

About an Integral of Legendre

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Abstract

In this note we show some formulas related with: Legendre Integral and Number Pi : $\pi=3.14159\dots$

Keyword: Legendre integral, Number Pi

I. Introduction: Legendre Integral

Recordamos una integral dada por Adrien Marie Legendre (1752-1833) :

$$\int_0^1 \frac{1}{\sqrt{1-x^8}} dx = \frac{1}{\sqrt{2}} K(\sqrt{2}-1) \quad (1)$$

donde

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2 \sin^2 x}} dx = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2 x^2)}} dx, \quad 0 < k < 1 \quad (2)$$

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(1/2)_n}{n!} \right)^2 k^{2n}, \quad 0 < k < 1 \quad (3)$$

II. Formulas

La integral (1) se puede escribir como:

$$\int_0^1 \frac{1}{\sqrt{1-x^8}} dx = 1 + \int_1^{\infty} \left(1 - \sqrt[8]{1-x^{-2}} \right) dx \quad (4)$$

de (4) se tiene:

$$\int_1^{\infty} \left(1 - \sqrt[8]{1-x^{-2}} \right) dx = \frac{1}{\sqrt{2}} K(\sqrt{2}-1) - 1 \quad (5)$$

para $0 < z < 1$, se tiene :

$$\int_0^1 \frac{1}{\sqrt{1-x^8}} dx = \int_0^z \frac{1}{\sqrt{1-x^8}} dx + \int_z^1 \frac{1}{\sqrt{1-x^8}} dx = \int_0^z \frac{1}{\sqrt{1-x^8}} dx + \frac{1-z}{\sqrt{1-z^8}} + \int_{1/\sqrt{1-z^8}}^{\infty} \left(1 - \sqrt[8]{1-x^{-2}} \right) dx \quad (6)$$

Desarrollando en serie las integrales de (6) y usando (3) se tiene:

$$\frac{1}{\pi} = \left((1/2\sqrt{2}) \sum_{n=0}^{\infty} ((1/2)_n/n!)^2 (\sqrt{2}-1)^{2n} \right) / \left(\frac{1-z}{\sqrt{1-z^8}} + \sum_{n=0}^{\infty} ((1/2)_n z^{8n+1}/n!(8n+1)) - \frac{1}{\sqrt{1-z^8}} \sum_{n=1}^{\infty} ((-1/8)_n (1-z^8)^n/n!(2n-1)) \right), \quad 0 < z < 1 \quad (7)$$

Para $z^8 = 1/2$, se tiene :

$$\frac{1}{\pi} = \left(\sum_{n=0}^{\infty} ((1/2)_n/n!)^2 (\sqrt{2}-1)^{2n} \right) / \left(2(2-2^{7/8}) + 2\sqrt[8]{8} \sum_{n=0}^{\infty} ((1/2)_n 2^{-n}/n!(8n+1)) - 4 \sum_{n=1}^{\infty} ((-1/8)_n 2^{-n}/n!(2n-1)) \right) \quad (8)$$

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