization of known one-sided CFTs [13, 20], or of the two-sided quaternion Fourier transformation (QFT) [12, 15], or two-sided space-time Fourier transform [12, 15, 25] to a general Clifford algebra setting. Most known CFTs (prior to [23]) used in their kernels specific square roots of -1, like bivectors, pseudoscalars, unit pure quaternions, or sets of coorthogonal blades (commuting or anticommuting blades) [5]. All those restrictions on the square roots of -1 used in a CFT do not apply in our definition below. Note that if the left or right phase angle is identical to zero, we get one-sided right or left sided CFTs, respectively.

Definition 4.1 (Steerable CFT with respect to two square roots of -1 [23]). Let $f, g \in Cl(p', q')$, $f^2 = g^2 = -1$, be any two square roots of -1. The general steerable two-sided Clifford Fourier transform[3] (CFT) of $h \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$, with respect to f, g is

$$\mathcal{F}\{h\}(\boldsymbol{\omega}) = \mathcal{F}^{f,g}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{p,q}} e^{-fu(\boldsymbol{x},\boldsymbol{\omega})} h(\boldsymbol{x})\, e^{-gv(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{x}, \tag{4.1}$$

where $d^n\boldsymbol{x} = dx_1 \dots dx_n$, $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^{p,q}$, and $u, v : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$.

Remark 4.2. *In order to avoid clutter we often drop the upper indexes f, g as in $\mathcal{F}\{h\} = \mathcal{F}^{f,g}\{h\}$, but in principle the two-sided CFT always depends on the particular choice f, g of the two square roots of -1. Since square roots of -1 in $Cl(p', q')$ populate continuous submanifolds in $Cl(p', q')$, the CFT of Definition* 4.1 *is generically steerable within these submanifolds. In Definition* 4.1*, the two square roots $f, g \in Cl(p', q')$ of -1, may be from the same (or different) conjugacy class and component, respectively.*

Linearity of the CFT integral (4.1) allows us to use the ± split $h = h_- + h_+$ of Definition 3.2 to obtain, see [23],

$$\begin{aligned}\mathcal{F}^{f,g}\{h\}(\boldsymbol{\omega}) &= \mathcal{F}^{f,g}\{h_-\}(\boldsymbol{\omega}) + \mathcal{F}^{f,g}\{h_+\}(\boldsymbol{\omega}) \\ &= \mathcal{F}_-^{f,g}\{h\}(\boldsymbol{\omega}) + \mathcal{F}_+^{f,g}\{h\}(\boldsymbol{\omega}),\end{aligned} \tag{4.2}$$

since by their construction the operators of the Clifford Fourier transformation $\mathcal{F}^{f,g}$, and of the ± split with respect to f, g commute. From (3.5) follows the next theorem.

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[3]The image Clifford algebra $Cl(p', q')$ can be identical to $Cl(p, q)$ over the domain vector space $\mathbb{R}^{p,q}$, but this is not necessary, and completely depends on the application context.

Theorem 4.3 (CFT of $h_\pm$, [23]). *The CFT of the $\pm$ split parts $h_\pm$, with respect to two square roots $f, g \in Cl(p', q')$ of -1 , of a Clifford function $h \in L^1(\mathbb{R}^{p,q};\ Cl(p', q'))$ have the quasi-complex forms*

$$\begin{aligned}\mathcal{F}^{f,g}_\pm\{h\} &= \mathcal{F}^{f,g}\{h_\pm\} \qquad (4.3)\\ &= \int_{\mathbb{R}^{p,q}} h_\pm\, e^{-g(v(\boldsymbol{x},\boldsymbol{\omega})\mp u(\boldsymbol{x},\boldsymbol{\omega}))} d^n x = \int_{\mathbb{R}^{p,q}} e^{-f(u(\boldsymbol{x},\boldsymbol{\omega})\mp v(\boldsymbol{x},\boldsymbol{\omega}))} h_\pm\, d^n x\ .\end{aligned}$$

Remark 4.4. *Theorem* 4.3 *establishes in combination with* (4.2) *a general method for how to compute a two-sided CFT in terms of two one-sided CFTs*[4]. *For special relations of two-sided and one-sided quaternionic Fourier transforms see* [12, 14, 15, 21].

Remark 4.5. *The quasi-complex forms in Theorem* 4.3 *allow to establish discretized and fast versions of the general two-sided CFT of Definition* 4.1 *as sums of complex discretized and fast Fourier transformations (FFT), respectively.*

For establishing an inversion formula, and other important CFT properties, certain *assumptions* about the phase functions $u(\boldsymbol{x},\boldsymbol{\omega})$, $v(\boldsymbol{x},\boldsymbol{\omega})$ need to be made. One possibility is, e.g. to arbitrarily partition the scalar product $\boldsymbol{x} * \widetilde{\boldsymbol{\omega}} = \sum_{l=1}^n x_l\omega_l = u(\boldsymbol{x},\boldsymbol{\omega}) + v(\boldsymbol{x},\boldsymbol{\omega})$, with

$$u(\boldsymbol{x},\boldsymbol{\omega}) = \sum_{l=1}^{k} x_l\omega_l, \qquad v(\boldsymbol{x},\boldsymbol{\omega}) = \sum_{l=k+1}^{n} x_l\omega_l, \qquad (4.4)$$

for any arbitrary but fixed $1 \le k \le n$. We could also include any subset $B_u \subseteq \{1,\ldots,n\}$ of coordinates in $u(\boldsymbol{x},\boldsymbol{\omega})$ and the complementary set $B_v = \{1,\ldots,n\}\setminus B_u$ of coordinates in $v(\boldsymbol{x},\boldsymbol{\omega})$, etc. (4.4) will be assumed whenever the inverse CFT (4.5) is applied. We then get the following *inversion* theorem.

Theorem 4.6 (CFT inversion [23]). *With the assumption* (4.4) *for u, v, we get for $h \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$, that*

$$h(\boldsymbol{x}) = \mathcal{F}_{-1}\{\mathcal{F}\{h\}\}(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{p,q}} e^{fu(\boldsymbol{x},\boldsymbol{\omega})}\mathcal{F}^{f,g}\{h\}(\boldsymbol{\omega})\, e^{gv(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{\omega}, \quad (4.5)$$

where $d^n\boldsymbol{\omega} = d\omega_1 \ldots d\omega_n$, $\boldsymbol{x},\boldsymbol{\omega} \in \mathbb{R}^{p,q}$. For the existence of (4.5) *we further need $\mathcal{F}^{f,g}\{h\} \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$.*

We further define for later use the following two mixed (steerable) *exponential-sine* Fourier transforms

$$\mathcal{F}^{f,\pm s}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{p,q}} e^{-fu(\boldsymbol{x},\boldsymbol{\omega})} h(\boldsymbol{x})(\pm 1)\sin(-v(\boldsymbol{x},\boldsymbol{\omega})\,)d^n\boldsymbol{x}, \qquad (4.6)$$

$$\mathcal{F}^{\pm s,g}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{p,q}} (\pm 1)\sin(-u(\boldsymbol{x},\boldsymbol{\omega}))h(\boldsymbol{x})e^{-gv(\boldsymbol{x},\boldsymbol{\omega})}\, d^n\boldsymbol{x}. \qquad (4.7)$$

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[4]For a general study of one-sided CFTs see [20].

With the help of

$$\sin(-u(\boldsymbol{x},\boldsymbol{\omega})) = \tfrac{f}{2}(e^{-fu(\boldsymbol{x},\boldsymbol{\omega})} - e^{ftu(\boldsymbol{x},\boldsymbol{\omega})}),$$
$$\sin(-v(\boldsymbol{x},\boldsymbol{\omega})) = \tfrac{g}{2}(e^{-gv(\boldsymbol{x},\boldsymbol{\omega})} - e^{gv(\boldsymbol{x},\boldsymbol{\omega})}), \tag{4.8}$$

we can rewrite the above mixed exponential-sine Fourier transforms in terms of the CFT of Definition 4.1 as

$$\mathcal{F}^{f,\pm s}\{h\} = \pm\tfrac{1}{2}(\mathcal{F}^{f,g}\{hg\} - \mathcal{F}^{f,-g}\{hg\}), \tag{4.9}$$
$$\mathcal{F}^{\pm s,g}\{h\} = \pm\tfrac{1}{2}(\mathcal{F}^{f,g}\{fh\} - \mathcal{F}^{-f,g}\{fh\}). \tag{4.10}$$

We further note the following useful relationships using the argument reflection of Notation 2.2

$$\mathcal{F}^{-f,g}\{h\} = \mathcal{F}^{f,g}\{h^{(1,0)}\} = \mathcal{F}\{h^{(1,0)}\}, \quad \mathcal{F}^{f,-g}\{h\} = \mathcal{F}\{h^{(0,1)}\}, \tag{4.11}$$

and similarly

$$\mathcal{F}^{f,-s}\{h\} = \mathcal{F}^{f,s}\{h^{(0,1)}\}, \quad \mathcal{F}^{-s,g}\{h\} = \mathcal{F}^{s,g}\{h^{(1,0)}\}. \tag{4.12}$$

The main properties of the CFT of Definition 4.1 have been studied in detail in [23].

5. Convolution and steerable Mustard convolution

We define the *convolution* of two Clifford (algebra) signals $a, b \in L^1(R^{p,q}; Cl(p',q'))$ as

$$(a \star b)(\boldsymbol{x}) = \int_{\mathbb{R}^2} a(\boldsymbol{y})b(\boldsymbol{x}-\boldsymbol{y})d^2\boldsymbol{y}, \tag{5.1}$$

provided that the integral exists.

The *Mustard* convolution [29] of two Clifford signals $a, b \in L^1(R^{p,q}; Cl(p',q'))$ is defined as

$$(a \star_M b)(\boldsymbol{x}) = (\mathcal{F}^{f,g})^{-1}(\mathcal{F}^{f,g}\{a\}\mathcal{F}^{f,g}\{b\}), \tag{5.2}$$

provided that the integral exists.

Remark 5.1. *The Mustard convolution has the conceptual and computational advantage to simply yield as spectrum in the CFT Fourier domain the point wise product of the CFTs of the two signals, just as for the classical complex Fourier transform. On the other hand, by its very definition, the Mustard convolution depends on the choice of the pair f, g, of multivector square roots of -1, used in the Definition* 4.1 *of the CFT. The Mustard convolution* (5.2) *is therefore a steerable operator, depending on the choice*[5] *of the pair f, g.*

We additionally define a further type of (steerable) *exponential-sine* Mustard convolution as

$$(a \star_{Ms} b)(\boldsymbol{x}) = (\mathcal{F}^{f,g})^{-1}(\mathcal{F}^{f,s}\{a\}\mathcal{F}^{s,g}\{b\}), \tag{5.3}$$

provided that the integral exists.

[5] For an example particularly relevant to relativistic physics see [25].

In the following two Subsections we will express the convolution (5.1) in terms of the Mustard convolution (5.2) and vice versa.

5.1. Expressing the convolution in terms of the Mustard convolution

In this Subsection we assume the use of the two-sided CFT with two general multivector square roots of -1, $f, g \in Cl(p', q')$. The definition of the classical convolution (5.1) is independent of the application of a CFT. The Mustard convolutions of (5.2) and (5.3) depend on the definition of the CFT and in particular on the choice of the two multivector square roots of -1, f, g. Therefore it is meaningful in the following to distinguish[6] the expression of the classical convolution in terms of Mustard convolutions for the three cases of general pairs f, g, of commuting f, g (i.e. $[f, g] = 0$), and of anticommuting f, g (i.e. $fg = -gf$), which we consequently state in different theorems and corollaries.

In [7] Theorem 4.1 on page 584 expresses the classical convolution of two quaternion functions with the help of the general left-sided QFT as a sum of 40 Mustard convolutions. Corresponding expressions have been established for the general two-sided QFT in [23], and the space-time Fourier transform in [25]. In our approach we use Theorem 5.12 on page 327 of [23], which expresses the convolution of two Clifford signal functions in the Clifford Fourier domain with the help of the CFT of Definition 4.1. Because of its importance, we restate this theorem here again.

Theorem 5.2 (CFT of convolution [23]**).** *Assuming a general pair of square roots of* -1, f, g, *the general two-sided CFT of the convolution* (5.1) *of two functions* $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q')$ *can then be expressed as*

$$\begin{aligned}\mathcal{F}^{f,g}\{a \star b\} = {} & \mathcal{F}^{f,g}\{a_{+f}\}\mathcal{F}^{f,g}\{b_{+g}\} + \mathcal{F}^{f,-g}\{a_{+f}\}\mathcal{F}^{f,g}\{b_{-g}\} \\ & + \mathcal{F}^{f,g}\{a_{-f}\}\mathcal{F}^{-f,g}\{b_{+g}\} + \mathcal{F}^{f,-g}\{a_{-f}\}\mathcal{F}^{-f,g}\{b_{-g}\} \\ & + \mathcal{F}^{f,s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{+g}\} + \mathcal{F}^{f,-s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{-g}\} \\ & + \mathcal{F}^{f,s}\{a_{-f}\}[f,g]\mathcal{F}^{-s,g}\{b_{+g}\} + \mathcal{F}^{f,-s}\{a_{-f}\}[f,g]\mathcal{F}^{-s,g}\{b_{-g}\}.\end{aligned} \tag{5.4}$$

Note that due to the commutation properties of (4.6) and (4.7) we can place the commutator $[f, g]$ also inside the exponential-sine transform terms as e.g. in

$$\begin{aligned}\mathcal{F}^{f,s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{+g}\} &= \mathcal{F}^{f,s}\{a_{+f}[f,g]\}\mathcal{F}^{s,g}\{b_{+g}\} \\ &= \mathcal{F}^{f,s}\{a_{+f}\}\mathcal{F}^{s,g}\{[f,g]b_{+g}\}.\end{aligned} \tag{5.5}$$

For the special case of a commuting pair of square roots of -1, $[f, g] = 0$, we obtain a much simpler equation.

Corollary 5.3 (CFT of convolution with commuting f, g: $fg = gf$). *Assuming a commuting pair of square roots of* -1, $[f, g] = 0$, *the general two-sided*

[6] This distinction is a direct consequence of the *steerability* of the Mustard convolutions (5.2) and (5.3) inherited from the two-sided CFT of Definition 4.1.

CFT of the convolution (5.1) *of two functions* $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$ *can be expressed as*

$$\mathcal{F}^{f,g}\{a \star b\} = \mathcal{F}^{f,g}\{a_{+f}\}\mathcal{F}^{f,g}\{b_{+g}\} + \mathcal{F}^{f,-g}\{a_{+f}\}\mathcal{F}^{f,g}\{b_{-g}\} + \mathcal{F}^{f,g}\{a_{-f}\}\mathcal{F}^{-f,g}\{b_{+g}\} + \mathcal{F}^{f,-g}\{a_{-f}\}\mathcal{F}^{-f,g}\{b_{-g}\}. \quad (5.6)$$

We can now easily express the convolution of two quaternion signals $\mathcal{F}^{f,g}\{a \star b\}(\boldsymbol{\omega})$ in terms of only eight Mustard convolutions (5.2) and (5.3).

Theorem 5.4 (Convolution in terms of two types of Mustard convolution). *Assuming a general pair of square roots of* -1, f, g, *the convolution* (5.1) *of two Clifford functions* $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$ *can be expressed in terms of four Mustard convolutions* (5.2) *and four exponential-sine Mustard convolutions* (5.3) *as*

$$\begin{aligned} a \star b = {} & a_{+f} \star_M b_{+g} + a_{+f}^{(0,1)} \star_M b_{-g} + a_{-f} \star_M b_{+g}^{(1,0)} + a_{-f}^{(0,1)} \star_M b_{-g}^{(1,0)} \\ & + a_{+f} \star_{Ms} [f,g]b_{+g} + a_{+f}^{(0,1)} \star_{Ms} [f,g]b_{-g} \\ & + a_{-f} \star_{Ms} [f,g]b_{+g}^{(1,0)} + a_{-f}^{(0,1)} \star_{Ms} [f,g]b_{-g}^{(1,0)}. \end{aligned} \quad (5.7)$$

Remark 5.5. *We use the convention, that terms such as* $a_{+f} \star_{Ms} [f,g]b_{+g}$, *should be understood with brackets* $a_{+f} \star_{Ms} ([f,g]b_{+g})$, *which are omitted to avoid clutter.*

Assuming commutation, $[f,g] = 0$, the standard Mustard convolution is sufficient to express the classical convolution.

Corollary 5.6 (Convolution in terms of Mustard convolution with commuting f, g). *Assuming a commuting pair of square roots of* -1, $[f,g] = 0$, *the convolution* (5.1) *of two Clifford functions* $a, b \in L^1(R^{p,q}; Cl(p', q'))$ *can be expressed in terms of four Mustard convolutions* (5.2) *as*

$$a \star b = a_{+f} \star_M b_{+f} + a_{+f}^{(0,1)} \star_M b_{-f} + a_{-f} \star_M b_{+f}^{(1,0)} + a_{-f}^{(0,1)} \star_M b_{-f}^{(1,0)} \quad (5.8)$$

Furthermore, applying (4.9) and (4.10), we can expand the terms in (5.4) with exponential-sine transforms into sums of products of CFTs. For example, the first term gives

$$\begin{aligned} & \mathcal{F}^{f,s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{+g}\} \\ & = \frac{1}{4}\left(\mathcal{F}^{f,g}\{a_{+f}g\} - \mathcal{F}^{f,-g}\{a_{+f}g\}\right)\left(\mathcal{F}^{f,g}\{f[f,g]b_{+g}\} - \mathcal{F}^{-f,g}\{f[f,g]b_{+g}\}\right) \\ & = \frac{1}{4}\Big(\mathcal{F}\{a_{+f}g\}\mathcal{F}\{f[f,g]b_{+g}\} - \mathcal{F}\{a_{+f}g\}\mathcal{F}\{f[f,g]b_{+g}^{(1,0)}\} \\ & \qquad - \mathcal{F}\{a_{+f}^{(0,1)}g\}\mathcal{F}\{f[f,g]b_{+g}\} + \mathcal{F}\{a_{+f}^{(0,1)}g\}\mathcal{F}\{f[f,g]b_{+g}^{(1,0)}\}\Big). \end{aligned} \quad (5.9)$$

For the important special case of anticommuting square roots of -1, $fg = -gf$, e.g. in quaternion algebra $f = \boldsymbol{i}$, $g = \boldsymbol{k}$ [8, 12, 14], or in the generalization to space-time $f = e_4 = e_t$, $g = e_1e_2e_3 = i_3$ in $Cl(3,1)$ [12, 14], equation (5.9) can be further simplified. Assuming $fg = -gf$, we have

$$[f,g] = 2fg \qquad f[f,g] = 2ffg = -2g, \quad (5.10)$$

and we can simplify (5.9) to

$$\begin{aligned}
&\mathcal{F}^{f,s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{+g}\} = 2\mathcal{F}^{f,s}\{a_{+f}\}fg\mathcal{F}^{s,g}\{b_{+g}\} \\
&= \frac{2}{4}\Big(\mathcal{F}\{a_{+f}g\}\mathcal{F}\{(-g)b_{+g}\} - \mathcal{F}\{a_{+f}g\}\mathcal{F}\{(-g)b_{+g}^{(1,0)}\} \\
&\qquad -\mathcal{F}\{a_{+f}^{(0,1)}g\}\mathcal{F}\{(-g)b_{+g}\} + \mathcal{F}\{a_{+f}^{(0,1)}g\}\mathcal{F}\{(-g)b_{+g}^{(1,0)}\}\Big) \\
&= \frac{1}{2}\Big(\mathcal{F}\{a_{+f}\}\mathcal{F}\{b_{+g}^{(1,0)}\} - \mathcal{F}\{a_{+f}\}\mathcal{F}\{b_{+g}\} \\
&\qquad -\mathcal{F}\{a_{+f}^{(0,1)}\}\mathcal{F}\{b_{+g}^{(1,0)}\} + \mathcal{F}\{a_{+f}^{(0,1)}\}\mathcal{F}\{b_{+g}\}\Big),
\end{aligned} \tag{5.11}$$

because

$$\mathcal{F}\{a_{+f}g\}\mathcal{F}\{(-g)b_{+g}\} = \mathcal{F}\{a_{+f}\}g(-g)\mathcal{F}\{b_{+g}^{(1,0)}\} = \mathcal{F}\{a_{+f}\}\mathcal{F}\{b_{+g}^{(1,0)}\}, \quad \text{etc.} \tag{5.12}$$

where we applied for the first equality that for $fg = -gf$,

$$e^{\alpha f}g = ge^{-\alpha f}. \tag{5.13}$$

In general equation (5.9) allows us in turn to express the quaternion signal convolution purely in terms of standard Mustard convolutions.

Theorem 5.7 (Convolution in terms of Mustard convolution). *Assuming a general pair of multivector square roots of* -1, f, g, *the convolution* (5.1) *of two quaternion functions* $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p',q'))$ *can be expressed in terms of twenty standard Mustard convolutions* (5.2) *as*

$$\begin{aligned}
a\star b &= a_{+f}\star_M b_{+g} + a_{+f}^{(0,1)}\star_M b_{-g} + a_{-f}\star_M b_{+g}^{(1,0)} + a_{-f}^{(0,1)}\star_M b_{-g}^{(1,0)} \\
&+\tfrac{1}{4}(\, a_{+f}g\star_M fcb_{+g} - a_{+f}g\star_M fcb_{+g}^{(1,0)} - a_{+f}^{(0,1)}g\star_M fcb_{+g} + a_{+f}^{(0,1)}g\star_M fcb_{+g}^{(1,0)} \\
&+ a_{+f}^{(0,1)}g\star_M fcb_{-g} - a_{+f}^{(0,1)}g\star_M fcb_{-g}^{(1,0)} - a_{+f}g\star_M fcb_{-g} + a_{+f}g\star_M fcb_{-g}^{(1,0)} \\
&+ a_{-f}g\star_M fcb_{+g}^{(1,0)} - a_{-f}g\star_M fcb_{+g} - a_{-f}^{(0,1)}g\star_M fcb_{+g}^{(1,0)} + a_{-f}^{(0,1)}g\star_M fcb_{+g} \\
&+ a_{-f}^{(0,1)}g\star_M fcb_{-g}^{(1,0)} - a_{-f}^{(0,1)}g\star_M fcb_{-g} - a_{-f}g\star_M fcb_{-g}^{(1,0)} + a_{-f}g\star_M fcb_{-g}\,),
\end{aligned} \tag{5.14}$$

with the abbreviation $c = [f,g]$.

Assuming anticommutation, $fg = -gf$, we can eliminate in Theorem 5.7 the commutators $c = [f,g]$ with the help of (5.11), which after cancellations leaves only sixteen terms.

$$\begin{aligned}
&a\star b = \\
&\tfrac{1}{2}(\, a_{+f}\star_M b_{+g}^{(1,0)} + a_{+f}\star_M b_{+g} - a_{+f}^{(0,1)}\star_M b_{+g}^{(1,0)} + a_{+f}^{(0,1)}\star_M b_{+g} \\
&\quad + a_{+f}^{(0,1)}\star_M b_{-g}^{(1,0)} + a_{+f}^{(0,1)}\star_M b_{-g} - a_{+f}\star_M b_{-g}^{(1,0)} + a_{+f}\star_M b_{-g} \\
&\quad + a_{-f}\star_M b_{+g} + a_{-f}\star_M b_{+g}^{(1,0)} - a_{-f}^{(0,1)}\star_M b_{+g} + a_{-f}^{(0,1)}\star_M b_{+g}^{(1,0)} \\
&\quad + a_{-f}^{(0,1)}\star_M b_{-g} + a_{-f}^{(0,1)}\star_M b_{-g}^{(1,0)} - a_{-f}\star_M b_{-g} + a_{-f}\star_M b_{-g}^{(1,0)}\,).
\end{aligned} \tag{5.15}$$

Furthermore, we can combine by Definition 3.2 four pairs of $\pm$ split terms, e.g.,

$$a_{+f} \star_M b_{+g} + a_{+f} \star_M b_{-g} = a_{+f} \star_M b, \text{ etc.} \tag{5.16}$$

Assuming $fg = -gf$, this leaves only twelve terms for expressing a classical convolution in terms of a Mustard convolution,

$$\begin{aligned} a \star b = \tfrac{1}{2}\big(& a_{+f} \star_M b_{+g}^{(1,0)} + a_{+f} \star_M b - a_{+f}^{(0,1)} \star_M b_{+g}^{(1,0)} + a_{+f}^{(0,1)} \star_M b \\ & + a_{+f}^{(0,1)} \star_M b_{-g}^{(1,0)} - a_{+f} \star_M b_{-g}^{(1,0)} \\ & + a_{-f} \star_M b_{+g} + a_{-f} \star_M b^{(1,0)} - a_{-f}^{(0,1)} \star_M b_{+g} + a_{-f}^{(0,1)} \star_M b^{(1,0)} \\ & + a_{-f}^{(0,1)} \star_M b_{-g} - a_{-f} \star_M b_{-g} \big). \end{aligned} \tag{5.17}$$

Moreover, we can combine with the help of the involution $f()g$ of (3.3) four pairs of terms like

$$\begin{aligned} a_{+f} \star_M b_{+g}^{(1,0)} - a_{+f} \star_M b_{-g}^{(1,0)} &= a_{+f} \star_M [b_{+g}^{(1,0)} - b_{-g}^{(1,0)}] \\ &= a_{+f} \star_M (f[b_{+g}^{(1,0)} + b_{-g}^{(1,0)}]g) = a_{+f} \star_M fb^{(1,0)}g, \end{aligned} \tag{5.18}$$

where in the final result we omit the round brackets, i.e. we understand $a_{+f} \star_M fb^{(1,0)}g = a_{+f} \star_M (fb^{(1,0)}g)$. This in turn leaves only eight terms for expressing a classical convolution in terms of a Mustard convolution, assuming $fg = -gf$.

$$\begin{aligned} a \star b = \tfrac{1}{2}\big(& a_{+f} \star_M fb^{(1,0)}g + a_{+f} \star_M b - a_{+f}^{(0,1)} \star_M fb^{(1,0)}g + a_{+f}^{(0,1)} \star_M b \\ & + a_{-f} \star_M fbg + a_{-f} \star_M b^{(1,0)} - a_{-f}^{(0,1)} \star_M fbg + a_{-f}^{(0,1)} \star_M b^{(1,0)} \big). \end{aligned} \tag{5.19}$$

Finally, we note, that (5.19) contains pairs of functions $a_{\pm f}$ with unreflected and reflected second argument. Adding these pairs leads to even $\oplus$ or odd $\ominus$ symmetry in the second argument. That is we combine

$$a_{+f}^{\oplus} = \tfrac{1}{2}(a_{+f} + a_{+f}^{(0,1)}), \qquad a_{+f}^{\ominus} = \tfrac{1}{2}(a_{+f} - a_{+f}^{(0,1)}), \qquad \text{etc.} \tag{5.20}$$

This allows us for $fg = -gf$, to write the classical convolution in terms of just four Mustard convolutions.

Theorem 5.8 (Convolution in terms of Mustard convolution with anticommuting f, g). *Assuming an anticommuting pair f, g, of multivector square roots of -1, with $fg = -gf$, the convolution* (5.1) *of two Clifford functions $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$ can be expressed in terms of four standard Mustard convolutions* (5.2) *as*

$$a \star b = \tfrac{1}{2}\big(a_{+f}^{\ominus} \star_M fb^{(1,0)}g + a_{+f}^{\oplus} \star_M b + a_{-f}^{\ominus} \star_M fbg + a_{-f}^{\oplus} \star_M b^{(1,0)} \big). \tag{5.21}$$

5.2. Expressing the Mustard convolution in terms of the convolution

Now we will simply write out the Mustard convolution (5.2) and simplify it until only standard convolutions (5.1) remain. In this Subsection we will use the general $\pm$ split of Definition 3.2. Our result should be compared, e.g, in the special case of the left-sided QFT with the Theorem 2.5 on page 584 of [7] with 32 classical convolutions for expressing the Mustard convolution of quaternion functions. Similar results to ours can be found in [23] for the two-sided QFT, and in [25] for the space-time Fourier transform.

We begin by writing the Mustard convolution (5.2) of two quaternion functions $a, b \in L^2(\mathbb{R}^{p,q}; Cl(p', q'))$

$$\begin{aligned}
a \star_M b(\boldsymbol{x}) &= \tfrac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} e^{fu(\boldsymbol{x},\boldsymbol{\omega})} \mathcal{F}\{a\}(\boldsymbol{\omega}) \mathcal{F}\{b\}(\boldsymbol{\omega}) e^{gv(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{\omega} \\
&= \tfrac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} e^{-fu(\boldsymbol{x},\boldsymbol{\omega})} \int_{\mathbb{R}^2} e^{-fu(\boldsymbol{y},\boldsymbol{\omega})} a(\boldsymbol{y}) e^{-gv(\boldsymbol{y},\boldsymbol{\omega})} d^n\boldsymbol{y} \\
&\qquad \int_{\mathbb{R}^{p,q}} e^{-fu(\boldsymbol{z},\boldsymbol{\omega})} b(\boldsymbol{z}) e^{-gv(\boldsymbol{z},\boldsymbol{\omega})} d^2\boldsymbol{z} e^{gv(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{\omega} \\
&= \tfrac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} e^{fu(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{\omega})} (a_+(\boldsymbol{y}) + a_-(\boldsymbol{y})) e^{-gv(\boldsymbol{y},\boldsymbol{\omega})} \\
&\qquad e^{-fu(\boldsymbol{z},\boldsymbol{\omega})} (b_+(\boldsymbol{z}) + b_-(\boldsymbol{z})) e^{gv(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}.
\end{aligned} \tag{5.22}$$

Next, we use the identities (3.5) in order to shift the inner factor $e^{-gv(\boldsymbol{y},\boldsymbol{\omega})}$ to the left and $e^{-fu(\boldsymbol{z},\boldsymbol{\omega})}$ to the right, respectively. We abbreviate $\int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}}$ to $\iiint$.

$$\begin{aligned}
&a \star_M b(\boldsymbol{x}) = \\
&= \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{\omega})} e^{fv(\boldsymbol{y},\boldsymbol{\omega})} a_+(\boldsymbol{y}) b_+(\boldsymbol{z}) e^{gu(\boldsymbol{z},\boldsymbol{\omega})} e^{gv(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega} \\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{\omega})} e^{fv(\boldsymbol{y},\boldsymbol{\omega})} a_+(\boldsymbol{y}) b_-(\boldsymbol{z}) e^{-gu(\boldsymbol{z},\boldsymbol{\omega})} e^{gv(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega} \\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{\omega})} e^{-fv(\boldsymbol{y},\boldsymbol{\omega})} a_-(\boldsymbol{y}) b_+(\boldsymbol{z}) e^{gu(\boldsymbol{z},\boldsymbol{\omega})} e^{gv(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega} \\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{\omega})} e^{-fv(\boldsymbol{y},\boldsymbol{\omega})} a_-(\boldsymbol{y}) b_-(\boldsymbol{z}) e^{-gu(\boldsymbol{z},\boldsymbol{\omega})} e^{gv(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}
\end{aligned} \tag{5.23}$$

Furthermore, we abbreviate the inner function products as $ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z}) := a_\pm(\boldsymbol{y}) b_\pm(\boldsymbol{z})$, and apply the general $\pm$ split of Definition 3.2 once again to obtain $ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z}) = [ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z})]_+ + [ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z})]_- = ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z})_+ + ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z})_-$. We omit the square brackets and use the convention that the final $\pm$ split indicated by the final $\pm$ index should be performed last. This allows to apply (3.5) again in order to shift the factors $e^{\pm gu(\boldsymbol{z},\boldsymbol{\omega})} e^{gv(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})}$ to the left. We end up with the following eight terms

$$
\begin{aligned}
&a \star_M b(\boldsymbol{x}) = \qquad (5.24)\\
&= \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z},\boldsymbol{\omega})} e^{fv(\boldsymbol{y}-(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} ab_{++}(\boldsymbol{y},\boldsymbol{z})_+ d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z},\boldsymbol{\omega})} e^{fv(\boldsymbol{y}+(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} ab_{++}(\boldsymbol{y},\boldsymbol{z})_- d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z},\boldsymbol{\omega})} e^{fv(\boldsymbol{y}-(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} ab_{+-}(\boldsymbol{y},\boldsymbol{z})_+ d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z},\boldsymbol{\omega})} e^{fv(\boldsymbol{y}+(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} ab_{+-}(\boldsymbol{y},\boldsymbol{z})_- d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z},\boldsymbol{\omega})} e^{fv(-\boldsymbol{y}-(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} ab_{-+}(\boldsymbol{y},\boldsymbol{z})_+ d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z},\boldsymbol{\omega})} e^{fv(-\boldsymbol{y}+(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} ab_{-+}(\boldsymbol{y},\boldsymbol{z})_- d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z},\boldsymbol{\omega})} e^{fv(-\boldsymbol{y}-(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} ab_{--}(\boldsymbol{y},\boldsymbol{z})_+ d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}\\
&+ \tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z},\boldsymbol{\omega})} e^{fv(-\boldsymbol{y}+(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} ab_{--}(\boldsymbol{y},\boldsymbol{z})_- d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}.
\end{aligned}
$$

We now only show explicitly how to simplify the second triple integral, the others follow the same pattern.

$$
\begin{aligned}
&\tfrac{1}{(2\pi)^n} \iiint e^{fu(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z},\boldsymbol{\omega})} e^{fv(\boldsymbol{y}+(\boldsymbol{x}-\boldsymbol{z}),\boldsymbol{\omega})} [a_+(\boldsymbol{y})b_+(\boldsymbol{z})]_- d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega}\\
&= \iint \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} e^{\sum_{l=1}^{k}(x_l-y_l+z_l)\omega_l} e^{\sum_{m=k+1}^{n}(y_m+(x_m-z_m))\omega_m} [a_+(\boldsymbol{y})b_+(\boldsymbol{z})]_- d^n\boldsymbol{\omega} d^n\boldsymbol{y} d^n\boldsymbol{z}\\
&= \iint \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \prod_{l=1}^{k} e^{f(x_l-y_l+z_l)\omega_l} \prod_{m=k+1}^{n} e^{f(y_m+(x_m-z_m))\omega_m} [a_+(\boldsymbol{y})b_+(\boldsymbol{z})]_- d^n\boldsymbol{\omega} d^n\boldsymbol{y} d^n\boldsymbol{z}\\
&= \iint \prod_{l=1}^{k} \delta(x_l-y_l+z_l) \prod_{m=k+1}^{n} \delta(y_m+(x_m-z_m)) [a_+(\boldsymbol{y})b_+(\boldsymbol{z}_k,\boldsymbol{z}_{(n-k)})]_- d^n\boldsymbol{y} d^n\boldsymbol{z}\\
&= \int_{\mathbb{R}^{p,q}} [a_+(\boldsymbol{y})b_+(-(\boldsymbol{x}_k-\boldsymbol{y}_k),\boldsymbol{x}_{(n-k)}+\boldsymbol{y}_{(n-k)})]_- d^n\boldsymbol{y}\\
&= \int_{\mathbb{R}^{p,q}} [a_+(\boldsymbol{y})b_+(-(\boldsymbol{x}_k-\boldsymbol{y}_k),-(-\boldsymbol{x}_{(n-k)}-\boldsymbol{y}_{(n-k)}))]_- d^n\boldsymbol{y}\\
&= \int_{\mathbb{R}^{p,q}} [a_+(\boldsymbol{y})b_+^{(1,1)}(\boldsymbol{x}_k-\boldsymbol{y}_k,-\boldsymbol{x}_{(n-k)}-\boldsymbol{y}_{(n-k)})]_- d^n\boldsymbol{y}\\
&= [a_+ \star b_+^{(1,1)}(\boldsymbol{x}_k,-\boldsymbol{x}_{(n-k)})]_-. \qquad (5.25)
\end{aligned}
$$

Note that $a_+ \star b_+^{(1,1)}(\boldsymbol{x}_k,-\boldsymbol{x}_{(n-k)})$ means to first apply the convolution to the pair of functions a_+ and $b_+^{(1,1)}$, and only then to evaluate them with the argument $(-\boldsymbol{x}_k,\boldsymbol{x}_{(n-k)})$. So in general $a_+ \star b_+^{(1,1)}(\boldsymbol{x}_k,-\boldsymbol{x}_{(n-k)}) \neq a_+ \star b_+(-\boldsymbol{x}_k,\boldsymbol{x}_{(n-k)})$. Simplifying the other seven triple integrals similarly we finally obtain the desired decomposition of the Mustard convolution (5.2) in terms of the classical convolution.

Theorem 5.9 (Mustard convolution in terms of standard convolution). *The Mustard convolution* (5.2) *of two quaternion functions* $a, b \in L^1(R^{p,q}; Cl(p',q'))$ *can be expressed in terms of eight standard convolutions* (5.1) *as*

$$\begin{aligned} a \star_M b(\boldsymbol{x}) &= \\ &= [a_+ \star b_+(\boldsymbol{x})]_+ + [a_+ \star b_+^{(1,1)}(x_1, -x_2)]_- \\ &\quad + [a_+ \star b_-^{(1,0)}(\boldsymbol{x})]_+ + [a_+ \star b_-^{(0,1)}(x_1, -x_2)]_- \\ &\quad + [a_- \star b_+^{(0,1)}(x_1, -x_2)]_+ + [a_- \star b_+^{(1,0)}(\boldsymbol{x})]_- \\ &\quad + [a_- \star b_-^{(1,1)}(x_1, -x_2)]_+ + [a_- \star b_-(\boldsymbol{x})]_-. \end{aligned} \tag{5.26}$$

Remark 5.10. *If we would explicitly insert according Definition* 3.2 $a_\pm = \frac{1}{2}(a \pm fag)$ *and* $b_\pm = \frac{1}{2}(b \pm fbg)$*, and similarly explicitly insert the second level* $\pm$ *split* $[\ldots]_\pm$*, we would obtain up to a maximum of 64 terms. It is therefore obvious how significant and efficient the use of the general* $\pm$ *split is in this context.*

6. Conclusion

In this paper we have briefly reviewed non-degenerate Clifford algebras, their manifolds of multivector square roots of -1, Clifford algebra decomposition with respect to a pair of square roots of -1, the general steerable two-sided Clifford Fourier transform, and introduced a pair of related steerable mixed exponential-sine Clifford Fourier transforms. We defined the notions of (classical non-steerable) convolution of two Clifford algebra valued functions over $\mathbb{R}^{p,q}$, the steerable Mustard convolution (with its CFT as the point wise product of the CFTs of the factor functions), and a special steerable Mustard convolution involving the point wise products of mixed exponential-sine CFTs.

The main results are: An efficient decomposition of the classical convolution of Clifford algebra signals in terms of eight Mustard type convolutions. For the special cases of two commuting (or anticommuting) multivector square roots of -1 axis in the CFT, only four terms of the standard Mustard convolution prove to be sufficient. Even in the case of two general multivector square roots of -1 axis in the CFT, the classical convolution of two Clifford algebra signals can always be fully expanded in terms of standard Mustard convolutions. Finally we showed how to fully generally expand the Mustard convolution of two Clifford algebra signals in terms of eight classical convolutions.

In view of the many potential applications of the CFT [4], including already its lower-dimensional realizations as QFT [24, introduction], and spacetime FT [25, introduction], we expect our new results to be of great interest in physics, pure and applied mathematics, and engineering, e.g., for filter design and feature extraction in multi-dimensional signal and (color) image processing. Finally, the CFT and all convolutions described above can be implemented for simulations and real data applications in the recently released Clifford Multivector Toolbox (for MATLAB) [31].

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