0-Branes of Lattice Gauge Theory: Explicit Monopole Dominance

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Abstract

The site reduction of U(1) lattice gauge theory is used to model the dynamics of magnetic monopoles. The reduced lattice theory is the 1D plane-rotator model of the angle-valued coordinates on the discrete world-line. The energy spectrum is obtained exactly, with a minimum in the ground-state at coupling $g_c = 1.125$. For $g < g_c$ and $T < T_c = 0.247/a$ the model exhibits two phases of real and imaginary velocities, like particles facing a potential barrier. In the gauge theory side the real velocity phase corresponds to magnetic energy exceeding the electric energy, indicating the dominance of monopole density. For $g > g_c$ or $T > T_c$ the monopoles always dominate.

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According to string theory the gauge fields and coordinates are interchanged upon the action of T-duality [1]. In particular, upon the compactifications some directions of space, the gauge fields arising from open strings would emerge as the transverse coordinates of Dp-branes in the dual compactified space, leading to the correspondence [1, 2]

$$A_i \longleftrightarrow X_i/l_s^2 \tag{1}$$

in which l_s is the string theory length. Dp-branes are proposed to represent the solutions of the effective field theory, possessing charge and mass proportional to the inverse string coupling λ_s, similar to gauge theory magnetic monopoles with $m \propto g_{YM}^{-2} = \lambda_s^{-1}$. The dynamics of the coordinates X_i's is captured by a theory resulted from dimensional reduction of the ordinary gauge theory for A_i's (now X_i's) to the world-volume of D-branes [1, 3]. In the case of D0-branes, all spatial components of the gauge field would appear as the time dependent space coordinates of D0-branes [1]. In the case of N Dp-branes, the transverse coordinates would appear as N dimensional hermitian matrices [3].

It is reasonable to look for the application of the correspondence between gauge fields and coordinates at strong coupling regime. In this way the lattice gauge theories are the natural candidates, as they have shown their capacity to capture the essential features expected at strong coupling limit [4]. Interestingly, in the lattice formulation of gauge theories the gauge fields appear to be periodic variables [4], as the same is expected generally for the coordinates of Dp-branes [1]. Accordingly, here the aim is to adapt the above correspondence for the lattice gauge theories and to look for possible implications. The pure gauge sector of compact U(1) theory on 4D Euclidean lattice is given by [4]:

$$S_{\text{gauge}} = \frac{1}{2g^2} \sum_{\vec{n}} \sum_{\mu\nu} \left(e^{\mathrm{i} f_{\vec{n},\mu\nu}} - 1 \right) \tag{2}$$

in which the basic object for each lattice plaquette of size a is defined by

$$e^{\mathrm{i} f_{\vec{n},\mu\nu}} := e^{\mathrm{i}\, a A_{\vec{n},\mu}} e^{\mathrm{i}\, a A_{\vec{n}+\hat{\mu},\nu}} e^{-\mathrm{i}\, a A_{\vec{n}+\hat{\nu},\mu}} e^{-\mathrm{i}\, a A_{\vec{n},\nu}}. \tag{3}$$

with $A_{\vec{n},\mu}$ as the gauge field in lattice site $\vec{n}$ in direction μ, and $\hat{\mu}$ as the unit-vector along direction μ. In the continuum limit $aA \ll 1$, defining $F_{\vec{n},\mu\nu} := f_{\vec{n},\mu\nu}/a^2$, the gauge sector reduces to [4]

$$S_{\text{gauge}} \simeq -\frac{1}{4g^2} a^4 \sum_{\vec{n}} F^2_{\vec{n},\mu\nu} \to -\frac{1}{4g^2} \int d^4x\, F^2_{\mu\nu}. \tag{4}$$

In the process of dimensional reduction the dependence on spatial directions is removed, and the spatial components of the gauge fields are interpreted as coordinates X_i's. Based on (1) we assume the following between dimensionless quantities:

$$a\, A^i \to x^i/R \tag{5}$$

leading to

$$f_{n,0i} = (x^i_{n+1} - x^i_n)/R, \qquad \exp(\mathrm{i}\, f_{n,ij}) = 1 \tag{6}$$

In above n represents the dependence on the discrete time, the only remaining coordinate of the original space-time lattice. By these, the action takes the form

$$S_0 = \frac{1}{g^2} \sum_{n,i} \left(\cos \frac{x^i_{n+1} - x^i_n}{R} - 1 \right) \tag{7}$$

which is the sum of three copies of the Hamiltonian of the 1D plane-rotator model of magnetic systems. In fact the close relation between lattice gauge theories and spin systems was recognized from the first appearance of lattice gauge theories [4,5], and has been used widely for better understanding the gauge theory side. In particular, the so-called Villain model [6], as an approximation to the plane-rotator model, was used for gauge theory purposes [7–9]. Here the model is interpreted as a discrete world-line endowed by the compact coordinates x^i's

$$-\pi R \leq x^i \leq \pi R \tag{8}$$

In the first place let us check the continuum limit defined by:

$$\begin{aligned} aA^i &= x^i/R \ll 1 \\ x_{n+1} - x_n &\to a\, \dot{x} \\ \sum_n &\to a^{-1} \int dt \end{aligned} \tag{9}$$

leading to

$$S_0 \simeq \frac{a}{2g^2R^2} \int dt\ \dot{x}_i^2 \tag{10}$$

This action describes the dynamics of a free particle with mass $m_0 = a/(g^2R^2)$. It is mentioned, as far as the dependence on coupling constant is concerned, this mass corresponds to that of a magnetic monopole. We will see shortly that the emergent dynamics by action (7) supports the correspondence with monopoles as well. Following [4] it is useful to define the new variables $y^i = x^i/R$ taking values in $[-\pi, \pi]$. By these, the action (7) takes the form

$$S_0 = \frac{1}{g^2} \sum_{n,i} \left(\cos(y^i_{n+1} - y^i_n) - 1\right) \tag{11}$$

As the action is separable for each direction, it is sufficient to consider only one copy in the following amplitude, setting $\kappa = g^{-2}$

$$\langle y_0, 0 | y_M, Ma\rangle = \mathcal{N} \prod_{m=1}^{M-1} \int_{-\pi}^{\pi} \frac{dy_m}{2\pi} \; \exp\left[\kappa \sum_{n=1}^{M-1} (\cos(y_{n+1} - y_n) - 1)\right] \tag{12}$$

which is in fact the transition amplitude between $|y_0\rangle$ at time 0 and $|y_M\rangle$ at time $\tau = Ma$. The normalization factor $\mathcal{N}$ will be fixed to match the above to the continuum limit; a free particle in uncompactified space. Using the identity for the modified Bessel function of the first kind [10]:

$$\exp[\kappa \cos(y' - y)] = \sum_{s=-\infty}^{\infty} I_p(\kappa) e^{\mathrm{i}\, s(y'-y)} \tag{13}$$

we have

$$\langle y_0, 0 | y_M, Ma\rangle = \mathcal{N} \sum_{s=-\infty}^{\infty} \left(e^{-\kappa} I_s(\kappa)\right)^{M-1} e^{\mathrm{i}\, s(y_0 - y_M)} \tag{14}$$

In the zero coupling limit $g^{-2} = \kappa \to \infty$ by the saddle point approximation we have

$$I_p(\kappa) = \lim_{\kappa\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} dy \exp(\kappa \cos y + \mathrm{i}\, sy) \simeq \frac{e^{\kappa}}{\sqrt{2\pi\kappa}} \exp\left(-\frac{s^2}{2\kappa}\right) \tag{15}$$

by which we find for (14)

$$\langle y_0, 0 | y_M, Ma\rangle \simeq \mathcal{N} \frac{1}{(2\pi\kappa)^{(M-1)/2}} \sum_{s=-\infty}^{\infty} \exp\left(-\frac{Ms^2}{2\,\kappa} + \mathrm{i}\, s(y_M - y_0)\right) \tag{16}$$

The sum in above can be approximated by the integral, leading to

$$\langle y_0, 0 | y_M, Ma\rangle \simeq \mathcal{N} \frac{1}{(2\pi\kappa)^{(M-1)/2}} \left(\frac{2\pi\kappa}{M}\right)^{1/2} \exp\left(-\frac{\kappa}{2M}(y_M - y_0)^2\right) \tag{17}$$

by which, using $y = x/R$, $Ma = \tau$ and $m_0 = a/(g^2R^2) = \kappa a/R^2$, we have

$$\langle x_0, 0|x_\tau, \tau\rangle \simeq \mathcal{N}\frac{1}{(2\pi\kappa)^{(M-1)/2}}\left(\frac{m_0}{2\pi\tau}\right)^{1/2}\exp\left(-\frac{m_0(x_\tau - x_0)^2}{2\,\tau}\right) \tag{18}$$

The above, provided that $\mathcal{N} = (2\pi\kappa)^{(M-1)/2}$, is in fact the propagator of a free particle in imaginary time formalism. By fixing the factor $\mathcal{N}$, one has for (14)

$$\langle y_0, 0|y_M, Ma\rangle = \sum_{p=-\infty}^{\infty}\Big((2\pi\kappa)^{1/2}e^{-\kappa}I_p(\kappa)\Big)^{M-1}e^{\mathrm{i}\,p(y_0-y_M)} \tag{19}$$

By comparing the above with ($\tau = Ma$)

$$\langle y_0, 0|y_M, \tau\rangle = \sum_{s=-\infty}^{\infty}\exp(-E_s\tau)\,\psi_s(y_0)\psi_s^*(y_M) \tag{20}$$

we read plane-wave $\psi_s(x) \propto \exp(\mathrm{i}\,s\,x/R)$ as energy eigenfunction with eigenvalue

$$E_s(\kappa) = -\frac{1}{a}\ln\left[(2\pi\kappa)^{1/2}e^{-\kappa}I_s(\kappa)\right] \tag{21}$$

By $I_s(z) = I_{-s}(z)$ we see the spectrum is doubly degenerate for $s \neq 0$. Using (15) one easily checks that at zero coupling limit $\kappa = g^{-2} \to \infty$

$$E_s \simeq \frac{s^2}{2a\kappa} \tag{22}$$

which is the energy of a free particle $E = p^2/(2m_0)$ with $m_0 = \kappa a/R^2$ and momentum $p = s/R$ along the compact direction. In the intermediate coupling the spectrum is discrete. In the strong coupling limit $\kappa = g^{-2} \to 0$, using

$$I_s(z) \simeq \frac{1}{s!}\left(\frac{z}{2}\right)^s, \qquad z \ll 1 \tag{23}$$

we have

$$E_s = (s + \frac{1}{2})\,\frac{\ln g^2}{a} + O(s\ln s) + O(g^{-2}) \tag{24}$$

in which the 2nd term is independent of the coupling constant and is relevant only for $s \gtrsim \ln g^2 \gg 1$. Also at strong coupling

$$E_{s+1} - E_s \simeq \frac{\ln g^2}{a} \gg \frac{1}{a} \tag{25}$$

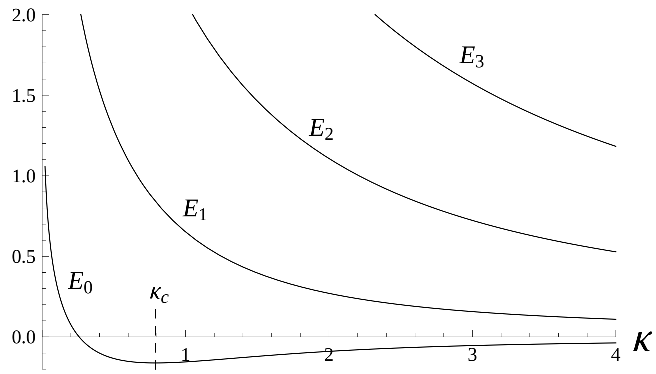

Figure 1: The few lowest energies by (21) versus κ (energy unit: a^{-1}).

Most interestingly about the spectrum (21) is the behavior of the energy of ground-state for different values of coupling constant g. In fact the lowest energy $E_0(\kappa)$ has a minimum at the critical value $\kappa_c = 0.790$ (see Fig. 1). The existence of this minimum implies that at sufficiently low temperature the mean-value of the conjugate variable would encounter a change in its sign for $\kappa \approx \kappa_c$. At temperatures $T \ll E_1(\kappa_c) - E_0(\kappa_c)$, in which even the first excited state is not accessible, the one-particle partition function takes the form

$$Z_1(\kappa) \simeq \exp(-E_0(\kappa)/T) \tag{26}$$

for which by $\kappa \approx \kappa$ we have $(E''(\kappa_c) > 0)$

$$E_0(\kappa) \simeq E_0(\kappa_c) + \frac{1}{2} E_0''(\kappa_c)\,(\kappa - \kappa_c)^2 \tag{27}$$

The mean-value of the conjugate variable of κ is defined by (T unit: a^{-1}):

$$\left\langle 1 - \cos\frac{a\dot{x}}{R} \right\rangle_{\kappa,T} = T\,\frac{\partial \ln Z_1(\kappa)}{\partial\,\kappa} \tag{28}$$

for which at low temperatures we find

$$\left\langle 1 - \cos\frac{a\dot{x}}{R} \right\rangle_{\substack{\kappa\approx\kappa_c \\ T\to 0}} \simeq -E_0''(\kappa_c)\,(\kappa - \kappa_c) \tag{29}$$

This result shows that for $\kappa > \kappa_c$ the velocity $\dot{x}$ is an imaginary number. This situation is quite similar to the case when particles inside a barrier have potential greater than the energy, leading to negative kinetic term and so imaginary velocity. As a consequence, although the particles have finite density inside the barrier, they are not detectable. For $\kappa < \kappa_c$ instead, the particles have the ordinary behavior and are directly detectable.

The implication of the above behavior on the original lattice gauge theory is as follows. By the conjugate variable of $\kappa = g^{-2}$ in the field theory side we expect

$$\Big\langle 1 - \cos f_{\mu\nu} \Big\rangle_{\substack{\kappa\approx\kappa_c \\ T\to 0}} \propto -(\kappa - \kappa_c) \tag{30}$$

Written in the ordinary gauge theory terms, by which

$$1 - \cos f_{\mu\nu} \simeq \frac{a^4}{2} F_{\mu\nu}^2 \propto (\vec{B}^2 - \vec{E}^2), \tag{31}$$

leads to ($g_c = 1/\sqrt{\kappa_c} = 1.125$)

$$\Big\langle \vec{B}^2 - \vec{E}^2 \Big\rangle_{\substack{g\approx g_c \\ T\to 0}} \propto \frac{1}{g_c^2} - \frac{1}{g^2} \tag{32}$$

In particular at $T \to 0$ we have

$$\Big\langle \vec{B}^2 \Big\rangle > \Big\langle \vec{E}^2 \Big\rangle \qquad \text{for} \qquad g > g_c \tag{33}$$

which is interpreted as the dominance of the density of magnetic monopoles in the defining vacuum of the theory. This is an evidence for the picture by which the reduced model is in fact describing the monopoles of the gauge theory. According to the dual Meissner effect scenario for the confined phase of gauge theories [11–13], the phase with (33) should be the confined one, in which the abundantly available monopoles prevent spreading of the electric fields originated from electric charges. In fact the theoretical studies [5, 7–9] as well as several lattice simulations [14] have found strong evidence for such a phase transition in U(1) compact lattice theory.

At higher temperatures the excited states make contributions to the partition function, leading to different behaviors in the mean-value (28) (see Fig. 2). There is a critical temperature $T_c = 0.246\, a^{-1}$ above which whatever the coupling is the mean-value is positive, meaning that at sufficiently high temperatures the monopoles always are detectable. One can present the whole picture in a T-g phase diagram for the mean-value (32) (see Fig. 3).

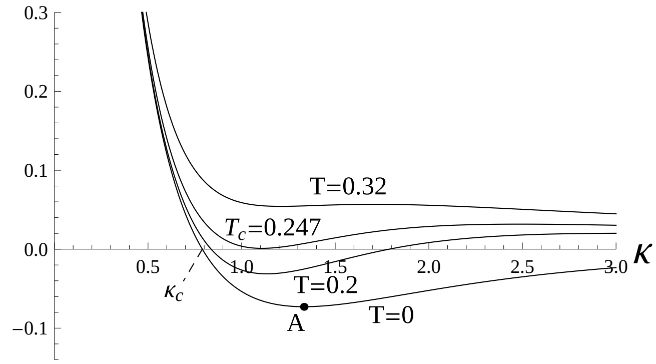

Figure 2: Four plots of $\langle 1 - \cos(a\dot{x}/R)\rangle$ versus κ (T unit: a^{-1}). The minimum A is at $(1.33, -0.073)$. The zero of T_c-curve is at $\kappa = 1.11$.

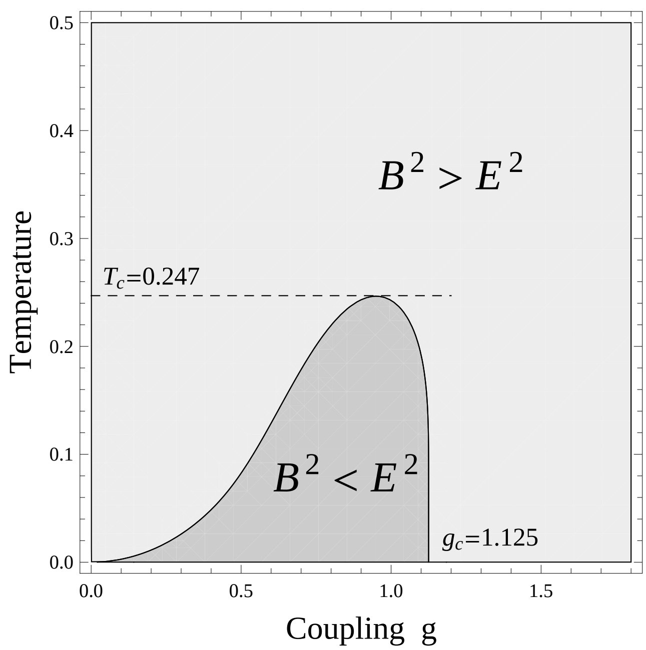

Figure 3: T-g phase diagram of the model (T unit: a^{-1}).

The extension of the model to the U(N) lattice gauge theory is possible as well, in which the coordinates resulted from the site reduction are hermitian matrices. In the reduced theory we have for (3)

$$e^{\mathrm{i}f_{n,0i}} = e^{\mathrm{i}\,aA_n^0}\, e^{\mathrm{i}\,X_{n+1}^i/R}\, e^{-\mathrm{i}\,aA_n^0}\, e^{-\mathrm{i}\,X_n^i/R}, \tag{34}$$

$$e^{\mathrm{i}f_{n,ij}} = e^{\mathrm{i}\,X_n^i/R}\, e^{\mathrm{i}\,X_n^j/R}\, e^{-\mathrm{i}\,X_n^i/R}\, e^{-\mathrm{i}\,X_n^j/R}, \tag{35}$$

leading to the action

$$S_0 = \frac{1}{g^2}\mathrm{Tr}\sum_{n,i}(\cos f_{n,0i} - 1) + \frac{1}{2g^2}\mathrm{Tr}\sum_{n,i,j}\left(e^{\mathrm{i}f_{n,ij}} - 1\right) \tag{36}$$

The model enjoys the gauge symmetry

$$X_n^i \to U_n\, X_n^i\, U_n^\dagger, \qquad e^{\mathrm{i}\,aA_n^0} \to U_n\, e^{\mathrm{i}\,aA_n^0}\, U_{n+1}^\dagger \tag{37}$$

in which U_n's are unitary matrices depending on discrete time. It is easy to check that (36) in the temporal gauge $A_n^0 \equiv 0$ and the continuum limit (9) reduces to

$$S_0 \simeq \frac{a}{g^2R^2}\int dt\ \mathrm{Tr}\left(\frac{1}{2}\dot{X}_i^2 + \frac{1}{4a^2R^2}[X^i, X^j]^2\right). \tag{38}$$

Setting $a^2R^2 \to l_s^4$, the above action is known as the matrix dynamics governing the dynamics of N 0-branes [1,3], with the interpretation that the $N^2 - N$ extra degrees encoded in matrices are capturing the dynamics of strings stretched between 0-branes. In a series of works, it has been suggested that the 0-brane matrix dynamics is used to model the bound-states of quarks and emergent QCD-strings [15–17]. It is argued that the matrix coordinate description of gauge variables might generate the stringy aspects expected from gauge theories, without the need to treat the world-sheet anomalies encountered in the non-critical space-time dimensions. The relevance of matrix coordinates to non-Abelian theories has been discussed in [17,18]. The symmetry aspects of the above picture is discussed in [19].

By the action (36) the analytic expressions as the U(1) case are not expected. Instead, the numerical analysis are certainly possible, and expected to give more information about the phase structure of non-Abelian gauge theories.

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