A Monte Carlo scheme for diffusion estimation

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***Abstract*—In this work, we design an efficient Monte Carlo scheme for diffusion estimation, where global and local parameters are involved in a unique inference problem. This scenario often appears in distributed inference problems in wireless sensor networks. The proposed scheme uses parallel local MCMC chains and then an importance sampling (IS) fusion for obtaining an efficient estimation of the global parameters. The resulting algorithm is simple and flexible. It can be easily applied iteratively, or extended in a sequential framework. In order to apply the novel scheme, the only assumption required about the model is that the measurements are conditionally independent given the related parameters.**

Keywords: *Diffusion estimation; Distributed inference; Parallel MCMC; Importance sampling.*

I. Introduction

Distributed inference have obtained a massive attention in the last decades, for different applications [1], [2], [3], [4], [5], [6], [7], [8]. For instance, in signal processing several algorithms have been proposed for solving inference problems in wireless sensor networks, where each node provides measurements about global and local parameters. Namely, all the received measurements are statistically related to a global parameter, whereas only a subset of the observations provides statistical information about the local variables. Furthermore, Bayesian methods have become very popular in signal processing during the last years and, with them, Monte Carlo (MC) techniques that are often necessary for the implementation of optimal a posteriori estimators [9], [10], [11]. Indeed, MC methods are powerful tools for numerical inference and optimization [12], [13], [14], [15], [16]. They are very flexible techniques. The only requirement needs for applying an MC technique is to be able to evaluate point-wise the posterior probability density function (pdf) [9], [10].

In this work, we introduce a simple and flexible approach for the diffusion estimation of global parameters, providing simultaneously the inference of the local parameters. The proposed solution employs parallel MCMC algorithms for analyzing the local features of the network and an importance sampling (IS) fusion for obtaining complete estimators of the global parameters. Each MCMC method addresses a different target posterior function, obtained by considering a subset of observations. This approach also presents several computational benefits from a Monte Carlo point of view (as remarked exhaustively in Section III-A). For instance, the mixing of the MCMC methods is facilitated by the reduced number of measurements involved in the partial posterior, since this partial posterior distribution is implicitly *tempered* [17], [18], [19], [16]. Furthermore, several parallel or related schemes, proposed in literature, could adapted for this framework [20], [21], [22], [18], [19], [23], [6], [7], [8]. Numerical simulations show the advantages of the proposed approach.

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II. Problem statement

In this work, we are interested in making inference the following variable of interest,

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\Theta}^{(G)} \\ \boldsymbol{\Theta}^{(L)} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_M \end{bmatrix} \in \mathcal{A} \subseteq \mathbb{R}^{D_\theta}, \qquad (1)$$

composed by the vectors $\mathbf{x} = [x_1, \dots, x_{d_X}]^\top \in \mathbb{R}^{d_X}$ and $\mathbf{v}_m = [v_{1,m}, \dots, v_{d_m,m}]^\top \in \mathbb{R}^{d_m}$, for $m = 1, \dots, M$. The vector $\boldsymbol{\Theta}^{(G)} = \mathbf{x}$ represents a global parameter, whereas $\boldsymbol{\Theta}^{(L)} = [\mathbf{v}_1, \dots, \mathbf{v}_M]^\top \in \mathbb{R}^L$ are local parameters. We receive a set of d_Y measurements, $\mathcal{Y} = \{z_1, z_2, \dots, z_{d_Y}\}$, with each $z_j \in \mathbb{R}$ (we assume z_j be scalar only for simplicity), related to variable of interest $\boldsymbol{\Theta}$. We consider M disjoint subset of $\mathcal{Y}$, i.e., we can write $\mathcal{Y} = \mathbf{y}_{1:M} = \bigcup_{m=1}^{M} \mathbf{y}_m$, and $\mathbf{y}_j \cap \mathbf{y}_k = \emptyset$, for all $j \neq k$. We assume that the observations are conditionally independent, i.e., the likelihood function can be factorized as

$$L(\mathbf{y}_{1:M}|\boldsymbol{\Theta}) = \prod_{m=1}^{M} \ell_m(\mathbf{y}_m|\mathbf{x}, \mathbf{v}_m). \qquad (2)$$

Considering a prior probability density function (pdf) $p(\boldsymbol{\Theta}) = p(\mathbf{x}, \mathbf{v}_m)$, the *complete posterior* pdf can written as

$$\Omega(\boldsymbol{\Theta}|\mathbf{y}_{1:M}) = \prod_{m=1}^{M} \bar{\pi}_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m), \qquad (3)$$

where the *partial posteriors* are

$$\bar{\pi}_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m) \propto$$
$$\pi_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m) = \ell_m(\mathbf{y}_m|\mathbf{x}, \mathbf{v}_m) \left[p(\mathbf{x}, \mathbf{v}_m)\right]^{1/M}. \qquad (4)$$

For the inference of $\mathbf{x}$, an important role is played by the marginal posterior density

$$G(\mathbf{x}|\mathbf{y}_{1:M}) = \int_{\mathbb{R}^L} \Omega(\mathbf{\Theta}|\mathbf{y}_{1:M}) d\mathbf{v}_1 \dots d\mathbf{v}_M \quad (5)$$

$$= \int_{\mathbb{R}^L} \prod_{m=1}^{M} \bar{\pi}_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m) d\mathbf{v}_1 \dots d\mathbf{v}_M \quad (6)$$

$$= \prod_{m=1}^{M} g_m(\mathbf{x}|\mathbf{y}_m), \quad (7)$$

where

$$g_m(\mathbf{x}|\mathbf{y}_m) = \int_{\mathbb{R}^{d_m}} \bar{\pi}_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m) d\mathbf{v}_m. \quad (8)$$

Note that the integrals above cannot be computed analytically, in general. This framework appears naturally in several applications, for instance in the so-called Node-Specific Parameter Estimation (NSPE) problem within a sensor network [4], [5], [24], [25] (see Figure 1). Furthermore, a similar approach is considered in Big Data context [23], [6], [7], [8].

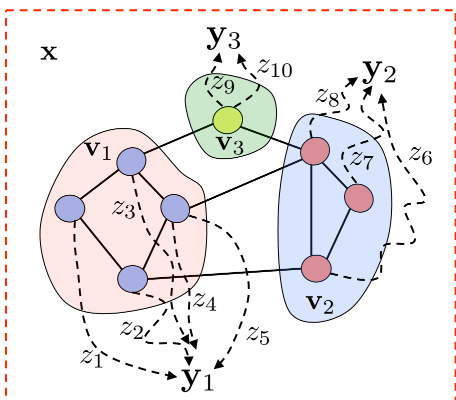

Figure 1. Diffusion estimation in a sensor network. In this graphical examples, the network is composed by 8 nodes, dividing in 3 clusters and providing 3 different vectors of measurements $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$. In this case, $\mathcal{Y} = \mathbf{y}_1 \cup \mathbf{y}_2 \cup \mathbf{y}_3$, $\mathbf{y}_j \cap \mathbf{y}_k \neq 0$ for $j \neq k$ (in this case, $d_{Y_1} = 5$, $d_{Y_2} = 3$, $d_{Y_3} = 2$). Each sensor can provide more than one measurement.

Remark. Note that we could assume easily that each partial posterior $\bar{\pi}_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m)$ depends also to other local variables, e.g., $\bar{\pi}_m(\mathbf{x}, \mathbf{v}_m, \mathbf{v}_k, \mathbf{v}_j|\mathbf{y}_m)$, with $k, j \neq m$. The algorithm proposed in this work can be automatically extended for this case. For simplicity, we have considered only one local variable in each partial posterior.

III. Bayesian inference

Our purpose is to make inference about $\mathbf{\Theta} = [\mathbf{x}, \mathbf{v}_{1:M}]$ given $\mathcal{Y} = \mathbf{y}_{1:M}$. For instance, we desire to compute the *Minimum Mean Square Error* (MMSE) estimators of $\mathbf{x}$ and $\mathbf{v}_{1:M}$, i.e.,

$$\widehat{\mathbf{x}} = \int_{\mathcal{A}} \mathbf{x} \; \Omega(\mathbf{x}, \mathbf{v}_{1:M}|\mathbf{y}_{1:M}) d\mathbf{x} d\mathbf{v}_{1:M}, \quad (9)$$

$$= \int_{\mathbb{R}^{d_X}} \mathbf{x} \; G(\mathbf{x}|\mathbf{y}_{1:M}) d\mathbf{x}, \quad (10)$$

and

$$\widehat{\mathbf{v}}_m = \int_{\mathcal{A}} \mathbf{v}_m \; \Omega(\mathbf{x}, \mathbf{v}_{1:M}|\mathbf{y}_{1:M}) d\mathbf{x} d\mathbf{v}_{1:M}, \quad (11)$$

for $m = 1, \dots, M$. In general, we are not able to calculate analytically the integrals above. Thus, we apply a Monte Carlo (MC) approach for computing approximately $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{v}}_{1:M}$.

A. Benefits of the parallel MC implementation

The previous factorization of the posterior pdf suggests the use of M parallel algorithms and then combine the corresponding outputs. This is convenient from a Monte Carlo point of view. Namely, the use of M parallel MC methods, each one addressing one partial posterior $\bar{\pi}_m$, presents several computational benefits:

a) Each partial posterior $\bar{\pi}_m$ is embedded in a state space of lower dimensionality, specifically, $\mathbf{x}, \mathbf{v}_m \in \mathbb{R}^{d_X \times d_m}$. This clearly helps the exploration of the space by the MC algorithm.

b) Each partial posterior involves a smaller number of measurements. This is an advantage since the mass of probability is in general more disperse than when a big number of observations is jointly considered, producing a *tempering effect* (data-tempering) [17], [16], [19]. This again helps the exploration of the state space (as suggested, e.g., in [19]).

c) This scenario automatically allows a possible parallel implementation.

IV. Distributed Monte Carlo Inference

Let us consider that we able to draw N samples $\boldsymbol{\theta}_m^{(n)} = [\mathbf{x}^{(n)}, \mathbf{v}_m^{(n)}]$, with $n = 1, \dots, N$, directly from each $\bar{\pi}_m$, i.e.,

$$\boldsymbol{\theta}_m^{(1)}, \dots, \boldsymbol{\theta}_m^{(N)} \sim \bar{\pi}_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m),$$

with $m = 1, \dots, M$. Due to the factorization of the complete target pdf $\Omega(\mathbf{\Theta}|\mathbf{y}_{1:M})$, for inferring the local variable $\mathbf{v}_m$, we use only the samples $\mathbf{v}_m^{(n)}$, $n = 1, \dots, N$, obtained from $\bar{\pi}_m$. For the global variable $\mathbf{x}$, we can build M different partial Monte Carlo estimators. However, all the information contained in the different partial posteriors should be employed for providing a more efficient unique estimator of $\mathbf{x}$. It can be done combining adequately the M partial Monte Carlo estimators of $\mathbf{x}$.

Let us consider that we are able to draw samples from each marginal pdfs $g_m(\mathbf{x}|\mathbf{y}_m)$ in Eq. (8) and also assume that we are able to evaluate it. In this case, we can use the following IS scheme:

1) Draw $\mathbf{x}_m^{(1)}, \dots, \mathbf{x}_m^{(N)} \sim g_m(\mathbf{x}|\mathbf{y}_m)$.
2) Assign the weight

$$w_m^{(n)} = \frac{G(\mathbf{x}_m^{(n)}|\mathbf{y}_{1:M})}{g_m(\mathbf{x}_m^{(n)}|\mathbf{y}_m)}, \quad (12)$$

$$= \prod_{k=1; k \neq m}^{M} g_k(\mathbf{x}_k^{(n)}|\mathbf{y}_k), \quad (13)$$

to each sample $\mathbf{x}_m^{(n)}$, for $n = 1, \dots, N$.

Then, the IS approximation of the MMSE estimator $\widehat{\mathbf{x}}$ is

$$\widetilde{\mathbf{x}} = \frac{1}{\sum_{m=1}^{M}\sum_{n=1}^{N} w_m^{(n)}} \sum_{m=1}^{M}\sum_{n=1}^{N} w_m^{(n)} \mathbf{x}_m^{(n)} \qquad (14)$$

The previous approach has two main problems:

- We are not able to draw from $g_m(\mathbf{x}|\mathbf{y}_m)$ in Eq. (8).
- It is not possible to evaluate $g_m(\mathbf{x}|\mathbf{y}_m)$ and $G(\mathbf{x}_m|\mathbf{y}_{1:M})$.

A. Proposed Algorithm

A possible approximate solution consists in the following procedure. First, we use parallel MCMC algorithms for drawing samples $\boldsymbol{\theta}_m^{(n)} = [\mathbf{x}^{(n)}, \mathbf{v}_m^{(n)}]$ from each partial posterior $\bar{\pi}_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m)$. Given an index $m \in \{1, \dots, M\}$, after that the chain converges, note that the samples $\mathbf{x}_m^{(n)}$ are distributed as $g_m(\mathbf{x}|\mathbf{y}_m)$ whereas $\boldsymbol{\theta}_m^{(n)}$ is distributed as $\bar{\pi}_m(\mathbf{x}, \mathbf{v}_m|\mathbf{y}_m)$. Then, we build a kernel density approximation [26],

$$\widehat{g}_m(\mathbf{x}|\mathbf{y}_m) = \frac{1}{N}\sum_{n=1}^{N} \varphi(\mathbf{x}|\mathbf{x}_m^{(n)}, \mathbf{C}), \qquad (15)$$

for each $m \in \{1, \dots, M\}$, using the generated samples $\mathbf{x}_m^{(n)}$, $n = 1, \dots, N$, as means of a kernel function φ with bandwidth matrix $\mathbf{C}$. In this way, we can compute an approximate weight

$$\widehat{w}_m^{(n)} = \prod_{k=1;k\neq m}^{M} \widehat{g}_k(\mathbf{x}_k^{(n)}|\mathbf{y}_k), \qquad (16)$$

so that the IS estimator $\widetilde{\mathbf{x}}$ in Eq. (14) can be approximated with

$$\bar{\mathbf{x}} = \frac{1}{\sum_{m=1}^{M}\sum_{n=1}^{N} \widehat{w}_m^{(n)}} \sum_{m=1}^{M}\sum_{n=1}^{N} \widehat{w}_m^{(n)} \mathbf{x}_m^{(n)}. \qquad (17)$$

Table I and Figure 2 summarizes the proposed algorithm. Table II summarized the notation about the estimators of $\mathbf{x}$.

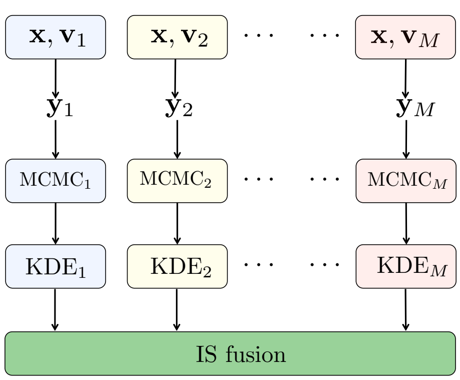

Figure 2. Graphical representation of Parallel Marginal Markov Importance Sampling scheme: M local MCMC chains are run, each one addressing one partial posterior $\bar{\pi}_m$. A kernel density estimation (KDE) for approximating the marginal partial posteriors $g_m(\mathbf{x}|\mathbf{y}_m)$ is performed locally. Then, an importance sampling fusion is employed for providing a global estimator of the global parameter $\mathbf{x}$.

Theoretical support. After a burn-in period, the MCMC chains converges to the invariant target pdf (for $N \to \infty$, the convergence is ensured [9], [10]). Namely after some iterations, the MCMC methods yields samples $\{\mathbf{x}_m^{(n)}, \mathbf{v}_m^{(n)}\}$ distributed according to $\bar{\pi}_m$, so that $\{\mathbf{x}_m^{(n)}\}$ are distributed as the marginal partial posteriors $g_m(\mathbf{x}|\mathbf{y}_m)$ [9]. There exists an optimal bandwidth $\mathbf{C}^*$ [26] such that

$$\widehat{g}_m(\mathbf{x}|\mathbf{y}_m) \to g_m(\mathbf{x}|\mathbf{y}_m), \quad \text{for} \quad N \to \infty.$$

As a consequence, we have that $\widehat{w}_m^{(n)} \to w_m^{(n)}$ and $\bar{\mathbf{x}} \to \widetilde{\mathbf{x}}$, for $N \to \infty$. Moreover, the IS estimator is consistent [9] so that $\bar{\mathbf{x}} \to \widetilde{\mathbf{x}} \to \widehat{\mathbf{x}}$, as $N \to \infty$. In general, the optimal bandwidth $\mathbf{C}^*$ is unknown. However, using a bandwidth $\mathbf{C} \neq \mathbf{C}^*$, Eq. (15) provides an estimator of $g_m(\mathbf{x}|\mathbf{y}_m)$ [26], in any case.

Alternative IS weights. Other *proper* IS weights can be employed in our framework, providing consistent estimators [27], [28], [29]. For instance, a full deterministic mixture approach [28], [30], [31] for multiple importance sampling (MIS) schemes can be used, i.e.,

$$w_m^{(n)} = \frac{G(\mathbf{x}_m^{(n)}|\mathbf{y}_{1:M})}{\frac{1}{M}\sum_{j=1}^{M} g_j(\mathbf{x}_m^{(n)}|\mathbf{y}_j)}, \qquad (22)$$

$$= \frac{\prod_{j=1}^{M} g_j(\mathbf{x}_m^{(n)}|\mathbf{y}_j)}{\frac{1}{M}\sum_{j=1}^{M} g_j(\mathbf{x}_m^{(n)}|\mathbf{y}_j)}. \qquad (23)$$

It is possible to show that the application of these DM-MIS weights provides more efficient IS estimators [29], [28] (i.e.,

Table I
PARALLEL MARGINAL MARKOV IMPORTANCE SAMPLER (PMMIS).

1. **Local MCMC samplers:** Generate M chains of length N, i.e.,
$$\boldsymbol{\theta}_m^{(1)} = [\mathbf{x}^{(1)}, \mathbf{v}_m^{(1)}], \dots, \boldsymbol{\theta}_m^{(N)} = [\mathbf{x}^{(N)}, \mathbf{v}_m^{(N)}],$$
with target pdf $\bar{\pi}_m(\mathbf{x}|\mathbf{y}_m) \propto \pi_m(\mathbf{x}|\mathbf{y}_m)$, $m = 1, \dots, M$.
2. **Kernel density estimation:** Build
$$\widehat{g}_m(\mathbf{x}|\mathbf{y}_m) = \frac{1}{N}\sum_{n=1}^{N} \varphi(\mathbf{x}|\mathbf{x}_m^{(n)}, \mathbf{C}), \qquad (18)$$
given a function φ and scale parameter $\mathbf{C}$.
3. **Global IS fusion:** Compute the weights
$$\widehat{w}_m^{(n)} = \prod_{k=1;k\neq m}^{M} \widehat{g}_k(\mathbf{x}_k^{(n)}|\mathbf{y}_k), \qquad (19)$$
for $m = 1, \dots, M$ and $n = 1, \dots, N$.
4. **Outputs:** Return the Monte Carlo estimators
$$\bar{\mathbf{x}} = \frac{1}{\sum_{m=1}^{M}\sum_{n=1}^{N} \widehat{w}_m^{(n)}} \sum_{m=1}^{M}\sum_{n=1}^{N} \widehat{w}_m^{(n)} \mathbf{x}_m^{(n)}, \qquad (20)$$
and
$$\widetilde{\mathbf{v}}_m = \frac{1}{N}\sum_{n=1}^{N} \mathbf{v}_m^{(n)}, \quad m = 1, \dots, M. \qquad (21)$$

Table II
NOTATION OF DIFFERENT ESTIMATORS OF $\mathbf{x}$.

Notation	Description
$\widehat{\mathbf{x}}$	MMSE estimator in Eq. (9).
$\widetilde{\mathbf{x}}$	Monte Carlo estimator (approximation of $\widehat{\mathbf{x}}$) in Eq. (14).
$\bar{\mathbf{x}}$	Approximate Monte Carlo estimator (approximation of $\widetilde{\mathbf{x}}$) in Eqs. (17)-(20).

with smaller variance).

B. Parallel Metropolis-Hastings algorithms

For simplicity, we consider Metropolis-Hastings (MH) methods in the first step of the novel scheme. However, more sophisticated algorithms can be employed. More specifically, starting with a randomly chosen $\boldsymbol{\theta}_m^{(0)}$, we perform the following steps:

For $m = 1, \dots, M$:

For $n = 1, \dots, N$:

1) Draw $\boldsymbol{\theta}'$ from a proposal pdf $q_m(\boldsymbol{\theta}'|\boldsymbol{\theta}_m^{(n-1)})$.
2) Set $\boldsymbol{\theta}_m^{(n)} = \boldsymbol{\theta}'$, with probability

$$\alpha = \min\left[1, \frac{\pi_m(\boldsymbol{\theta}'|\mathbf{y}_m)q_m(\boldsymbol{\theta}_m^{(n-1)}|\boldsymbol{\theta}')}{\pi_m(\boldsymbol{\theta}_m^{(n-1)}|\mathbf{y}_m)q_m(\boldsymbol{\theta}'|\boldsymbol{\theta}_m^{(n-1)})}\right], \quad (24)$$

otherwise set $\boldsymbol{\theta}_m^{(n)} = \boldsymbol{\theta}_m^{(n-1)}$ (with probability $1-\alpha$).

V. Numerical simulations

In order to test PMMIS, we consider a Gaussian likelihoods

$$f_m(\mathbf{y}_m|x, v_m) = \prod_{j=1}^{d_{Y_m}} \mathcal{N}(z_j|x, v_m, \boldsymbol{\Sigma}_m), \quad (25)$$

with $\mathbf{y}_m = [z_1, \dots, z_{d_{Y_m}}]^\top$, $m = 1, \dots, M$, and where $z_j, x, v_m \in \mathbb{R}$. Note that $d_{Y_m} = |\mathbf{y}_m|$. We consider flat improper priors over x and v_m, $m = 1, \dots, M$. We set $M = 10$, so that we have 10 different partial target pdfs $\bar{\pi}_m(x, v_m|\mathbf{y}_m) \propto f_m(\mathbf{y}_m|x, v_m)$. The covariance matrices are $\boldsymbol{\Sigma}_m = [\sigma_{1,m}^2 \ \rho_m; \rho_m \ \sigma_{2,m}^2]$ with ,

$$\begin{aligned}
\sigma_{1,1:M} &= \left[\frac{1}{2}, \frac{3}{2}, 4, \frac{5}{2}, 3, \frac{7}{2}, 3, \frac{5}{2}, 2, \frac{1}{2}\right], \\
\sigma_{2,1:M} &= \left[\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}\right], \\
\rho_{1:M} &= \left[0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}\right]
\end{aligned}$$

We set $x = -1$ and $v_{1:M} = [-5, -4, -3, -2, -1, 0, 1, 2, 3, 4]$ as "true" values of the parameters. Then, given these values, we generate (according to the model in Eq. (25)) different numbers of measurements for each partial likelihood, specifically, $d_{Y_{1:M}} = [2, 2, 50, 2, 5, 20, 5, 100, 2, 10]$. Given a set $\mathcal{Y} = \mathbf{y}_{1:M}$ of generated observations, in this toy example we can compute the MMSE estimator $\widehat{x} = -0.911$ by a costly deterministic numerical procedure using a thin grid (for approximating the marginal posterior and then the corresponding expected value). Thus, we apply PMMIS in 400 independent runs and compare the obtained estimator $\bar{x}$ with $\widehat{x} = -0.911$, computing the corresponding MSE.

We consider Gaussian functions φ for the kernel density estimation, with the optimal bandwidth suggested in [26]. For the proposal pdfs q_m's of the MH algorithms, we employ standard Gaussian random walk proposals (with identity covariance matrix $[1\ 0; 0\ 1]$). We test PMMIS for different values of the length of the chains, N, from $N = 15$ to $N = 2000$. Furthermore, at each run, we also compute a trivial Monte Carlo approximation of $\widehat{x}$, given by

$$\bar{x}_{trivial} = \frac{1}{M}\sum_{m=1}^{M} \tilde{x}_m = \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} x_m^{(n)},$$

where $\tilde{x}_m = \frac{1}{N}\sum_{n=1}^{N} x_m^{(n)}$ is the Monte Carlo approximation of $\widehat{x}_m$, obtained using the samples of m-th chain. The results are shown in Figure 3, in terms of MSE versus N (length of the chains, i.e., number of MH iterations for each partial target). Three curves are shown, corresponding to the use of standard IS weights (dashed line), multiple IS weights (solid line) and the trivial solution (dotted-dashed line). We can observe that PMMIS provides good results outperforming the trivial solution for $N > 40$. This means that, for $N \leq 40$, the samples generated by the MH methods still belong to the "burn-in" period and the convergence to the invariant pdf is not reached. However, with a adequate number of iterations of the chains, PMMIS provides good results. In the example, the standard IS and DM-MIS weights perform similarly (with a slight advantages for the DM-MIS weights).

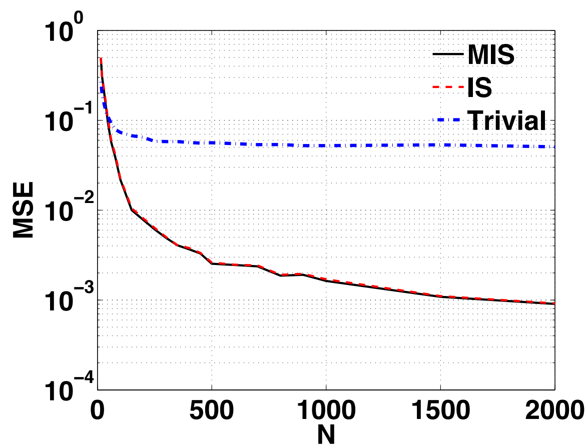

Figure 3. MSE in the estimation of $\hat{x}$ obtaining by the Monte Carlo approximation $\bar{x}$ by PMMIS using standard IS weights (dashed line), DM-MIS weights (solid line) and the trivial weights $\frac{1}{MN}$ (dotted-dashed line), for the final fusion (semilog scale representation).

VI. Conclusions

In this work, we have introduced a novel Monte Carlo scheme in order to obtain an efficient distributed estimation. The new Bayesian method provides a simultaneous estimation of local and global parameters. The estimation of the global parameters takes into account all the possible statistical information. The proposed algorithm is an importance sampler that assigns weights to the samples obtained by the application of parallel MCMC methods. Each MCMC addresses a different partial target distribution, considering only a subset of measurements. As future line, we consider the possible design of an iterative implementation of the proposed scheme, where the proposal pdfs employed by the MCMC algorithms are adapted online, generating in this way an interaction among the parallel chains.

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