An axiomatic algebraico-functional theory of an *n*-dimensional Euclidean affine space

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Abstract

A concise rigorous *axiomatic algebraico-functional theory* (*AAFT*) of a *real affine Euclidean space of any given dimension* $n\geq 1$ (*nDRAfES)*, $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, is developed from an algebraic system $\dot{\boldsymbol{E}}^{\mathrm{g}}$, called an *affine additive group* (*AAG*). The lattera consists of a certain *underlying set of points* $\dot{E}$, called an *affine additive group manifold* (*AAGM*), and of a certain *commutative* [*abstract*] *additive group* (*CAG*) $\hat{\boldsymbol{E}}^{\mathrm{g}}$, called the *adjoint group of* $\dot{\boldsymbol{E}}^{\mathrm{g}}$, whose underlying set $\hat{E}$ of elements, called *vectors*, is related to $\dot{E}$ by a *binary surjection* $\hat{V}:\dot{E}\times\dot{E}\to\hat{E}$, satisfying the appropriate version of the *Chasle*, or *triangle*, *law*, according to which any three points $\dot{x}$, $\dot{y}$, and $\dot{z}$ of $\dot{E}$ (the apices of a triangle) satisfy the equality $\hat{V}(\dot{x},\dot{y})\hat{+}\hat{V}(\dot{y},\dot{z})\hat{+}\hat{V}(\dot{z},\dot{x})=\hat{0}$, where $\hat{0}$ is the null-vector of $\hat{E}$. The prepositive qualifier "*real*" to "space" is concurrent to the postpositive qualifier "*over the field of real numbers* $\boldsymbol{R}$". When $\hat{\boldsymbol{E}}^{\mathrm{g}}$ is successively supplemented by the appropriate additional attributes to become first a *real abstract vector* (*linear*) *space* (*RAbVS*) $\hat{\boldsymbol{E}}(\boldsymbol{R})$ and ultimately an *n-dimensional* (*nD*) *real abstract vector Euclidean space* (*nDRAbVES*) $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, $\dot{\boldsymbol{E}}^{\mathrm{g}}$ is automatically self-adjusted to all current metamorphoses of its adjoint group to become first a *real affine space* (*RAfS*) $\dot{\boldsymbol{E}}(\boldsymbol{R})$ and ultimately an *n*DRAfES $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, of which the above $\hat{\boldsymbol{E}}(\boldsymbol{R})$ and $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ are *adjoint*. A new consistent method of logographically denoting various algebraic systems is suggested. Relative to its any *orthonormal basis*, $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, adjoint of $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, is *isomorphic* to the *nD real arithmetical vector Euclidean space* (*nDRArVES*) $\overline{\boldsymbol{E}}_n(\boldsymbol{R})$, whose underlying set $\overline{E}_n$ consists of *ordered n-tuples of real numbers*, being coordinates of the respective abstract vectors of the underlying vector set $\hat{E}$ of $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$. A hypothetical *time continuum* that is

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associated with $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$ or by analogy with any $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ is a special interpretation of $\dot{\boldsymbol{E}}_1(\boldsymbol{R})$, which is denoted by ‘$\dot{\boldsymbol{T}}(\boldsymbol{R})$’, so that the pertinent interpretations of $\hat{\boldsymbol{E}}_1$, $\overline{\boldsymbol{E}}_1$, $\dot{E}_1$, $\hat{E}_1$, $\overline{E}_1$ are denoted respectively by ‘$\hat{\boldsymbol{T}}$’, ‘$\overline{\boldsymbol{T}}$’, ‘$\dot{T}$’, ‘$\hat{T}$’, ‘$\overline{T}$’. In the framework of the AAFT, a real-valued functional form that is initially defined on a certain region of $\dot{E}_n$ or $\dot{T} \times \dot{E}_n$ can rigorously be transduced into (mapped onto) a certain real-valued functional form defined on a certain region of $\overline{E}_n$ or $\overline{T} \times \overline{E}_n$ and vice versa. Therefore, the AAFT can serve as an *underlying discipline of differential and integral calculi* and hence it is a formal interface between *any hypothetical physical processes in* $\dot{\boldsymbol{T}}(\boldsymbol{R}) \times \dot{\boldsymbol{E}}_n(\boldsymbol{R})$ and *their mathematical descriptions in* $\overline{\boldsymbol{T}}(\boldsymbol{R}) \times \overline{\boldsymbol{E}}_n(\boldsymbol{R})$. Particularly, the AAFT is the underlying discipline of the theory that has been developed in Iosilevskii [2015]. By way of an example of AAG’s, an *n*-dimensional primitive (Bravais) affine lattice in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ is discussed in subsection 5.7.

1. Introduction

1) In a cosmologically small spatial scale as a linear size of the solar planetary system during a cosmologically small span of time as that of the life time of the Earth, the *receptacle of Nature*, i.e. the receptacle of matter along all metamorphoses, which occur to matter in *time* and which are called physical, chemical, biological, etc processes, is commonly regarded as a certain *3-dimensional affine Euclidean space* $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$ (briefly $\dot{\boldsymbol{E}}_3$) *over the field* $\boldsymbol{R}$ *of real numbers*, called also an *affine real Euclidean 3-space*. *Time* is a hypothetical *non-spatial 1-dimensional continuum* that can be regarded as a special version (interpretation) of a *1-dimensional affine Euclidean space* $\dot{\boldsymbol{E}}_1(\boldsymbol{R})$ (briefly $\dot{\boldsymbol{E}}_1$) *over* $\boldsymbol{R}$, to be denoted by ‘$\dot{\boldsymbol{T}}(\boldsymbol{R})$’ (briefly ‘$\dot{\boldsymbol{T}}$’), in which *the above processes* go on in the irreversible direction from past through present to future. It is postulated that, via those processes, $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$ is united with $\dot{\boldsymbol{T}}(\boldsymbol{R})$ to form a 4-dimensional pseudo-Euclidean real affine space of index 1 – the *space-time of special theory of relativity*, which is called the *Minkowski space* and which will be denoted by ‘$\dot{\boldsymbol{M}}_4(\boldsymbol{R})$’. The presence of gravitating masses in the hypothetic $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$, along with the inseparable gravitational processes going on in and expanded across $\dot{\boldsymbol{T}}(\boldsymbol{R})$, change the known metric properties of $\dot{\boldsymbol{M}}_4(\boldsymbol{R})$, so that it is replaced by, i.e. as if turns into, the *Riemannian space* $\mathcal{R}$ *of general theory of relativity*.

2) A physical process occurs in a certain region of the direct product $\dot{\boldsymbol{T}}(\boldsymbol{R})\times\dot{\boldsymbol{E}}_3(\boldsymbol{R})$, while both $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$ and $\dot{\boldsymbol{T}}(\boldsymbol{R})$ comprise *points* and not vectors or numbers. Therefore, when appropriate, a physical process should be described by a certain real-valued or complex-valued functional form defined in $\dot{\boldsymbol{T}}(\boldsymbol{R})\times\dot{\boldsymbol{E}}_3(\boldsymbol{R})$. At the same time, a presently common way to treat physical processes theoretically with the purpose to create their concise rigorous concepts is to describe them by certain real-valued or complex-valued functional forms defined on appropriate regions of the direct product $\overline{\boldsymbol{T}}(\boldsymbol{R})\times\overline{\boldsymbol{E}}_3(\boldsymbol{R})$ of a *1-dimensional real arithmetical vector Euclidean space* $\overline{\boldsymbol{T}}(\boldsymbol{R})$ and a *3-dimensional real arithmetical vector Euclidean space* $\overline{\boldsymbol{E}}_3(\boldsymbol{R})$, i.e. actually by functional forms depending on four *independent real-valued variables*, e.g. 'x_0', 'x_1', 'x_2', and 'x_3'. In this case, the latter functional forms are treated in the framework of modern differential and integral calculus.

3) A theoretical physicist usually metamorphoses a functional form, which is supposedly defined in $\dot{\boldsymbol{T}}(\boldsymbol{R})\times\dot{\boldsymbol{E}}_3(\boldsymbol{R})$, into the respective functional form, defined in $\overline{\boldsymbol{T}}(\boldsymbol{R})\times\overline{\boldsymbol{E}}_3(\boldsymbol{R})$, by choosing, actually or imaginarily (mentally), the appropriate *laboratory coordinate systems* $\dot{c}_{\{3\}}$ and $\dot{c}_{\{1\}}$ in $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$ and $\dot{\boldsymbol{T}}(\boldsymbol{R})$ respectively, relative to which a point of $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$ is characterized by the corresponding ordered triple $\langle x_1, x_2, x_3\rangle$ of real numbers, being a vector in $\overline{\boldsymbol{E}}_3(\boldsymbol{R})$, and a point of $\dot{\boldsymbol{T}}(\boldsymbol{R})$ is characterized by the corresponding ordered single $\langle x_0\rangle$ of a real number, being a vector in $\overline{\boldsymbol{T}}(\boldsymbol{R})$; $\langle x_1, x_2, x_3\rangle$ is the repeated ordered pair $\langle\langle x_1, x_2\rangle, x_3\rangle$ subject to

$$\langle x_1, x_2\rangle \doteqdot \{\{x_1\}, \{x_1, x_2\}\} \tag{1.1}$$

(see, e.g., Halmos [1960, pp. 22–25]), so that

$$\langle\langle x_1, x_2\rangle, x_3\rangle = \{\{\langle x_1, x_2\rangle\}, \{\langle x_1, x_2\rangle, x_3\}\}, \tag{1.2}$$

whereas $\langle x_0\rangle$ is the singleton $\{x_0\}$, i.e. $\langle x_0\rangle \doteqdot \{x_0\}$. It is understood that $\dot{c}_{\{3\}}$ is a Galilelian orthonormal, i.e. normal orthogonal (rectangular), rectilinear or curvilinear, coordinate system, whose origin $\dot{0}_{\{3\}}$ is a certain fixed point of the underlying set $\dot{E}_3$ of $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$ and whose basis is an ordered triple of orthonormal *position vectors* $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$, which are the bijective images in the power set $\mathsf{P}(\dot{E}_3)$ of orthonormal *free basis vectors* $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ belonging to the underlying set $\hat{E}_3$ of a certain *3-dimensional real abstract vector Euclidean* *space* $\hat{\boldsymbol{E}}_3(\boldsymbol{R})$ that is called the *abstract vector space adjoint of* $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$. Therefore, besides $\dot{c}_{\{3\}}$, there is a much simpler and much more effective coordinate system $c_{\{3\}}$, which consists of the same origin $\dot{0}_{\{3\}}$ belonging to $\dot{E}_3$ and of the ordered triple of the vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ belonging to $\hat{E}_3$. The system $c_{\{3\}}$ has the properties that the coordinates of any point of $\dot{E}_3$ relative to it are the same as the coordinates of that point relative to $\dot{c}_{\{3\}}$, and that, in addition, a vector of $\hat{E}_3$ is expandable into the basis vectors of $c_{\{3\}}$, while it is not expandable into the basis vectors of $\dot{c}_{\{3\}}$, Therefore, in fact, a certain $c_{\{3\}}$, and not and $\dot{c}_{\{3\}}$, is used. At the same time, $\dot{c}_{\{1\}}$ is a «*clock*» whose origin (initial instant of time) $\dot{0}_{\{1\}}$, denoted also by '$\dot{\theta}$', is a certain fixed point of the underlying set $\dot{T}$ of $\dot{\boldsymbol{T}}(\boldsymbol{R})$ and whose basis is the unit *position vector* $\vec{\tau}$, directed from past to future, which is the bijective image in the power set $\mathsf{P}(\dot{T})$ of the *unit free basis vector* $\hat{\tau}$ belonging to the underlying set $\hat{T}$ of a certain 1-*dimensional real abstract vector Euclidean space* $\hat{\boldsymbol{T}}(\boldsymbol{R})$ that is called the *abstract vector space adjoint of* $\dot{\boldsymbol{T}}(\boldsymbol{R})$. Therefore, besides $\dot{c}_{\{1\}}$, there is a much simpler and much more effective coordinate system $c_{\{1\}}$, which consists of the same origin $\dot{0}_{\{1\}}$, or $\dot{\theta}$, belonging to $\dot{T}$ and of the vector $\hat{\tau}$ belonging to $\hat{T}$, and which has the properties that the coordinate of any point of $\dot{T}$ relative to $c_{\{1\}}$ is the same as the coordinate of that point relative to $\dot{c}_{\{1\}}$ and that, in addition, any vector of $\hat{T}$ is expandable into $\hat{\tau}$, while it is not expandable into $\vec{\tau}$. Therefore, $c_{\{1\}}$, and not $\dot{c}_{\{1\}}$, is, in fact, used. The «clocks» $\dot{c}_{\{1\}}$ and $c_{\{1\}}$ are alternatively denoted by '$\dot{\omega}$' and 'ω' respectively, the understanding being that mnemonically 'ω' is the first letter of the Greek noun '*ωρολόγιον*' \orológion\ meaning *a clock*. A global coordinate system in $\dot{\boldsymbol{T}}(\boldsymbol{R})$ is called a *system of chronology* or briefly *chronology*.

4) In order to continue this discussion conveniently, I shall summarize and generalize the above notation as follows.

i) $\boldsymbol{R}$ is the field of real numbers.

ii) R is the underlying set of $\boldsymbol{R}$, so that $R = (-\infty, \infty)$.

iii) $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, or briefly $\dot{\boldsymbol{E}}_n$, is an *n*-dimensional Euclidean affine (point) space over $\boldsymbol{R}$, called also a Euclidean real affine *n*-space.

iv) $\dot{E}_n$ is the underlying set of points of $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$.

v) $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, or briefly $\hat{\boldsymbol{E}}_n$, is an *n*-dimensional Euclidean abstract linear, or abstract vector, space over $\boldsymbol{R}$, adjoint of $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, called also a Euclidean real abstract linear, or abstract vector, *n*-space.

vi) $\hat{E}_n$ is the underlying set of real abstract vectors of $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$.

vii) $\overline{\boldsymbol{E}}_n(\boldsymbol{R})$, or briefly $\overline{\boldsymbol{E}}_n$, is an *n*-dimensional Euclidean arithmetical vector space over $\boldsymbol{R}$, isomorphic to $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, called also a Euclidean real arithmetical vector *n*-space.

viii) $\overline{E}_n$ is the underlying set of real arithmetical vectors of $\overline{\boldsymbol{E}}_n(\boldsymbol{R})$.

ix) $\dot{\boldsymbol{T}}(\boldsymbol{R})$, $\hat{\boldsymbol{T}}(\boldsymbol{R})$, $\overline{\boldsymbol{T}}(\boldsymbol{R})$, $\dot{T}$, $\hat{T}$, and $\overline{T}$ are time-relevant interpretations of $\dot{\boldsymbol{E}}_1(\boldsymbol{R})$, $\hat{\boldsymbol{E}}_1(\boldsymbol{R})$, $\overline{\boldsymbol{E}}_1(\boldsymbol{R})$, $\dot{E}_1$, $\hat{E}_1$, and $\overline{E}_1$ respectively.

5) As far as modern mathematical analysis, including differential and integral calculi, is concerned, this is from the very beginning developed for real-valued or complex-valued functional forms or, in other words, for the associated functions of such forms, while the real-valued functional forms, e.g., are defined in $\overline{E}_n$ with various natural $n \geq 1$. In this case, rigorous relations between $\overline{E}_n$ and $\dot{E}_n$ are not, as a rule, explicated. Moreover, I dare suggest that the book *Analyse mathtématiǫue* by Laurent Schwartz [1967, Part I, chapter 1III, §1] is the only exposition, in which the necessity of developing differential and integral calculi from Euclidean affine spaces at scratch was made explicit by the very fact of including a theory of affine spaces under the heading «*Differential calculus*» of chapter III of Part I. There is a translation of this book into Russian, but it has not, likely, been translated into English. As follows from its subtitle and also from the fact that all its formulas are handwritten, Schwartz's book is a collection of handouts of his lectures, which leaves its mark of sloppiness in the interpretation of some relevant notations and in the organization of the pertinent subject matter of the book. Still, broadly speaking and putting aside some minor inconsistencies, Schwartz employs a system of conventional *geometric* notations to develop an original *algebraic* theory of Euclidean affine spaces along with some *functional* relations relevant to differential calculus. At the same time, the conventional geometric notations employed and extended by Schwartz are *visual* and in their most part they are nearly *iconographic* (*pictographic*) and not pure *symbolic* (*ideographic*) that should be associated with the pertinent denotata by *abstract association*. Although some elements of his system of notation are unavoidably equivocal, the notation as a whole is convenient and mnemonically

justified. In fact, however, the conventional visual geometric notations can be employed only if they are used for construction of a pure *algebraic* theory of affine spaces. In developing a rigorous functional theory of affine spaces, these notations turn out to be cumbersome and inflexible and therefore inappropriate. This is likely the reason that has prevented Schwartz from explicating certain fundamental functional relations between, say, a function that is defined in $\overline{E}_n$ or in $\overline{T}\times\overline{E}_n$ and its successive predecessors that are defined first in $\dot{E}_n$ or in $\dot{T}\times\dot{E}_n$ and then in $\hat{E}_n$ or in $\hat{T}\times\hat{E}_n$ – the relations, which should turn a theory of Euclidean affine spaces into an underlying discipline of differential and integral calculus. Some peculiarities of Schwartz's theory and his system of notation, which illustrate the above said, are explicated below with the help of the pertinent symbolic notation of this exposition.

6) Schwartz defines an affine space by two axioms *in that order*: 1º) the *Chasle law* and 2º) the *law of a bijection* $\{a\}\times\dot{E}_n\to\hat{E}_n$, where *a* is a fixed point of $\dot{E}_n$. In this case, the vector of $\hat{E}_n$, being the result of the concrete act of the bijection from the ordered pair of points *a* and *x* of $\dot{E}_n$ into $\hat{E}_n$ is denoted in the book by '$\overrightarrow{ax}$', so that the overarrow is tacitly turned out to be the *functional constant* that denotes the bijection itself, i.e. $\rightarrow:\{a\}\times\dot{E}_n\to\hat{E}_n$ At the same time, the bijection $\{a\}\times\dot{E}_n\to\hat{E}_n$ is a *restriction* of the surjection $\dot{E}_n\times\dot{E}_n\to\hat{E}_n$ and conversely the latter is the *ultimate extension* (*continuation*) of the former. Accordingly, the surjection can, more naturally, be postulated instead of the bijction. In agreement with this fact but without mentioning it and without mentioning the fact of the very existence of the surjection, the Chasle law is written in the Schwartz book as $\overrightarrow{ab}+\overrightarrow{bc}+\overrightarrow{ca}=\vec{0}$, where *a, b,* and *c* are assumed to be arbitrary points of $\dot{E}_n$, whereas $\overrightarrow{ab}$, $\overrightarrow{bc}$, $\overrightarrow{ca}$, and $\vec{0}$ are stated to be arbitrary vectors of $\hat{E}_n$ and $\vec{0}$ to be the null-vector of $\hat{E}_n$. Hence, in these occurrences an overarrow equivocally denotes the above surjection, i.e. $\rightarrow:\dot{E}_n\times\dot{E}_n\to\hat{E}_n$. The two different functions, which are thus equivocally denoted by '$\rightarrow$', are syntactically indistinguishable. In addition, Schwartz equivocally interprets $\overrightarrow{ab}$ (e.g.) as the vector with the initial point *a* and terminal point *b*, i.e. as the *position vector* of the point *b* relative to the point *a*. Therefore, he alternatively denotes $\overrightarrow{ab}$ by '$\overrightarrow{b-a}$'. However, elements of $\hat{E}_n$ are *free* and only *free* vectors,

being *translations* of $\dot{E}_n$, whereas a position vector is an element of the power set $\mathrm{P}(\dot{E}_n)$, i.e. an unmovable manifold (subset) of the set $\dot{E}_n$. The surjection $\vec{\ }:\{a\}\times\dot{E}_n\to\hat{E}_n$ implies the inverse bijectiion $\{a\}\times\hat{E}_n\to\dot{E}_n$, which Schwartz equivocally denotes by the sign '+', the same that he uses for denoting the binary composition operation $\hat{E}_n\times\hat{E}_n\to\hat{E}_n$ of vectors in $\hat{E}_n$. That is to say, if *a* is a given point of $\dot{E}_n$ and $\vec{h}$ is a vector of $\hat{E}_n$ then $a+\vec{h}$ is a point of $\dot{E}_n$. To compare, in accordance with the system of symbolic (ideographic) notation that I use in this exposition, the surjection $\vec{\ }:\dot{E}_n\times\dot{E}_n\to\hat{E}_n$ is denoted by '$\hat{V}$', the bijection $\vec{\ }:\{a\}\times\dot{E}_n\to\hat{E}_n$ by '$\hat{V}_a$', and its inverse by '$\hat{V}_a^{-1}$'.

7) Schwartz is not the only mathematician who treats the bijection $\{a\}\times\hat{E}_n\to\dot{E}_n$ as the addition of a vector in $\hat{E}_n$ to a point in $\dot{E}_n$ that results in another point in $\dot{E}_n$. For instance, Mac Lane and Birkhoff [1967, p. 420] employ a like notation to suggest the following essentially different definition of an affine space.

> «DEFINITION. An affine space *P* over *F* is a non-void set for which there exists a finite-dimensional vector space *V* and a function $V\times P\to P$, written as $(v,p)\mapsto v+p$, such that
>
> (i) For all vectors $v,w\in V$ and all points $p\in P$,
>
> $$0+p=p,\qquad (v+w)+p=v+(w+p).$$
>
> (ii) For any two points $p,q\in P$ there is exactly one vector $v\in V$ with
>
> $$v+q=p.$$
>
> The dimension of *P* is the vector space dimension of *V*.»

The surjection $+:\dot{E}_n\times\hat{E}_n\to\dot{E}_n$ that is implicitly defined by axiom (ii) is the extension of analogous to the extension of Schwartz's axiom 2^{o}), but axiom (i) is completely different from Schwartz's axiom 1^{o}). In the first identity of axiom (i), 0 is evidently the null-vector of *V*. The second identity of axiom (i) is a peculiar associative law for the sign '+' with the proviso that the function denoted by the first occurrence of '+' on the left-hand side of this identity is completely different from the function denoted by the first occurrence of '+' on the right-hand side of this identity.

8) In this exposition, a system of the appropriate symbolic (ideographic, not iconographic, pure abstract) notation is employed for developing a rigorous an *algebraico-*

functional theory (*AAFT*) of an *n*-dimensional real affine Euclidean space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ is developed from an *affine additive group* (*AAG*) as has been described in Abstract.

9) An affine space $\dot{\boldsymbol{E}}(\boldsymbol{R})$ is one of the most complex algebraic systems, which involves a few simpler algebraic systems. In this exposition, I suggest and employ a consistent method of naming algebraic systems instead of the two presently common ones, because these are inconsistent as explicated below. In order to maintain distinction between an algebraic system and its underlying set formally, many writers on mathematics construe the former as an ordered multiple, the first coordinate of which is the underlying set of the system and the other coordinates are some or all other attributes of the system. Accordingly, the symbol of the ordered multiple is used as a name of the algebraic system (see, e.g., MacLane and Birkhoff [1967, pp. 61, 63, 118, etc]). This method of naming algebraic systems is, however, never used systematically, because it leads to insuperable notational conflicts and turns out to be paradoxical after all. Indeed, an ordered *n*-tuple with $n \geq 2$ is an $(n–1)$-fold reiterated ordered pair defined as

$$\langle x_1, x_2, ..., x_{n-1}, x_n \rangle \doteq \underbrace{\langle\langle..}_{n-1}\langle x_1, x_2\rangle, x_3\rangle, ..., x_{n-1}\rangle, x_n\rangle \tag{1.3}$$

subject to (1.1) (cf. (1.2)) and therefore it is a complicated composite set whose complexity rapidly increases with *n*. Therefore, even most basic and simplest set theoretic relations such as a relation of belonging an element (as a vector) to a given algebraic system (as a vector space) or a relation of inclusion between an algebraic system and its subsystem (as that between a vector space and some one of its subspaces) are inexpressible in terms of ordered multiple names as names of algebraic systems. Following the above method, the vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ should have been denoted, for instance, by the ordered quadruple name ‘$\langle \hat{E}, \boldsymbol{R}, \hat{+}, \hat{-}, \hat{\cdot} \rangle$’, or $\langle$‘$\hat{E}$’, ‘$\boldsymbol{R}$’, ‘$\hat{+}$’, ‘$\hat{-}$’, ‘$\hat{\cdot}$’$\rangle$, that contains as its constituents the name ‘$\hat{E}$’ of the underlying set of $\hat{\boldsymbol{E}}(\boldsymbol{R})$, the name ‘$\boldsymbol{R}$’ of the field of real numbers, and the names ‘$\hat{+}$’, ‘$\hat{-}$’, and ‘$\hat{\cdot}$’ of three operations, of which $\hat{+}$ is the binary operation of addition of vectors of $\hat{E}$, $\hat{-}$ the singulary operation of additive inversion of a vector of $\hat{E}$, and $\hat{\cdot}$ is the binary operation of multiplication of a scalar of R and a vector of $\hat{E}$ in either order. In this case, in order to be consistent in denoting algebraic systems by using ordered multiple names, $\boldsymbol{R}$ should in turn be denoted by the ordered multiple name, whose coordinates are the name ‘R’ of the underlying set of $\boldsymbol{R}$ and also the logographic names of all algebraic operations on R. By (1.3),

$$\langle \hat{E}, \boldsymbol{R}, \hat{+}, \hat{-}, \hat{\cdot} \rangle \doteqdot \langle\langle\langle\langle \hat{E}, \boldsymbol{R} \rangle, \hat{+} \rangle, \hat{-} \rangle, \hat{\cdot} \rangle ,$$

and therefore '$\langle \hat{E}, \boldsymbol{R}, \hat{+}, \hat{-}, \hat{\cdot} \rangle$' is actually a name of an extremely complicated set, which has nothing to do with a vector space. This is why I do not use ordered multiple names as names of algebraic systems and prefer to indicate «*togetherness*» of the sets (including both the underlying sets and the functions), forming an algebraic system, as the union of those sets, provided of course that they do not mutually intersect. For the same reason, I do not follow the popular method, according to which a relation in general and a function, i.e. a functional (single-valued) relation, in particular is considered as the ordered triple of the graph, domain of definition, and domain of variation (or domain of arrival) of the relation and is denoted accordingly (see, e.g., Bourbaki [1960, chapter II, §3]).

10) Use of names of ordered multiples as names of algebraic systems or of relations is not only inconsistent logically, but it is also paradoxical psychologically. For instance, if '$\langle \hat{E}, \boldsymbol{R}, \hat{+}, \hat{-}, \hat{\cdot} \rangle$' is used as a name of an *n*-dimensional real vector space then the underlying set $\hat{E}$ of vectors, the field $\boldsymbol{R}$, and the operations (functions) $\hat{+}$, $\hat{-}$, and $\hat{\cdot}$ of the vector space are *simultaneous denotata* of the constituent names '$\hat{E}$', '+', '·', '-' of the above ordered multiple name, i.e. they are *simultaneous objects* of an interpreter of that name, while the field $\boldsymbol{R}$ and the operations $\hat{+}$, $\hat{-}$, and $\hat{\cdot}$ are in fact *conceptual properties of elements of* $\hat{E}$ with respect to the interpreter rather than to be his objects simultaneous with $\hat{E}$. Therefore, there is in use an alternative method of naming an algebraic system, according to which the name of the dominant underlying set of that system is equivocally used as a name of the system itself, while properties of elements of the underlying set, – such properties, e.g., as functions, – are hided as connotata (connotations values) of that name. For instance, MacLane and Birkhoff [1967] say: «Hence a *group G* is a set *G* together with the binary operation $G\times G\to G$, written $(a,b)\mapsto ab$, such that …» (*ibid.* p. 71) and also «A *ring* $R=(R,+,\cdot,1)$ is a set *R* with two binary operations, addition and multiplication, and a nullary operation, "select 1", such that … » (*ibid.* p. 118). According to this onomatological method, a logographic symbol such as '$\hat{E}$', which is initially is introduced, e.g., as a name of the underlying set of a commutative abstract additive group $\hat{\boldsymbol{E}}^{\mathrm{g}}$, becomes after all a homograph (homographonym, homonym) of the group. Consequently, if $\hat{\boldsymbol{E}}^{\mathrm{g}}$ is developed so as to become an abstract vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ then '$\hat{E}$' becomes after all a homonym of $\hat{\boldsymbol{E}}(\boldsymbol{R})$. Such equivocality of '$\hat{E}$' is confusing and hence inacceptable. Therefore, the latter onomatological method is not used in this treatise either.

2. Linear (vector) spaces

2.1. Underlying meta-definitions

For convenience in the subsequent discussion, I begin from restating Definitions 2.1, 2.4, and 2.5 and Comment 2.1 of Iosilevskii [2015] as the following Definitions 2.1–2.3 and Comment 2.1 respectively.

Definition 2.1. 1) The signs $\overset{\rightarrow}{=}$ and $\overset{\leftarrow}{=}$ are indiscriminately called *the asymmetric*, or *one-sided*, *equality signs by definition* or, discriminately, *the rightward equality sign by definition* and *the leftward equality sign by definition* respectively. A binary figure, in which either sign $\overset{\rightarrow}{=}$ or $\overset{\leftarrow}{=}$ is used *assertively*, is called a *formal binary asymmetric synonymic definition* (*FBASD*). In making a FBASD, at the head of an arrow I shall write the *material definiens* – the substantive (substance-valued expression), which is already known either from a previous definition or from another source; at the base of the arrow I shall write the *material definiendum* – the new substantive, which is being introduced by the definition and which is designed to be used instead of or interchangeably with the definiens *in the scope of the FBASD*. Therefore, the sign $\overset{\rightarrow}{=}$ is rendered into ordinary language thus: "*is to stand as a synonym for*" or straightforwardly "*is the synonymous definiendum of*", and $\overset{\leftarrow}{=}$ thus: "*can be used instead of interchangeably with*" or straightforwardly "*is the synonymous definiens of*". The [material] definiendum and [material] definiens of a FBASD are indiscriminately called the *terms* of the definition. Neither the definiendum nor the definiens of an FBASD should involve any *function symbols*, particularly any outermost (enclosing) quotation marks, that are not their constituent parts and that are therefore used but not mentioned with the following proviso. If it is necessary to indicate the integrity of the definiendum or of the definiens then that term of the definition can be enclosed in *square brackets as metalinguistic punctuation marks*, which do not, by definition, belong to the enclosed term and which are therefore used but not mentioned. In the scope of a FBASD, which does not include the FBASD itself, tokens of the terms of the FBASD can be related by the ordinary *reflexive*, *symmetric*, and *transitive sign of equality* =. In contrast to =, either sign $\overset{\rightarrow}{=}$ or $\overset{\leftarrow}{=}$ is *transitive*, but *not reflexive* and *not symmetric*.

2) In order to state formally that two old or two new substantives are to be used interchangeably (synonymously), I shall write the substantives, without any quotation marks that are not their constituent parts, in either order on both sides of the two-sided sign $\overset{\leftrightarrow}{=}$. Such a relation is called a *formal binary symmetric synonymic definition* (*FBSSD*), whereas the sign

$\doteqdot$ is called the *symmetric*, or *two-sided, equality sign by definition*. In this case, $\doteqdot$ is read as "*is to be concurrent to*" or, alternatively, **"**—$\doteqdot$ …**"** is read as **"**— *and … are to be concurrent***"** or as **"**— *and … are to be synonyms***"**, where alike ellipses should be replaced alike. In the scope of an FBSSD, tokens of the terms of the FBSSD can be related by the ordinary sign of equality =.

3) In stating synonymic definitions of substantives, the arrows →, ←, and ↔ can be used instead of $\risingdotseq$, $\fallingdotseq$, and $\doteqdot$ respectively, the understanding being that the arrows are general definition signs, which can apply to relations and not only to subctantives.•

Definition 2.2. 1) 'ω_0' *denotes*, i.e. ω_0 *is*, the set of all *natural numbers* from 0 to infinity.

2) Given $n \in \omega_0$,

$$\omega_n \risingdotseq \{i | i \in \omega_0 \text{ and } i \geq n\}, \tag{2.1}$$

i.e. 'ω_1', 'ω_2', etc *denote* the sets of natural numbers from 1, 2, etc respectively to infinity.

3) Given $m \in \omega_0$, given $n \in \omega_m$,

$$\omega_{m,n} \risingdotseq \{i | i \in \omega_0 \text{ and } n \geq i \geq m\}, \tag{2.2}$$

i.e. '$\omega_{m,n}$' *denotes* the set of natural numbers from a given number *m* to another given number *n* subject to *n*≥*m*. It is understood that

$$\omega_{m,m} = \{m\},\ \omega_{m,\infty} = \omega_m,\ \omega_{m,n} = \varnothing \text{ if } m \geq n. \tag{2.3}$$ •

Definition 2.3. 1) '$I_{-\infty,\infty}$' *denotes*, i.e. $I_{-\infty,\infty}$ *is*, the set of all *natural integers* (*natural integral numbers*) – strictly positive, strictly negative, and zero.

2) Given $n \in I_{-\infty,\infty}$,

$$I_{n,\infty} = I_{\infty,n} \risingdotseq \{i | i \in I_{-\infty,\infty} \text{ and } i \geq n\}, \tag{2.4}$$

$$I_{-\infty,n} = I_{n,-\infty} \risingdotseq \{i | i \in I_{-\infty,\infty} \text{ and } i \leq n\}, \tag{2.5}$$

i.e. $I_{n,\infty}$ or $I_{\infty,n}$ is the set of all natural integers greater than or equal to *n*, and $I_{-\infty,n}$ or $I_{n,-\infty}$ is the set of all natural integers less than or equal to *n*.

3) Given $m \in I_{-\infty,\infty}$: given $n \in I_{m,\infty}$:

$$I_{m,n} \risingdotseq \{i | i \in I_{-\infty,\infty} \text{ and } n \geq i \geq m\}, \tag{2.6}$$

i.e. $I_{m,n}$` is the set of all natural integers that are greater than or equal to *m* and less than or equal to *n*.•

Comment 2.1. Definitions 2.1(1) and 2.2(1) are *explicative* ones. A theory of natural integers in particular, and a theory of any numbers (as rational, real, or complex ones) in general can consistently be deduced from the five Peano axioms, which are, in turn, theorems of an axiomatic set theory (see, e.g., Halmos [1960, pp. 46–53], Burrill [1967], Feferman [1964]).•

2.2. An *n*-dimensional Euclidean linear (vector) space via its underlying algebraic systems

Definition 2.4. 1) A *commutative*, or *Abelian*, [*abstract*] *additive group* $\hat{\boldsymbol{E}}^{\mathrm{g}}$, called also a *linear*, or *vector*, *group,* is an underlying set $\hat{E}$ of its elements, which may sometimes be identified with $\hat{\boldsymbol{E}}^{\mathrm{g}}$, *together* with two *primary* (*postulated*) *functions*: a *surjective commutative* (*symmetrical*) *and associative binary addition function* $\hat{+}:\hat{E}\times\hat{E}\to\hat{E}$ and a *bijective singulary additive inversion function* $\hat{-}:\hat{E}\to\hat{E}$ with respect to *the null* (*additive identity*) *element* $\hat{0}\in\hat{E}$. Elements of $\hat{E}$, called *vectors*, are denoted by the *variables* '$\hat{x}$', '$\hat{y}$', and '$\hat{z}$', any of which can be furnished with an Arabic numeral subscript '$_1$', '$_2$', etc or with any other label (as an asterisk or any number of primes) or with both, thus becoming another variable with the same range. The primary functions of $\hat{\boldsymbol{E}}^{\mathrm{g}}$ satisfy the following axioms, called the *Commutative Additive Group Axioms* (*CAGA's*):

CAGA1: The closure law. For each $(\hat{x},\hat{y})\in\hat{E}\times\hat{E}$: there is exactly one $\hat{z}\in\hat{E}$ such that $\hat{z}=\hat{x}\,\hat{+}\,\hat{y}$.

CAGA2: The associative law. For each $((\hat{x},\hat{y}),\hat{z})\in\lfloor\hat{E}\times\hat{E}\rfloor\times\hat{E}$ *:*

$$\hat{x}\,\hat{+}\,(\hat{y}\,\hat{+}\,\hat{z})=(\hat{x}\,\hat{+}\,\hat{y})\,\hat{+}\,\hat{z}\,. \tag{2.7}$$

CAGA3: The identity law. There exists a unique element $\hat{0}\in\hat{E}$, which is called *the null*, or *additive identity, element of* $\hat{E}$, such that for $\hat{x}\in\hat{E}$ *:*

$$\hat{0}\,\hat{+}\,\hat{x}=\hat{x}\,\hat{+}\,\hat{0}=\hat{x}\,. \tag{2.8}$$

*CAGA4: The additive inverse law***.** For each $\hat{x}\in\hat{E}$: there is exactly one element $\hat{-}\,\hat{x}\in\hat{E}$, which is called *the additive inverse*, or *additive reciprocal*, or *opposite of* $\hat{x}$, such that

$$\hat{x}\,\hat{+}\,(\hat{-}\,\hat{x})=(\hat{-}\,\hat{x})\,\hat{+}\,\hat{x}=\hat{0}\,. \tag{2.9}$$

CAGA5: The commutative (symmetrical) law. For each $(\hat{x},\hat{y})\in\hat{E}\times\hat{E}$:

$$\hat{x}\,\hat{+}\,\hat{y}=\hat{y}\,\hat{+}\,\hat{x}\,. \tag{2.10}$$

2) Besides the above primary functions, there is in $\hat{\boldsymbol{E}}^{g}$ a *secondary* (*defined*, *composite*) *surjective binary subtraction function* $\hat{-}:\hat{E}\times\hat{E}\to\hat{E}$ such that for each $(\hat{x},\hat{y})\in\hat{E}\times\hat{E}$:

$$\hat{x}\hat{-}\hat{y}\doteq\hat{x}\hat{+}(\hat{-}\hat{y}); \tag{2.11}$$

i.e. $\hat{-}\doteq\hat{+}\circ\hat{-}$, where ‘ $\circ$ ’ denotes the operation of composition of functions.

3) Here, and generally in what follows, «*togetherness*» as stated in the item 1 is understood as *the union of the pertinent sets* (*regular classes*, *small classes*), so that $\hat{\boldsymbol{E}}^{g}$ can formally be defined as:

$$\hat{\boldsymbol{E}}^{g}\doteq\hat{E}\cup\hat{+}\cup\hat{-}. \tag{2.12}$$ •

Definition 2.5. The field $\boldsymbol{R}$ of real numbers is the underlying set R of real numbers, which may sometimes be identified with $\boldsymbol{R}$. *together* with the following *primary* (*postulated*) functions: a *surjective commutative and associative binary addition function* $+:R\times R\to R$, a *surjective commutative and associative binary multiplication function* $\cdot:R\times R\to R$ that is *distributive over* + *relative to* =, a *bijective singulary additive inversion function* $-:R\to R$ with respect to *the null* (*additive identity*) *element* $0\in R$, and a *bijective singulary multiplicative inversion function* $^{-1}:R-\{0\}\to R-\{0\}$ with respect to *the unity* (*multiplicative identity*) *element* $1\in R$. In addition, there are in $\boldsymbol{R}$ two *secondary* (*defined*, *composite*) functions, namely a *surjective binary subtraction function* $-:R\times R\to R$, defined as: $-\doteq+\circ-$, and a *binary division function* $/:R\times[R-\{0\}]\to R$, defined as: $/\doteq\cdot\circ{}^{-1}$. Elements of R, called *scalars*, are denoted by small italic letters of the Latin alphabet without any overscript, as ‘*a*’, ‘*b*’, ‘*c*’, etc, any of which can be furnished with an Arabic numeral subscript ‘$_1$’, ‘$_2$’, etc or with any other label (as an asterisk or any number of primes) or with both, thus becoming another *variable* with the same range.•

Definition 2.6. 1) An *abstract* (*not arithmetical*) *linear*, or *vector*, *space* $\hat{\boldsymbol{E}}(\boldsymbol{R})$ or briefly $\hat{\boldsymbol{E}}$ over the field $\boldsymbol{R}$ of *real* numbers, called also a *real* abstract linear (vector) space, is a *commutative* (*Abelian*) *additive group* $\hat{\boldsymbol{E}}^{g}$ of elements, which are comprised in its underlying set $\hat{E}$ and which are called *vectors*, *together* with $\boldsymbol{R}$ and also together with an additional *surjective commutative and associative binary function* (*operation*) $\hat{\cdot}:[R\times\hat{E}]\cup[\hat{E}\times R]\to\hat{E}$ *of multiplication of a scalar by a vector* or *of a vector by a scalar*. The latter function is interrelated with the functions $\hat{+}$, +, and $\cdot$ by the following axioms that are called the *Vector Space Supplementary Axioms* (*VSSA’s*):

VSSA1: The closure and symmetry (*commutative*) *law*. For each $(\hat{x},a)\in\hat{E}\times R$, there is exactly one $\hat{y}\in\hat{E}$ such that

$$\hat{y}=a\,\hat{\cdot}\,\hat{x}=\hat{x}\,\hat{\cdot}\,a. \tag{2.13}$$

VSSA2: The distributive laws over $\hat{+}$ *and* +. For each $(\hat{x},\hat{y})\in\hat{E}\times\hat{E}$, for each $(a,b)\in R\times R$:

$$a\,\hat{\cdot}\,(\hat{x}\,\hat{+}\,\hat{y})=a\,\hat{\cdot}\,\hat{x}\,\hat{+}\,a\,\hat{\cdot}\,\hat{y}, \tag{2.14}$$

$$(a+b)\,\hat{\cdot}\,\hat{x}=a\,\hat{\cdot}\,\hat{x}\,\hat{+}\,b\,\hat{\cdot}\,\hat{x}, \tag{2.15}$$

VSSA3: The combined associative law. For each $\hat{x}\in\hat{E}$, for each $(a,b)\in R\times R$:

$$(a\cdot b)\,\hat{\cdot}\,\hat{x}=a\,\hat{\cdot}\,(b\,\hat{\cdot}\,\hat{x}). \tag{2.16}$$

VSSA4: The identity law for scalar multiplication. For each $\hat{x}\in\hat{E}$:

$$1\,\hat{\cdot}\,\hat{x}=\hat{x}. \tag{2.17}$$

2) In analogy with (2.12), «*togetherness*» as stated in the item 1 means that $\hat{\boldsymbol{E}}$ can formally be defined as:

$$\hat{\boldsymbol{E}}\doteqdot\hat{\boldsymbol{E}}(\boldsymbol{R})\doteqdot\hat{\boldsymbol{E}}^{\mathrm{g}}\cup\boldsymbol{R}\cup\hat{\cdot}\,. \tag{2.18}$$

subject to VSSA1–VSSA4. The set $\hat{E}$ is called the *principal*, or *major*, *underlying set of* $\hat{\boldsymbol{E}}$, while the set R is called the *minor underlying set of* $\hat{\boldsymbol{E}}$.•

Definition 2.7. Given $n\in\omega_1$, an *n-dimensional projective* (*not metric*, *not Euclidean*) *abstract* (*not arithmetical*) *linear* (*vector*) *space* $\hat{\boldsymbol{E}}_n^{\mathrm{p}}(\boldsymbol{R})$ or briefly $\hat{\boldsymbol{E}}_n^{\mathrm{p}}$ over the field $\boldsymbol{R}$ is an abstract linear space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ or $\hat{\boldsymbol{E}}$ together with an additional *axiom of the dimension of* $\hat{\boldsymbol{E}}(\boldsymbol{R})$. According to this axiom, $\hat{\boldsymbol{E}}(\boldsymbol{R})$ and hence its underlying set $\hat{E}$ has at most n linearly independent vectors (to be explicated in the subsection 2.5 below), in terms of which any other vector of $\hat{\boldsymbol{E}}(\boldsymbol{R})$ can be expressed.•

Definition 2.8. A *metric* (*inner product*, *Euclidean*) *abstract* (*not arithmetical*) *linear* (*vector*) *space* $\hat{\boldsymbol{E}}^{\mathrm{m}}(\boldsymbol{R})$ or briefly $\hat{\boldsymbol{E}}^{\mathrm{m}}$ over the field $\boldsymbol{R}$ is an *abstract linear space* $\hat{\boldsymbol{E}}(\boldsymbol{R})$ or $\hat{\boldsymbol{E}}$ *together* with an additional *axiom inner product of vectors of* $\hat{\boldsymbol{E}}(\boldsymbol{R})$. According to this axiom, there is in $\hat{\boldsymbol{E}}(\boldsymbol{R})$ a *commutative* (*symmetrical*) and *associative* (*distributive*) *binary function of inner multiplication of vectors* $\hat{\bullet}:\hat{E}\times\hat{E}\to R$, which is *positively definite* in the sense that for each $\hat{x}\in\hat{E}$:

$$\hat{x}\hat{\bullet}\,\hat{x} > 0 \text{ if } \hat{x} \neq \hat{0} \text{ or } \hat{x}\hat{\bullet}\,\hat{x} = 0 \text{ if } \hat{x} = \hat{0}. \qquad (2.19)\bullet$$

Definition 2.9. Given $n \in \omega_1$, an n-dimensional abstract (not arithmetical) linear (vector) Euclidean (metric, inner product) space $\hat{\boldsymbol{E}}_{\{n\}}(\boldsymbol{R})$ or briefly $\hat{\boldsymbol{E}}_{\{n\}}$ over the field $\boldsymbol{R}$ is an abstract linear (or vector) space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ or briefly $\hat{\boldsymbol{E}}$ over the field $\boldsymbol{R}$ together with both the axiom of the dimension of $\hat{\boldsymbol{E}}(\boldsymbol{R})$ and the axiom inner product of vectors of $\hat{\boldsymbol{E}}(\boldsymbol{R})$. Thus, equivalently, $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ is $\hat{\boldsymbol{E}}_n^{\mathrm{p}}(\boldsymbol{R})$ together with the later axiom or $\hat{\boldsymbol{E}}^{\mathrm{m}}(\boldsymbol{R})$ together with the former axiom. For more clarity, the underlying set of vectors of $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ will be denoted by '$\hat{E}_n$' so that $\hat{E}_n$ may sometimes be identified with $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$.•

Comment 2.2. Conventional definitions of all algebraic systems that has been mentioned above in this subsection can be found, e.g., in Birkhoff and Mac Lane [1965].•

2.3. Ordered *n*-tuples

2.3.1. General remarks

1) Besides the sets of *natural*, *integer* (*integral*), *rational*, *real*, and *complex numbers*, which are denoted by '*N*', '*I*', '*Q*', '*R*', and '*C*' in that order and which are called *scalars*, and also besides various algebraic systems as those mentioned in the previous subsection, mathematics and physics (especially theoretical physics) deal with *hypernumbers* of various kinds (classes) such as quaternions, tensors of various valences, and matrices. A hypernumber is *synecdochically* called a *holor* (from the Greek adjective "ὅλος" \ólos\ meaning *all* or *the whole*), the understanding being that a holor is generally a conceptual object that consists of several elements of a certain class (set) or certain classes (sets), which are called the *merates* (from the Greek noun "μέρος" \méros\ meaning *a part*), and also *coordinates* or *components*, of the holor (see, e.g. Moon and Spencer [1965, pp. 1, 14]). In this case, a hypernumber is a holor whose merates are numbers of a certain set and therefore it can alternatively (synonymously) be called a *numeric holor.* Particularly, a complex number is in fact a *two-component holor of real numbers*. However, besides numbers, merates of a holor can, e.g., be *points*, *vectors*, or *other holors*. A holor is said to be *univalent*, *bivalent*, *trivalent*, *quadrivalent*, etc if its merates are labeled respectively with one, two, three, four, etc, subscripts or superscripts. A scalar is alternatively called a *nilvalent holor*. In any *conventional set theory*, an n-component univalent holor $\bar{x}_{[1,n]}$ of elements $x_1, x_2, ..., x_n$ of a

given set X in that order is called an *ordered n-tuple* of those elements and it is defined as a *repeated*, (n–1)-*fold ordered pair* such as

$$\begin{aligned}\bar{x}_{[1,n]} &\rightleftharpoons \langle x_i \rangle_{i\in\omega_{1,n}} \rightleftharpoons \langle x_1, x_2, ..., x_{n-1}, x_n \rangle \\ &\rightleftharpoons \langle \bar{x}_{[1,n-1]}, x_n \rangle = \underbrace{\langle\langle..}_{n-1}\langle x_1, x_2\rangle, x_3\rangle, ..., x_{n-1}\rangle, x_n\rangle.\end{aligned} \tag{2.20}$$

More specifically, an ordered n-tuple that is defined by the formula (2.20) is called *the left-associated repeated* (or *reiterative*) (n–1)-*fold* (or (n–1)-*ary*) *ordered pair of* x_1, x_2, …, x_n *in that order*.

2) In the general case, a single (simple) ordered pair $\langle x_1, x_2 \rangle$ of elements x_1 and x_2 of any given sets X_1 and X_2 in that order is by definition the set $\{\{x_1\}, \{x_1, x_2\}\}$, i.e.

$$\bar{x}_{[1,2]} \rightleftharpoons \langle x_1, x_2 \rangle \rightleftharpoons \{\{x_1\}, \{x_1, x_2\}\}, \tag{2.21}$$

(see, e.g., Halmos [1960, pp. 22–25]). Ordered pairs satisfy the theorem (*ibid.*) such that for any elements x_1, x_2, x_1', and x_2':

$$\langle x_1, x_2 \rangle = \langle x_1', x_2' \rangle \text{ if and only if } x_1 = x_1' \text{ and } x_2 = x_2'. \tag{2.22}$$

Accordingly, for any n elements x_1, x_2,…, x_n and for any n elements x_1', x_2', …, x_n':

$$\langle x_1, x_2, ..., x_n \rangle = \langle x_1', x_2', ..., x_n' \rangle \text{ if and only if } x_1 = x_1', x_2 = x_2', ..., x_n = x_n'. \tag{2.23}$$

Also, it is useful for making some general statements to introduce a *one-component univalent holor* – a conceptual object, which is denoted by '$\bar{x}_{[1,1]}$' or '$\langle x_1 \rangle$' and which can therefore be also called an *ordered one-tuple*, or *ordered single*, the understanding being that such an object *is distinct from a scalar* (*nilvalent holor*) and that it *can have a scalar as its only component*. For instance, an element of a one-dimensional arithmetical vector space $\bar{\boldsymbol{E}}_1^{\mathrm{p}}(\boldsymbol{R})$ or $\bar{\boldsymbol{E}}_1(\boldsymbol{R})$ over the field $\boldsymbol{R}$ of real numbers (scalars) is a *one-component univalent holor* (*ordered one-tuple*) $\langle x \rangle$ of a real number (scalar) x, which is not the real number itself. Without loss of generality, $\bar{x}_{[1,1]}$ or $\langle x_1 \rangle$ can be identified with the *singleton* $\{x_1\}$ – the set having x_1 as its only member (element), so that

$$\bar{x}_{[1,1]} \rightleftharpoons \langle x_1 \rangle = \{x_1\}. \tag{2.24}$$

At the same time, a set of n elements with $n \in \omega_2$ can alternatively be called an *unordered n-tuple*. Therefore, $\langle x_1 \rangle$ as defined by (2.24) can be regarded as *an ordered one-tuple and as an unordered one-tuple simultaneously*. An ordered n-tuple with any $n \in \omega_2$ is indiscriminately called an *ordered multiple*. Thus, for any $n \in \omega_1$ an ordered *n*-tuple, i.e. an *n*-component univalent holor, is a nonempty set and is *not a nonempty individual*. It is worthy of recalling that, in contrast to an ordered multiple, an ordered set is a set that serves as a *domain of definition of the liner order relation* (*predicate*) ≤.

3) An ordered pair $\langle x_1, x_2 \rangle$ is conventionally denoted as '(x_1, x_2)' and accordingly an ordered *n*-tuple $\langle x_1, x_2, ..., x_{n-1}, x_n \rangle$ is denoted as '$(x_1, x_2, ..., x_{n-1}, x_n)$'. Particularly, in the Clairaut-Euler placeholders '$f(x_1, x_2)$' and '$f(x_1, x_2, ..., x_{n-1}, x_n)$', '$(x_1, x_2)$' is a placeholder for an ordered pair, whereas '$(x_1, x_2, ..., x_{n-1}, x_n)$' is a placeholder for an ordered *n*-tuple. However, if x_1 and x_2 are real numbers then the symbol '(x_1, x_2)' is ambiguous, for it may stand either for the ordered pair of those numbers in that order or for the open interval (x_1, x_2). Therefore, in denoting ordered pairs and ordered multiples, I shall use angle brackets and round brackets interchangeably, while in most general conceptual statements preference will be given to the former without any comments.

2.3.2. Definitions

Definition 2.10. Given $n \in \omega_1$, given *n* sets X_1, X_2,…, X_n, the set of ordered *n*-tuples defined as:

$$\begin{aligned}\bigtimes_{i=1}^{n} X_i &\doteqdot \bigtimes_{i \in I_{1,n}} X_i \doteqdot X_1 \times X_2 \times ... \times X_{n-1} \times X_n = \\ &\doteqdot \left[\bigtimes_{i \in I_{1,n-1}} X_i\right] \times X_n \doteqdot \underbrace{[[...[}_{n-2} X_1 \times X_2] \times X_3] \times ...] \times X_{n-1}] \times X_n \\ &\doteqdot \left\{\langle x_1, x_2, ..., x_{n-1}, x_n \rangle \middle| x_1 \in X_1, x_2 \in X_2, ..., x_{n-1} \in X_{n-1}, x_n \in X_n \right\}\end{aligned} \tag{2.25}$$

subject to (2.20) is called *the left-associated repeated* (or *reiterative*) (*n*–1)-*fold* (or (*n*–1)-*ary*) *direct* (or *Cartesian*) *product of* X_1, X_2,…, X_n *in that order*.•

Definition 2.11. Given $n \in \omega_1$, given a set *X*, if $X_1 = X_2 = ... = X_n = X$, the set of ordered *n*-tuples defined as:

$$\begin{aligned}X^{n\times} &\doteqdot \underbrace{X \times X \times ... \times X \times X}_{n \text{ times } X} \\ &\doteqdot X^{(n-1)\times} \times X \doteqdot \underbrace{[[...[}_{n-2} X \times X] \times X] \times ...] \times X] \times X \\ &\doteqdot \left\{\langle x_1, x_2, ..., x_{n-1}, x_n \rangle \middle| x_1 \in X, x_2 \in X, ..., x_{n-1} \in X, x_n \in X \right\}\end{aligned} \tag{2.26}$$

subject to (2.20), i.e. *the left-associated repeated* (or *reiterative*) (n–1)-*fold* (or (n–1)-*ary*) *direct* (or *Cartesian*) *product of X by itself*, is called *the left-associated* nth *direct* (or *Cartesian*) *power of X*, the understanding being that

$$X^{1\times} \doteqdot \{\langle x_1\rangle | x_1 \in X\} = \{\{x_1\} | x_1 \in X\} \neq X \ . \qquad (2.27)\bullet$$

2.4. Repeated binary operations

Definition 2.12. 1) Given $m \in \omega_1$, let ξ_1, ..., ξ_m be any m objects, to which a binary operation $*$, denoted by the placeholder '$*$', applies repeatedly (iteratively) m–1 times in the successive order starting from ξ_1 and ξ_2. Then

$$\begin{aligned} \ast(\xi_1,\xi_2,...,\xi_{m-1},\xi_m) &\doteqdot \mathop{\ast}_{i=1}^{m}\xi_i \doteqdot \xi_1 * \xi_2 * ... * \xi_{m-1} * \xi_m \doteqdot \left[\mathop{\ast}_{i=1}^{m-1}\xi_i\right] * \xi_m \\ &\doteqdot \underbrace{[...[}_{m-2}[\xi_1 * \xi_2] * \xi_3] * ... * \xi_{m-2}] * \xi_{m-1}] * \xi_m \end{aligned} \qquad (2.28)$$

and in general

$$\begin{aligned} \ast(\xi_{j_1},\xi_{j_2},...,\xi_{j_{m-1}},\xi_{j_m}) &\doteqdot \mathop{\ast}_{i=1}^{m}\xi_{j_i} \doteqdot \xi_{j_1} * \xi_{j_2} * ... * \xi_{j_{m-1}} * \xi_{j_m} \doteqdot \left[\mathop{\ast}_{i=1}^{m-1}\xi_{j_i}\right] * \xi_{j_m} \\ &\doteqdot \underbrace{[...[}_{m-2}[\xi_{j_1} * \xi_{j_2}] * \xi_{j_3}] * ... * \xi_{j_{m-2}}] * \xi_{j_{m-1}}] * \xi_{j_m}, \end{aligned} \qquad (2.29)$$

where the sequence $\langle j_1, j_2,..., j_{m-1}, j_m\rangle$ is any permutation of the sequence $\langle 1,2,...,m-1,m\rangle$.

2) If the operation $*$ is *associative* and *commutative* then

$$\begin{aligned} \ast(\xi_{j_1},\xi_{j_2},...,\xi_{j_{m-1}},\xi_{j_m}) &= \mathop{\ast}_{i=1}^{m}\xi_{j_i} \doteqdot \xi_{j_1} * \xi_{j_2} * ... * \xi_{j_{m-1}} * \xi_{j_m} \\ &= \xi_1 * \xi_2 + ... * \xi_{m-1} * \xi_m = \mathop{\ast}_{i=1}^{m}\xi_i = \ast(\xi_1,\xi_2,...,\xi_{m-1},\xi_m). \end{aligned} \qquad (2.30)$$

3) The symbols '$\mathop{\ast}\limits_{i=1}^{m}$' and '$\ast_{i=1}^{i=m}$', e.g., can be used interchangeably.

4) '$*$' and '$\ast$' is a pair of *proportional* (*homolographic*) *placeholders*, which should be replaced by a pair of proportional tokens of the respective sizes of any desired binary functional constant as '+', '·', '×', '∩', '∪', etc and also as '+′', $\hat{+}$, '$\hat{+}'$', '+′', '·′', '$\hat{\cdot}$', '$\hat{\cdot}'$', etc. Thus, if an initial binary functional constant **'$*$'** is furnished with some labels then **'$\ast$'** should be furnished with the same labels.•

Comment 2.3. 1) In accordance with Definition 2.12, if the symbol '+', e.g., is provided with some labels (as one or more primes, a caret, an overbar, a tilde, etc) then the symbol

‘$+$ ’ is provided with the same labels. It is therefore understood that if the convention of equivocal use of the sign ‘+’ instead of each one of the plus signs such as ‘$+'$’, $\hat{+}$, ‘$\hat{+}'$’, etc is adopted, tacitly or explicitly, then the sign ‘$+$ ’ should be used instead of any one of the signs ‘$+'$ ’, ‘$\hat{+}$ ’, ‘$\hat{+}'$ ’, etc. In this case, the denotatum of the operator ‘$+$ ’ depends on the type of its summand (operatum). It is also understood that if the conventional symbol ‘$\sum$ ’ is employed instead of ‘$+$ ’ then the symbols ‘$\hat{\sum}$ ’, ‘$\sum'$ ’, and ‘$\hat{\sum}'$ should be employed instead of ‘$\hat{+}$ ’, ‘$+'$ ’, ‘$\hat{+}'$ respectively; and similarly with ‘$\prod$’ and ‘$.$ ’ in place of ‘$\sum$ ’ and ‘$+$ ’. Thus, Definition 2.12 makes obvious that the conventional symbol ‘$\sum$ ’, or ‘$\prod$’, is equivocal and that for avoidance of confusion it should be provided with additional labels to connote the functional constant, which denotes the binary addition, or multiplication, operation, underlying the sequence of repeated binary addition, or multiplication, operations equivocally denoted by ‘$\sum$ ’, or ‘$\prod$’, respectively.

2) In the sequel, the ‘$+$ ’-symbols and the respective ‘$\sum$ ’-symbols will be used interchangeably.•

2.5. Linear superpositions of vectors and dimensions of real vector spaces

Definition 2.13. A *subspace* $\hat{\boldsymbol{E}}'(\boldsymbol{R})$ of a vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ is a subset $\hat{E}'$ of $\hat{E}$, which contains the null-vector $\hat{0}$ and which *together* with the field $\boldsymbol{R}$ and *together* with the pertinent restrictions

$$\hat{+}':(\hat{E}'\times\hat{E}')\to\hat{E}' \;,\; \hat{-}':\hat{E}'\to\hat{E}' .\; \hat{\cdot}':(\hat{E}'\times R)\cup(R\times\hat{E}')\to\hat{E}' \tag{2.31}$$

of the functions

$$\hat{+}:(\hat{E}\times\hat{E})\to\hat{E}\,,\;\; \hat{-}:\hat{E}\to\hat{E}\,,\; \hat{\cdot}:(\hat{E}\times R)\cup(R\times\hat{E})\to\hat{E} \tag{2.32}$$

of $\hat{\boldsymbol{E}}(\boldsymbol{R})$ is itself a vector space over the field $\boldsymbol{R}$. If $\hat{\boldsymbol{E}}'(\boldsymbol{R})$ is a subspace of $\hat{\boldsymbol{E}}(\boldsymbol{R})$ then the latter is called a *superspace* of the former and vice versa. If the relations

$$\hat{E}'\subset\hat{E}\,,\;\; \hat{+}'\subset\hat{+}\,,\; \hat{-}'\subset\hat{-}\,,\; \hat{\cdot}'\subset\hat{\cdot} \tag{2.33}$$

definitely hold rather than their variants with ‘$\subseteq$’ in place of ‘$\subset$’ then $\hat{\boldsymbol{E}}'(\boldsymbol{R})$ is called a *strict subspace of* $\hat{\boldsymbol{E}}(\boldsymbol{R})$, while the latter is called a *strict superspace* of the former and vice versa.

The operations of a subspace are conventionally denoted by the same signs as those denoting the operations of its superspace.•

Definition 2.14. Given a vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$, given $m \in \omega_1$, given m non-zero vectors $\hat{x}_1$, ..., $\hat{x}_m$ in $\hat{E}$, given m scalars a_1, ..., a_m in *R*, the vector $\hat{+}_{i=1}^{m}(a_i \hat{\cdot} \hat{x}_i)$, defined by

$$\hat{+}_{i=1}^{m}(a_i \hat{\cdot} \hat{x}_i) \doteq (a_1 \hat{\cdot} \hat{x}_1) \hat{+} (a_2 \hat{\cdot} \hat{x}_2) \hat{+} ... \hat{+} (a_n \hat{\cdot} \hat{x}_n) \tag{2.34}$$

in accordance with Definitions 2.4–2.6, is said to be a *linear combination*, or *linear superposition*, of the vectors $\hat{x}_1$, ..., $\hat{x}_m$.•

Theorem 2.1. Under Definition 2.14, let

$$\hat{\boldsymbol{L}}(\hat{x}_1,...,\hat{x}_m) \doteq \bigcup\left(\hat{L}(\hat{x}_1,...,\hat{x}_m), \boldsymbol{R}, \hat{+}, \hat{-}, \hat{\cdot}\right) \doteq \hat{L}(\hat{x}_1,...,\hat{x}_m) \cup \boldsymbol{R} \cup [\hat{+} \cup \hat{-} \cup \hat{\cdot}] \tag{2.35}$$

(cf. Definition 2.6), where

$$\hat{L}(\hat{x}_1,...,\hat{x}_m) \doteq \left\{ \hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i \middle| \textit{for each } j \in \omega_{1,m} : a_j \in R \right\} \subseteq \hat{E}, \tag{2.36}$$

i.e. $\hat{L}(\hat{x}_1, \ldots, \hat{x}_m)$ is the set of all linear combinations of the vectors $\hat{x}_1$, ..., $\hat{x}_m$ of the space $\hat{\boldsymbol{E}}(\boldsymbol{R})$. Then the set $\hat{\boldsymbol{L}}(\hat{x}_1,...,\hat{x}_m)$ is *the smallest subspace of* $\hat{\boldsymbol{E}}(\boldsymbol{R})$ that contains all the vectors $\hat{x}_1$, ..., $\hat{x}_m$. Accordingly, $\hat{\boldsymbol{L}}(\hat{x}_1,...,\hat{x}_m)$ is said to be *the space generated*, or *spanned*, *by the vectors* $\hat{x}_1$, ..., $\hat{x}_m$, or *the linear shell of the vectors* $\hat{x}_1$, ..., $\hat{x}_m$. It is understood that it can particularly happen that $\hat{\boldsymbol{L}}(\hat{x}_1,...,\hat{x}_m) = \hat{\boldsymbol{E}}(\boldsymbol{R})$.

Proof: Let $\hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i$, $\hat{+}_{i=1}^{m} a_i' \hat{\cdot} \hat{x}_i$ and $\hat{+}_{i=1}^{m} a_i'' \hat{\cdot} \hat{x}_i$ be linear combinations of the vectors $\hat{x}_1$, ..., $\hat{x}_m$, while b is any element in *R*. Then, by the pertinent rules established earlier, it follows that

$$\begin{aligned} \left(\hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i\right) \hat{+} \left(\hat{+}_{i=1}^{m} a_i' \hat{\cdot} \hat{x}_i\right) &= \hat{+}_{i=1}^{m} (a_i + a_i') \hat{\cdot} \hat{x}_i \\ = \hat{+}_{i=1}^{m} (a_i' + a_i) \hat{\cdot} \hat{x}_i &= \left(\hat{+}_{i=1}^{m} a_i' \hat{\cdot} \hat{x}_i\right) \hat{+} \left(\hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i\right), \end{aligned} \tag{2.37}$$

$$\begin{aligned} \left[\left(\hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i\right) \hat{+} \left(\hat{+}_{i=1}^{m} a_i' \hat{\cdot} \hat{x}_i\right)\right] \hat{+} \left(\hat{+}_{i=1}^{m} a_i'' \hat{\cdot} \hat{x}_i\right) &= \hat{+}_{i=1}^{m} [(a_i + a_i') + a_i''] \hat{\cdot} \hat{x}_i \\ = \hat{+}_{i=1}^{m} [a_i + (a_i' + a_i'')] \hat{\cdot} \hat{x}_i &= \left(\hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i\right) \hat{+} \left[\left(\hat{+}_{i=1}^{m} a_i' \hat{\cdot} \hat{x}_i\right) \hat{+} \left(\hat{+}_{i=1}^{m} a_i'' \hat{\cdot} \hat{x}_i\right)\right], \end{aligned} \tag{2.38}$$

$$\hat{\underset{i=1}{\overset{m}{+}}} 0 \,\hat{\cdot}\, \hat{x}_i = \hat{\underset{i=1}{\overset{m}{+}}} \hat{0} = \hat{0}\,, \tag{2.39}$$

$$\begin{aligned} &\left(\hat{\underset{i=1}{\overset{m}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i\right) + (-1)\cdot\left(\hat{\underset{i=1}{\overset{m}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i\right) = \left(\hat{\underset{i=1}{\overset{m}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i\right) + \left(\hat{\underset{i=1}{\overset{m}{+}}} ((-1)\cdot a_i) \,\hat{\cdot}\, \hat{x}_i\right) \\ &= \hat{\underset{i=1}{\overset{m}{+}}} [a_i + (-a_i)] \,\hat{\cdot}\, \hat{x}_i = \hat{\underset{i=1}{\overset{m}{+}}} 0 \,\hat{\cdot}\, \hat{x}_i = \hat{\underset{i=1}{\overset{m}{+}}} \hat{0} = \hat{0}, \end{aligned} \tag{2.40}$$

$$b\cdot\left(\hat{\underset{i=1}{\overset{m}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i\right) = \hat{\underset{i=1}{\overset{m}{+}}} (b\cdot a_i) \,\hat{\cdot}\, \hat{x}_i = \hat{\underset{i=1}{\overset{m}{+}}} (a_i \cdot b) \,\hat{\cdot}\, \hat{x} = \left(\hat{\underset{i=1}{\overset{m}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i\right)\cdot b\,. \tag{2.41}$$

For $b \equiv 1$, equation (2.41) yields

$$1\cdot\left(\hat{\underset{i=1}{\overset{m}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i\right) = \hat{\underset{i=1}{\overset{m}{+}}} (1\cdot a_i) \,\hat{\cdot}\, \hat{x}_i = \hat{\underset{i=1}{\overset{m}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i\,. \tag{2.42}$$

Equations (2.37)-(2.42) are the pertinent variants of the defining axioms of a vector space over the field $\boldsymbol{R}$ and therefore they prove that the set $\hat{\boldsymbol{L}}(\hat{x}_1,...,\hat{x}_m)$ as defined by (2.35) subject to (2.36) is a vector space. By Definition 2.13, this vector space is a subspace of $\hat{\boldsymbol{E}}(\boldsymbol{R})$. QED.•

Definition 2.15. Given a vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$, given $m \in \omega_1$, given m non-zero vectors $\hat{x}_1$, ..., $\hat{x}_m$ in $\hat{E}$, given m scalars a_1, ..., a_m in *R*, the vectors $\hat{x}_1$, ..., $\hat{x}_m$ are said to be *linearly independent* if and only if

$$\hat{\underset{i=1}{\overset{m}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i = \hat{0} \text{ only if } a_j = 0 \text{ for each } j \in \omega_{1,m}\,. \tag{2.43}$$

If the vectors $\hat{x}_1$, ..., $\hat{x}_m$ are not linearly independent then they are said to be *linearly dependent*.•

Theorem 2.2. Given a vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$, given $m \in \omega_1$, any m given non-zero vectors $\hat{x}_1$, ..., $\hat{x}_m$ in $\hat{E}$ are linearly dependent if and only if some one of the vectors is a linear combination of the preceding ones.

Proof: Assume, first, that for some $j \in \omega_{1,m}$: there are non-zero scalars a_1, ..., a_{j-1} in S such that

$$\hat{x}_j = \hat{\underset{i=1}{\overset{j-1}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i\,. \tag{2.44}$$

Hence,

$$\hat{\underset{i=1}{\overset{j-1}{+}}} a_i \,\hat{\cdot}\, \hat{x}_i + (-1) \,\hat{\cdot}\, \hat{x}_j = \hat{0}\,. \tag{2.45}$$

This equation can be rewritten as

$$\hat{\underset{i=1}{\overset{m}{+}}} a_i \hat{\cdot} \hat{x}_i = \hat{0}\,, \tag{2.46}$$

where

$$\begin{gathered}\text{either } a_j = \text{-}1 \text{ and } a_{j+1} = a_{j+2} = ... = a_m \,\bar{\bar{=}}\, 0 \text{ if } j < m \\ \text{or } a_m = a_j \,\bar{\bar{=}}\, \text{-}1 \text{ if } j = m.\end{gathered} \tag{2.47}$$

Thus, the '*if*''-part of the theorem is proved. In order to prove the '*only-if*'-part of the theorem, let us assume that the vectors $\hat{x}_1$, ..., $\hat{x}_m$ are linearly dependent. Thus, by Definition 2.15, we assume that there are m scalars a_1, ..., a_m in R such that *at least one of them does not equal zero* but (2.46) holds. Consequently, there is a unique number $j \in \omega_{1,m}$ such that

$$\begin{gathered}\text{either } a_j \neq 0 \text{ and } a_{j+1} = a_{j+2} = ... = a_m \,\bar{\bar{=}}\, 0 \text{ if } j < m \\ \text{or } a_j = a_m \neq 0 \text{ if } j = m.\end{gathered} \tag{2.48}$$

If $j = 1$ then, by (2.48), equation (2.46) reduces to $a_1 \hat{\cdot} \hat{x}_1 = \hat{0}$, whence $\hat{x}_1 = \hat{0}$ because $a_1 \neq 0$. However, the above result contradicts the hypothesis of the theorem that none of the vectors $\hat{x}_1$, ..., $\hat{x}_m$ equals zero. Hence, $j \geq 2$. Therefore, under either one of two alternative conditions (2.48), equation (2.46) can be solved with respect to '$\hat{x}_j$' so that

$$\hat{x}_j = \hat{\underset{i=1}{\overset{j-1}{+}}} \left(\text{-}\, a_j^{\text{-}1} \cdot a_i\right) \hat{\cdot}\, \hat{x}_i\,. \tag{2.49}$$

This equation expresses $\hat{x}_j$ as a linear combination of the preceding vectors (cf. (2.44). QED.•

Definition 2.16. Given a vector space $\hat{\boldsymbol{E}}$ over a field $\boldsymbol{R}$, let for some $n \in \omega_1$: $\{\hat{e}_1,...,\hat{e}_n\} \subset \hat{E}$ be a set of n *linearly independent vectors* that generate (`span) the whole space $\hat{\boldsymbol{E}}$; that is, $\hat{\boldsymbol{L}}(\hat{e}_1,...,\hat{e}_n) = \hat{\boldsymbol{E}}$ where $\hat{\boldsymbol{L}}(\hat{e}_1,...,\hat{e}_n)$ is defined by the variants of (2.35) and (2.36) with 'n' in place of 'm' and 'e' in place of 'x'.

a) The ordered n-tuple $\bar{\hat{e}}_{[1,n]}$, defined as

$$\bar{\hat{e}}_{[1,n]} \,\bar{\bar{=}}\, \langle \hat{e}_1,...,\hat{e}_n \rangle \in \hat{E}^{n\times}\,, \tag{2.50}$$

is said to be a *basis* of the vector space $\hat{\boldsymbol{E}}$.

b) The number n of the basis vectors is said to be the *dimension* of the vector space $\hat{\boldsymbol{E}}$. Accordingly, the latter vector space is denoted by '$\hat{\boldsymbol{E}}_n$' and is called an n*-dimensional* one, while the underlying set of $\hat{\boldsymbol{E}}_n$ is denoted by 'E_n'.

c) A vector space $\hat{\boldsymbol{E}}$ is said to be *finite-dimensional* if and only if it has a finite basis. Otherwise, $\hat{\boldsymbol{E}}$ is said to be *infinite-dimensional*. In other words, if there exists in $\hat{\boldsymbol{E}}$ a set of as many linearly independent vectors as one pleases then the vector space $\hat{\boldsymbol{E}}$ is an infinite-dimensional one, denoted by '$\hat{\boldsymbol{E}}_\infty$'.•

Corollary 2.1. Given $n \in \omega_1$, let $\hat{\boldsymbol{E}}_n$ be an n-dimensional vector space with a basis $\bar{\hat{e}}_{[1,n]}$ given by (2.50). Then for each $\hat{\boldsymbol{E}}_n$: there is a unique ordered n-tuple

$$\bar{x}_{[1,n]} \stackrel{\rightarrow}{=} \langle x_1,...,x_n \rangle \in R^{n\times} \tag{2.51}$$

such that the vector $\hat{x}$ is given by the equation

$$\hat{x} = \hat{x}_{[1,n]} \stackrel{\rightarrow}{=} \hat{\sum_{i=1}^{n}} x_i \hat{\cdot} \hat{e}_i \,. \tag{2.52}$$

Conversely, for each $\bar{x}_{[1,n]}$, satisfying (2.51): there is exactly one vector $\hat{x} \in \hat{E}_n$ given by (2.52). The ordered n-tuple $\bar{x}_{[1,n]}$, defined by (2.51), is said to be *the ordered n-tuple of the coordinates of the vector $\hat{x}$, defined by (2.52), relative to the basis* $\bar{\hat{e}}_{[1,n]}$.

Proof: Given a vector $\hat{x} \in \hat{E}_n$, the existence of at least one linear combination of the vectors $\hat{e}_1,...,\hat{e}_n$, which represents $\hat{x}$ in accordance with (2.52), immediately follows from Definition 2.15 by Theorem 2.1. Suppose that, besides $\bar{x}_{[1,n]}$ given by (2.51), there is another n-tuple

$$\bar{x}'_{[1,n]} \stackrel{\rightarrow}{=} \langle x'_1,...,x'_n \rangle \in R^{n\times}, \tag{2.53}$$

such that

$$\hat{x}' = \hat{x}'_{[1,n]} \stackrel{\rightarrow}{=} \hat{\sum_{i=1}^{n}} x'_i \hat{\cdot} \hat{e}_i \,. \tag{2.54}$$

Subtraction of the last equation from (2.52) yields

$$\hat{x}_{[1,n]} - \hat{x}'_{[1,n]} = \hat{\sum_{i=1}^{n}} (x_i - x'_i) \hat{\cdot} \hat{e}_i = \hat{0} \,. \tag{2.55}$$

By Definition 2.16, the vectors $\hat{e}_1,...,\hat{e}_n$ are linearly independent. Hence, by Definition 2.15, it follows from the pertinent variant of (2.43) that $x'_i = x_i$ for each $i \in \omega_{1,n}$, so that $\bar{x}'_{[1,n]} = \bar{x}_{[1,n]}$. Conversely, by (2.23), the equation $\bar{x}'_{[1,n]} = \bar{x}_{[1,n]}$ implies that $x'_i = x_i$ for each $i \in \omega_{1,n}$. At the same time, it is evident that given $\bar{x}_{[1,n]}$ satisfying (2.51): the vector $\hat{x}$ is uniquely determined by equation (2.52). QED.•

Corollary 2.2. Given $n \in \omega_1$, given an n-dimensional vector space $\hat{\boldsymbol{E}}_n$ with a basis $\overline{\hat{e}}_{[1,n]}$ given by (2.50), for each $\hat{x} \in \hat{E}_n$, for each $\hat{x} \in \hat{E}_n$, for each $a \in R$:

$$\hat{x} \hat{+} \hat{y} = \left(\hat{\sum_{i=1}^{n}} x_i \hat{\cdot} \hat{e}_i \right) \hat{+} \left(\hat{\sum_{i=1}^{n}} y_i \hat{\cdot} \hat{e}_i \right) = \hat{\sum_{i=1}^{n}} (x_i + y_i) \hat{\cdot} \hat{e}_i, \tag{2.56}$$

$$a \hat{\cdot} \hat{x} = \hat{\sum_{i=1}^{n}} (a \cdot x_i) \hat{\cdot} \hat{e}_i, \tag{2.57}$$

subject to (2.52) and also subject to

$$\hat{y} = \hat{y}_{[1,n]} \doteqdot \hat{\sum_{i=1}^{n}} y_i \hat{\cdot} \hat{e}_i. \tag{2.58}$$

Proof: The corollary follows from Definition 2.14 by Theorem 2.1 (cf. (2.37)) or Corollary 2.1.•

Comment 2.4. According to Corollary 2.2, given a basis of a vector space $\hat{\boldsymbol{E}}_n$, both the binary operations of addition of vectors of $\hat{E}_{\{n\}}$ and the binary operation of multiplication of a vector of $\hat{E}_n$ by a scalar of *R*, which are initially defined in abstract form, reduce to the corresponding operations on the scalars of *R*, which are coordinates of the vectors relative to the basis. In most cases occurring in practice, *R* is either the field of real numbers or the field of complex numbers, so that use of a basis becomes especially effective.•

Corollary 2.3. Given $n \in \omega_1$: any $n+1$ vectors $\hat{x}_1$, ..., $\hat{x}_{n+1}$ of an n-dimensional vector space $\hat{E}_n$ are linearly dependent.

Proof: According to Definition 2.16, the vectors $\hat{x}_1$, ..., $\hat{x}_{n+1}$ are linearly dependent if and only if there are $n+1$ scalars a_1, .., a_{n+1} in *R* such that *some of them do not equal zero*, while

$$\hat{\sum_{i=1}^{n+1}} a_i \hat{\cdot} \hat{x}_i = \hat{0}. \tag{2.59}$$

At the same time, by Corollary 2.1, there are $n+1$ n-tuples:

$$\overline{x}_{i[1,n]} \doteqdot \langle x_{i1}, ..., x_{in} \rangle \in R^{n\times} \text{ with } i \in \omega_{1,n+1}, \tag{2.60}$$

such that

$$\hat{x}_i = \hat{x}_{i[1,n]} \doteqdot \hat{\sum_{j=1}^{n}} x_{ij} \hat{\cdot} \hat{e}_j \text{ for each } i \in \omega_{1,n+1}. \tag{2.61}$$

Substitution of (2.61) into (2.59) yields

$$\hat{\underset{j=1}{\overset{n}{+}}} b_j \hat{\cdot} \hat{e}_j = \hat{0}, \tag{2.62}$$

where

$$b_j \doteq \underset{i=1}{\overset{n+1}{+}} a_i \cdot x_{ij} \text{ for each } j \in \omega_{1,n}. \tag{2.63}$$

However, by Definition 2.16, the vectors $\hat{e}_1, \ldots, \hat{e}_n$ are linearly independent. Therefore, by Definition 2.15, equation (2.62) holds if and only if $b_j = 0$ for each $j \in \omega_{1,n}$. Hence, by (2.63),

$$\underset{i=1}{\overset{n+1}{+}} a_i \cdot x_{ij} = 0 \text{ for each } j \in \omega_{1,n}. \tag{2.64}$$

Relation (2.64) can be regarded as a set of n homogeneous linear algebraic equations with respect to $n+1$ unknowns 'a_i' with $i \in \omega_{1,n+1}$. It is known from algebra that such a set always has a nontrivial solution for the ordered $(n+1)$-tuple $\langle$'a_1', … , 'a_{n+1}'$\rangle$ of variables. QED.•

3. Real Euclidean vector spaces

3.1. Real Euclidean abstract vector spaces

Definition 3.1. A real abstract vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$, i.e. an abstract vector space $\hat{\boldsymbol{E}}$ over the field $\boldsymbol{R}$ of real numbers, is called a Eu*clidean* one if and only if there is a real-valued binary function $\hat{\bullet}: \hat{E} \times \hat{E} \to R$, which is called the *inner*, or *scalar*, *multiplication function on* $\hat{E}$ and which satisfies the following axioms ("*IMA*" is an abbreviation for "*Inner Multiplication Axiom*").

IMA1: The functionality law. For each $\langle \hat{x}, \hat{y} \rangle \in \hat{E} \times \hat{E}$, there is a unique real number denoted by '$\hat{x} \hat{\bullet} \hat{y}$', which is called *the inner*, or *scalar, product of* $\hat{x}$ *and* $\hat{y}$.

IMA2: The commutative, or symmetrical, law. For each $\langle \hat{x}, \hat{y} \rangle \in \hat{E} \times \hat{E}$:

$$\hat{x} \hat{\bullet} \hat{y} = \hat{y} \hat{\bullet} \hat{x}.$$

IMA3: The distributive law over $\hat{+}$. For each $\langle \hat{x}, \hat{y}, \hat{z} \rangle \in \hat{E}^{3\times}$:

$$\hat{x} \hat{\bullet} (\hat{y} \hat{+} \hat{z}) = \hat{x} \hat{\bullet} \hat{y} \hat{+} \hat{x} \hat{\bullet} \hat{z}.$$

IMA4: The combined associative law. For each $\langle \hat{x}, \hat{y} \rangle \in \hat{E} \times \hat{E}$: for each $a \in R$

$$(a \hat{\cdot} \hat{x}) \hat{\bullet} \hat{y} = a \hat{\cdot} (\hat{x} \hat{\bullet} \hat{y}).$$

IMA5: The positive definiteness law. For each $\hat{x} \in \hat{E}$:

$$\hat{x} \,\hat{\bullet}\, \hat{x} > 0 \text{ if } \hat{x} \neq \hat{0} \text{ or } \hat{x} \,\hat{\bullet}\, \hat{x} = 0 \text{ if } \hat{x} = \hat{0}.$$

According IMA1–IMA5, the inner product of two vectors in $\hat{E}$ is given by a symmetric, positive definite, homogeneous, bilinear functional form, and conversely, any functional form possessing the above properties can be selected to represent the inner product of two vectors in $\hat{E}$.•

Definition 3.2. For each $\hat{x} \in \hat{E}$, the real number $|\hat{x}|$ defined as

$$|\hat{x}| \doteqdot (\hat{x} \,\hat{\bullet}\, \hat{x})^{\frac{1}{2}} = \sqrt{\hat{x} \,\hat{\bullet}\, \hat{x}} \geq 0 \qquad (3.1)$$

is called *the length of the vector* $\hat{x}$. The associated function of the functional form '$|\hat{x}|$' is said to be *the length function on* $\hat{E}$.•

Theorem 3.1. In any Euclidean space $\boldsymbol{\hat{E}(R)}$, the length function has the following properties ("*LFT*" is an abbreviation for "*Length Function Theorem*").

LFT1: The positive definiteness law. $|\hat{0}| = 0$ and for each $\hat{x} \in \hat{E} - \{\hat{0}\}$: $|\hat{x}| > 0$.

LFT2: The homogeneity law. For each $\hat{x} \in \hat{E}$ and for each $a \in R$: $|a \,\hat{\cdot}\, \hat{x}| = |a| \cdot |\hat{x}|$.

LFT3: The Cauchy-Schwartz inequality. For each $\langle \hat{x}, \hat{y} \rangle \in \hat{E} \times \hat{E}$: $|\hat{x} \,\hat{\bullet}\, \hat{y}| \leq |\hat{x}| \cdot |\hat{y}|$.

LFT4: The triangle inequality. For each $\langle \hat{x}, \hat{y} \rangle \in \hat{E} \times \hat{E}$: $|\hat{x} \,\hat{+}\, \hat{y}| \leq |\hat{x}| + |\hat{y}|$.

Proof: The theorem follows from Definition 3.1 by Definition 3.2. Indeed, LFT1 is an immediate corollary of IMA5. In accordance with IMA4, $|(a \,\hat{\cdot}\, \hat{x}) \,\hat{\bullet}\, (a \,\hat{\cdot}\, \hat{x})| = a^2 \cdot |\hat{x}|^2$, which yields LFT2 by Definition 3.2. By Definition 3.2, it also follows that for each $\langle \hat{x}, \hat{y} \rangle \in [\hat{E} - \{\hat{0}\}] \times [\hat{E} - \{\hat{0}\}]$ and each $\langle a, b \rangle \in [R - \{0\}] \times [R - \{0\}]$:

$$0 \leq (a \,\hat{\cdot}\, \hat{x} \,\hat{-}\, b \,\hat{\cdot}\, \hat{y}) \,\hat{\bullet}\, (a \,\hat{\cdot}\, \hat{x} \,\hat{-}\, b \,\hat{\cdot}\, \hat{y}) = a^2 \cdot |\hat{x}|^2 \,\hat{-}\, 2a \cdot b \cdot (\hat{x} \,\hat{\bullet}\, \hat{y}) + b^2 \cdot |\hat{y}|^2.$$

At $a = |\hat{y}|$ and $b = |\hat{x}|$, this relation reduces to $(\hat{x} \,\hat{\bullet}\, \hat{y}) \leq |\hat{x}| \cdot |\hat{y}|$. On the other hand, if $\hat{x} = \hat{0}$ or $\hat{y} = \hat{0}$, then LFT3 reduces to $0 \leq 0$, which is also true. Thus, LFT3 is established. LFT4 follows from LFT3, for

$$|\hat{x} \,\hat{+}\, \hat{y}|^2 = (\hat{x} \,\hat{+}\, \hat{y}) \,\hat{\bullet}\, (\hat{x} \,\hat{+}\, \hat{y}) = |\hat{x}|^2 + 2(\hat{x} \,\hat{\bullet}\, \hat{y}) + |\hat{y}|^2 \leq |\hat{x}|^2 + 2|\hat{x}| \cdot |\hat{y}| + |\hat{y}|^2 = (|\hat{x}| + |\hat{y}|)^2,$$

where use of LFT3 has been made. QED.•

Definition 3.3. For each $\langle \hat{x}, \hat{y} \rangle \in \hat{E} \times \hat{E}$, the real number $|\hat{x} \hat{-} \hat{y}|$ is called *the distance between the vectors* $\hat{x}$ *and* $\hat{y}$ *in* $\hat{E}$ The associated function of the functional form '$|\hat{x} \hat{-} \hat{y}|$' is called *the distance function in* $\hat{E}$ **.•**

Theorem 3.2. In any Euclidean space $\hat{\boldsymbol{E}}(\boldsymbol{R})$, the distance function has the following properties ("*DFT*" is an abbreviation for "*Distance Function Theorem*")*:*

DFT1. For each $\langle \hat{x}, \hat{y} \rangle \in \hat{E} \times \hat{E}$: (i) $|\hat{x} \hat{-} \hat{y}| > 0$ if $\hat{x} \neq \hat{y}$ or (ii) $|\hat{x} \hat{-} \hat{x}| = |\hat{0}| = 0$ if $\hat{x} = \hat{y}$.

DFT2. For each $\langle \hat{x}, \hat{y} \rangle \in \hat{E} \times \hat{E}$: $|\hat{x} \hat{-} \hat{y}| = |\hat{y} \hat{-} \hat{x}|$.

DFT3. For each $\langle \hat{x}, \hat{y}, \hat{z} \rangle \in \hat{E}^{3\times}$: $|\hat{x} \hat{-} \hat{y}| + |\hat{y} \hat{-} \hat{z}| \geq |\hat{x} \hat{-} \hat{z}|$.

Proof: The theorem follows from Definition 3.3 by Definition 3.2 and by Theorem 3.1. Specifically, DFT1 immediately follows from LFT1. Then, by LFT2,

$$|\hat{x} \hat{-} \hat{y}| = |\hat{-}(\hat{y} \hat{-} \hat{x})| = |-1| \cdot |\hat{y} \hat{-} \hat{x}| = |\hat{y} \hat{-} \hat{x}|,$$

which proves DFT2. Lastly, DFT3 follows from LFT4, for

$$|\hat{x} \hat{-} \hat{y}| + |\hat{y} \hat{-} \hat{z}| \geq |(\hat{x} \hat{-} \hat{y}) \hat{+} (\hat{y} \hat{-} \hat{z})| = |\hat{x} \hat{-} \hat{z}|.$$

QED.•

Theorem 3.3. For each $\langle \hat{x}, \hat{y} \rangle \in [\hat{E} - \{\hat{0}\}] \times [\hat{E} - \{\hat{0}\}]$, there is exactly one real number $\alpha \in [0, \pi]$ such that

$$\cos \alpha = \frac{\hat{x} \hat{\bullet} \hat{y}}{|\hat{x}| \cdot |\hat{y}|}, \tag{3.2}$$

whence

$$\alpha = \angle(\hat{x}, \hat{y}) \overset{\rightarrow}{=} \arccos \frac{\hat{x} \hat{\bullet} \hat{y}}{|\hat{x}| \cdot |\hat{y}|}. \tag{3.3}$$

The number $\angle(\hat{x}, \hat{y})$ is called *the angle between the vectors* $\hat{x}$ *and* $\hat{y}$.

Proof: The theorem immediately follows from item LFT3 of Theorem 3.1.•

Definition 3.4. For each $\langle \hat{x}, \hat{y} \rangle \in [\hat{E} - \{\hat{0}\}] \times [\hat{E} - \{\hat{0}\}]$, the vectors $\hat{x}$ and $\hat{y}$ are said to be *orthogonal*, which is expressed logographically either as '$\hat{x} \perp \hat{y}$' or as '$\hat{y} \perp \hat{x}$', if and only if $\angle(\hat{x}, \hat{y}) = \frac{\pi}{2}$ or equivalently $\hat{x} \hat{\bullet} \hat{y} = 0$; that is, if $\hat{x} \neq 0$ and $\hat{y} \neq \hat{0}$ then

$$\hat{x} \perp \hat{y} \Leftrightarrow \left(\angle(\hat{x}, \hat{y}) = \frac{\pi}{2} \right) \Leftrightarrow (\hat{x} \hat{\bullet} \hat{y} = 0), \tag{3.4}$$

where, and generally in what follows, '$\Leftrightarrow$' means *if and only if*.

Comment 3.1. The relation $\langle \hat{x}, \hat{y}\rangle \in \left[\hat{E} - \{\hat{0}\}\right] \times \left[\hat{E} - \{\hat{0}\}\right]$ means that $\hat{x} \neq \hat{0}$ and $\hat{y} \neq \hat{0}$. Therefore, under the above condition, the equations '$\angle(\hat{x}, \hat{y}) = \frac{\pi}{2}$' and '$\hat{x} \,\hat{\bullet}\, \hat{y} = 0$' are equivalent, by (3.3). Still, the equation '$\hat{x} \,\hat{\bullet}\, \hat{y} = 0$' also holds if $\hat{x} = \hat{0}$ or $\hat{y} = \hat{0}$. Therefore, some writers extend the notion of orthogonal vectors to the last case as well. As a consequence, $\hat{0}$ becomes the only vector that is orthogonal to any vector in $\hat{E}$ including itself. Also, in this case, the expression on the right-hand side of equation (3.2) becomes an indeterminate functional form of the type '$0/0$', so that equation (3.3) is also meaningless. Thus, there are forcible arguments for excluding the case where $\hat{x} = \hat{0}$ or $\hat{y} = \hat{0}$ from the definition of orthogonal vectors.•

Corollary 3.1: For each $\langle \hat{x}, \hat{y}\rangle \in \left[\hat{E} - \{\hat{0}\}\right] \times \left[\hat{E} - \{\hat{0}\}\right]$ and for each $\langle a, b\rangle \in [R - \{0\}] \times [R - \{0\}]$:

$$\hat{x} \perp \hat{y} \text{ if and only if } (a \,\hat{\cdot}\, \hat{x}) \perp (b \,\hat{\cdot}\, \hat{y}). \tag{3.5}$$

Proof: By IMA3 of Definition 3.1, it follows that

$$(a \,\hat{\cdot}\, \hat{x}) \,\hat{\bullet}\, (b \,\hat{\cdot}\, \hat{y}) = (a \cdot b) \,\hat{\cdot}\, (\hat{x} \,\hat{\bullet}\, \hat{y}). \tag{3.6}$$

The corollary follows from (3.6) by (3.4) because $a \neq 0$, $b \neq 0$, $\hat{x} \neq \hat{0}$, and $\hat{y} \neq \hat{0}$, by the hypothesis of the corollary.•

Lemma 3.1. Given $m \in \omega_2$, given m non-zero mutually orthogonal vectors $\hat{x}_1, \ldots, \hat{x}_m$ in $\hat{E}$:

$$\hat{x}_i \,\hat{\bullet}\, \hat{x}_j = \hat{x}_j \,\hat{\bullet}\, \hat{x}_i = |\hat{x}_i|^2 \delta_{ij} \text{ for each } i \in \omega_{1,m} \text{ and each } j \in \omega_{1,m}, \tag{3.7}$$

where 'δ_{ij}' is the Kronecker delta-symbol.

Proof: The lemma follows from Definitions 3.2 and 3.4.•

Theorem 3.4: ***A generalized Pythagorean theorem.*** Given $m \in \omega_2$, given m non-zero mutually orthogonal vectors $\hat{x}_1, \ldots, \hat{x}_m$ in $\hat{E}$:

$$\left| \hat{\sum_{i=1}^{m}} \hat{x}_i \right|^2 = \sum_{i=1}^{m} |\hat{x}_i|^2. \tag{3.8}$$

Proof: By (2.28), IMA3, and (3.7), it follows from the variant of (3.1) with '$\hat{\sum_{i=1}^{m}} \hat{x}_i$' in place of '$\hat{x}$' that

$$\left| \hat{\sum_{i=1}^{m}} \hat{x}_i \right|^2 = \left(\hat{\sum_{i=1}^{m}} \hat{x}_i \right) \hat{\bullet} \left(\hat{\sum_{j=1}^{m}} \hat{x}_j \right) = \sum_{i=1}^{m} \sum_{j=1}^{m} (\hat{x}_i \,\hat{\bullet}\, \hat{x}_j) = \sum_{i=1}^{m} \sum_{j=1}^{m} (\hat{x}_i \,\hat{\bullet}\, \hat{x}_i) \delta_{ij} = \sum_{i=1}^{m} |\hat{x}_i|^2.$$

QED.•

Theorem 3.5. Given $m \in \omega_1$, if a vector $\hat{y} \in \hat{E} - \{\hat{0}\}$ is orthogonal to each one of m given non-zero vectors $\hat{x}_1, \ldots, \hat{x}_m$ in $\hat{E}$ then it is orthogonal to each non-zero vector in the space $\hat{\boldsymbol{L}}(\hat{x}_1, \ldots, \hat{x}_m)$ spanned by $\hat{x}_1, \ldots, \hat{x}_m$.

Proof: In compliance with (2.36), let

$$\hat{x} = \hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i \neq \hat{0}, \tag{3.9}$$

where $a_1, \ldots, a_m$ are m arbitrary scalars in *R*. In this case, by items IMA2 and IMA3 of Definition 3.1, it follows from the hypothesis of the theorem that

$$\hat{y} \hat{\bullet} \hat{x} = \hat{y} \hat{\bullet} \left(\hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i \right) = +_{i=1}^{m} \hat{y} \hat{\bullet} (a_i \hat{\cdot} \hat{x}_i) = +_{i=1}^{m} a_i \cdot (\hat{y} \hat{\bullet} \hat{x}_i) = +_{i=1}^{m} a_i \cdot 0 = 0.$$

This equation proves the theorem by Definition 3.4.•

Definition 3.5. Given $m \in \omega_2$, m non-zero vectors $\hat{x}_1, \ldots, \hat{x}_m$ in a Euclidean vector space $\hat{\boldsymbol{E}}$ are said to be *normal orthogonal* or *orthonormal* if and only if

$$\hat{x}_i \hat{\bullet} \hat{x}_j = \delta_{ij} \text{ for each } i \in \omega_{1,m} \text{ and each } j \in \omega_{1,m} \tag{3.10}$$

(cf. (3.7)).•

Corollary 3.2. For each $m \in \omega_2$, m non-zero mutually orthogonal vectors $\hat{x}_1, \ldots, \hat{x}_m$ in a Euclidean vector space $\hat{\boldsymbol{E}}$ are linearly independent.

Proof: Assume that there are some m scalars $a_1, \ldots, a_m$ in *R*, not all equal zero, such that:

$$\hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i = \hat{0}.$$

Hence, by Definitions 3.1, 3.4, and 3.5, it follows that

$$0 = \hat{x}_j \hat{\bullet} \hat{0} = \hat{x}_j \hat{\bullet} \left(\hat{+}_{i=1}^{m} a_i \hat{\cdot} \hat{x}_i \right) = +_{i=1}^{m} \hat{x}_j \hat{\bullet} (a_i \hat{\cdot} \hat{x}_i)$$

$$= +_{i=1}^{m} a_i \cdot (\hat{x}_j \hat{\bullet} \hat{x}_i) = +_{i=1}^{m} a_i \delta_{ij} = a_j \text{ for each } j \in \omega_{1,m}.$$

This relation proves the corollary by Definition 2.15.•

3.2. Euclidean real vector spaces of a finite dimension

Definition 3.6. Given $n \in \omega_2$, if n normal orthogonal vectors $\hat{e}_1, \ldots, \hat{e}_n$ of a Euclidean real vector space $\hat{\boldsymbol{E}}$ span the space then the ordered n-tuple of those vectors is, in accordance with Definition 2.16(a), a basis of $\hat{E}$. This basis is said to be a *normal orthogonal*, or *orthonormal*, *basis* (briefly *NOB* or *ONB*) *of* $\hat{\boldsymbol{E}}$, – in agreement with Definition 3.5. It goes without saying that such a Euclidean space is an *n-dimensional* one, and hence a *finite-dimensional one*, – in accordance with Definition 2.16(c).•

Theorem 3.6: ***The Gram-Schmidt orthogonalization theorem.*** Given $n \in \omega_2$, given $m \in \omega_{1,n}$, let $\hat{x}_1, \ldots, \hat{x}_m$ be a sequence of linearly independent vectors in a Euclidean vector space $\hat{\boldsymbol{E}}$. Then there exists a sequence of m orthogonal non-zero vectors $\hat{y}_1, \ldots, \hat{y}_m$ which span the same subspace of $\hat{\boldsymbol{E}}$ as that spanned by the vectors $\hat{x}_1, \ldots, \hat{x}_m$, i.e.

$$\hat{\boldsymbol{L}}(\hat{y}_1, \ldots, \hat{y}_m) = \hat{\boldsymbol{L}}(\hat{x}_1, \ldots, \hat{x}_m) \subseteq \hat{\boldsymbol{E}}\,. \tag{3.11}$$

A sequence $\hat{y}_1, \ldots, \hat{y}_m$ having the above property can be written as

$$\hat{y}_1 \stackrel{\rightarrow}{=} \hat{x}_1\,, \tag{3.12}$$

$$\hat{y}_i = \hat{x}_i \,\hat{-}\, \hat{\sum_{j=1}^{i-1}} a_{ij} \,\hat{\cdot}\, \hat{x}_j \text{ for each } i \in \omega_{2,m}\,, \tag{3.13}$$

where a_{ij} with $i \in \omega_{2,m}$ and $j \in \omega_{1,i-1}$ are certain scalars in *R*.

Proof: Given two linearly independent vectors $\hat{x}_1$ and $\hat{x}_2$ in $\hat{\boldsymbol{E}}$, let

$$\hat{y}_2 = \hat{x}_2 \,\hat{-}\, a_{21} \,\hat{\cdot}\, \hat{x}_1 \tag{3.14}$$

where

$$a_{21} \stackrel{\rightarrow}{=} \frac{\hat{x}_2 \,\hat{\bullet}\, \hat{x}_1}{\hat{x}_1 \,\hat{\bullet}\, \hat{x}_1}\,. \tag{3.15}$$

In this case, $\hat{y}_1 \,\hat{\bullet}\, \hat{y}_2 = 0$. Therefore, $\hat{y}_1$ and $\hat{y}_2$ are linearly independent by Corollary 3.2, and also

$$\hat{\boldsymbol{L}}(\hat{y}_1, \hat{y}_2) = \hat{\boldsymbol{L}}(\hat{x}_1, \hat{x}_2) \subseteq \hat{\boldsymbol{E}} \tag{3.16}$$

by Theorem 2.1. Hence, for $m = 2$ the theorem is true. As *the induction hypothesis on 'm'*, let us therefore assume that for some $m \in \omega_{2,\infty}$: the scalars a_{ij} with $i \in \omega_{1,m-1}$ and $j \in \omega_{1,i-1}$ have already been constructed in such a way that $m-1$ vectors $\hat{y}_1, \ldots, \hat{y}_{m-1}$ of the form of (3.12) and (3.13) are mutually orthogonal non-zero vectors, which span the same subspace of $\hat{\boldsymbol{E}}$ as that spanned by the $m-1$ vectors $\hat{x}_1, \ldots, \hat{x}_{m-1}$. Thus, it is particularly assumed that

$$\hat{y}_i \,\hat{\bullet}\, \hat{y}_j = (\hat{y}_i \,\hat{\bullet}\, \hat{y}_i)\delta_{ij} \text{ for each } i \in \omega_{1,m-1} \text{ and each } j \in \omega_{1,i-1} \tag{3.17}$$

(cf. (3.7)) and that

$$\hat{\boldsymbol{L}}(\hat{y}_1, \ldots, \hat{y}_{m-1}) = \hat{\boldsymbol{L}}(\hat{x}_1, \ldots, \hat{x}_{m-1}) \subseteq \hat{\boldsymbol{E}}\,. \tag{3.18}$$

Let then

$$\hat{y}_m \doteq \hat{x}_m \mathbin{\hat{-}} \mathop{\hat{+}}_{i=1}^{m-1} b_{mi} \mathbin{\hat{\cdot}} \hat{y}_i \,, \tag{3.19}$$

where

$$b_{mi} \doteq \frac{\hat{x}_m \mathbin{\hat{\bullet}} \hat{y}_i}{\hat{y}_i \mathbin{\hat{\bullet}} \hat{y}_i} \text{ for each } i \in \omega_{1,m-1} \,. \tag{3.20}$$

In this case,

$$\begin{aligned} \hat{y}_m \mathbin{\hat{\bullet}} \hat{y}_j &= \hat{x}_m \mathbin{\hat{\bullet}} \hat{y}_j - \left[\mathop{+}_{i=1}^{m-1} b_{mj} \cdot \left(\hat{y}_i \mathbin{\hat{\bullet}} \hat{y}_j \right) \right] \\ &= \hat{x}_m \mathbin{\hat{\bullet}} \hat{y}_j - b_{mj} \left(\hat{y}_j \mathbin{\hat{\bullet}} \hat{y}_j \right) = 0 \text{ for each } j \in \omega_{1,m-1}, \end{aligned} \tag{3.21}$$

where use of (3.17) has been made. Thus, either the vector $\hat{y}_m$ is orthogonal to each one of the $m-1$ vectors $\hat{y}_1, \dots, \hat{y}_{m-1}$ or $\hat{y}_m = \hat{0}$. By (3.13) with $i \in \omega_{1,m-1}$, which is true by the induction hypothesis, equation (3.19) becomes

$$\hat{y}_m = \hat{x}_m \mathbin{\hat{-}} \mathop{\hat{+}}_{i=1}^{m-1} b_{mi} \mathbin{\hat{\cdot}} \hat{x}_i \mathbin{\hat{+}} \mathop{\hat{+}}_{i=2}^{m-1} b_{mi} \mathbin{\hat{\cdot}} \left(\mathop{\hat{+}}_{j=1}^{i-1} a_{ij} \mathbin{\hat{\cdot}} \hat{x}_j \right). \tag{3.22}$$

Upon exchanging the indices ‘ i ’ and ‘ j ’, the last term in (3.22) can be developed thus:

$$\mathop{\hat{+}}_{i=2}^{m-1} b_{mi} \mathbin{\hat{\cdot}} \left(\mathop{\hat{+}}_{j=1}^{i-1} a_{ij} \mathbin{\hat{\cdot}} \hat{x}_j \right) = \mathop{\hat{+}}_{j=2}^{m-1} b_{mj} \mathbin{\hat{\cdot}} \left(\mathop{\hat{+}}_{i=1}^{j-1} a_{ji} \mathbin{\hat{\cdot}} \hat{x}_i \right) = \mathop{\hat{+}}_{j=2}^{m-1} \mathop{\hat{+}}_{i=1}^{j-1} \left(b_{mj} \cdot a_{ji} \right) \mathbin{\hat{\cdot}} \hat{x}_i \,. \tag{3.23}$$

By the induction hypothesis, a_{ji} have so far been specified for each $j \in \omega_{2,m-1}$ and each $i \in \omega_{1,j-1}$. Therefore, given $j \in \omega_{2,m-1}$, one may, without loss of generality, set that

$$a_{ji} \doteq 0 \text{ for each } i \in \omega_{j,m-1} \,. \tag{3.24}$$

In this case, equation (3.22) subject to (3.23) and (3.24) takes the form of (3.13) at $i = m$, provided that

$$a_{mi} \doteq b_{mi} + \mathop{\hat{+}}_{j=2}^{m-1} b_{mj} \cdot a_{ji} \,. \tag{3.25}$$

Since $\hat{x}_m$ is independent of the vectors $\hat{x}_1, \dots, \hat{x}_{m-1}$, therefore $\hat{x}_m \notin \hat{L}\left(\hat{x}_1, \dots, \hat{x}_{m-1} \right)$ and hence $\hat{y}_m \neq \hat{0}$. Thus, the m vectors $\hat{y}_1, \dots, \hat{y}_m$ are mutually orthogonal by the induction hypothesis and by (3.21), and hence they are linearly independent by Corollary 3.2. By Theorem 2.1, the vectors $\hat{y}_1, \dots, \hat{y}_m$ span the same subspace of as that spanned by the vectors $\hat{x}_1, \dots, \hat{x}_m$, which proves (3.11). QED.•

Corollary 3.3. For each $n \in \omega_2$, each *n*-dimensional Euclidean real vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ has a normal orthogonal basis

$$\bar{\hat{e}}_{[1,n]} \doteqdot \langle \hat{e}_1,...,\hat{e}_n \rangle \in \hat{E}_n^{\;n\times} \tag{3.26}$$

(cf. (2.50)) so that

$$\hat{e}_i \,\hat{\bullet}\, \hat{e}_j = \delta_{ij} \text{ for each } i \in \omega_{1,n} \text{ and for each } j \in \omega_{1,n}. \tag{3.27}$$

Proof: In accordance with Definition 2.16(a), the ordered *n*-tuple

$$\bar{\hat{x}}_{[1,n]} \doteqdot \langle \hat{x}_1,...,\hat{x}_n \rangle \in \hat{E}_n^{\;n\times}$$

of any *n* linearly independent vectors $\hat{x}_1,...,\hat{x}_n$ of an *n*-dimensional vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ is a basis of the space. The ordered *n*-tuple

$$\bar{\hat{y}}_{[1,n]} \doteqdot \langle \hat{y}_1,...,\hat{y}_n \rangle \in \hat{E}_n^{\;n\times}$$

of the *n* mutually orthogonal vectors $\hat{y}_1,\ldots,\hat{y}_m$, which are constructed as linear superpositions of the vectors $\hat{x}_1,...,\hat{x}_n$ in accordance with the recursive *Gram-Schmidt orthogonalization procedure* of Theorem 3.6, is another, *orthogonal*, basis of $\hat{E}_{\{n\}}$, by Corollary 3.3. The *n* vectors $\hat{e}_1,...,\hat{e}_n$, defined as

$$\hat{e}_i \doteqdot |\hat{y}_i|^{-1} \,\hat{\cdot}\, \hat{y}_i \text{ for each } i \in \omega_{1,n}, \tag{3.28}$$

satisfy (3.27), by the pertinent variants of (3.1) and (3.17). QED.•

Corollary 3.4. Given $n \in \omega_1$, given an n-dimensional Euclidean [real] vector space $\hat{\boldsymbol{E}}_{\{n\}}(\boldsymbol{R})$ with a normal orthogonal basis (3.26) subject to (3.27), for each $\hat{x} \in \hat{E}_{\{n\}}$ as given relative to the basis by (2.52):

$$x_i = \hat{x} \,\hat{\bullet}\, \hat{e}_i \text{ for each } i \in \omega_{1,n}. \tag{3.29}$$

Proof: Given $j \in \omega_{1,n}$, it follows from (2.47) by Definition 3.1 and Corollary 3.3 that

$$\hat{e}_j \,\hat{\bullet}\, \hat{x} = \hat{e}_j \,\hat{\bullet} \left(\hat{\sum_{i=1}^{n}} x_i \,\hat{\cdot}\, \hat{e}_i \right) = \sum_{i=1}^{n} x_i \cdot (\hat{e}_j \,\hat{\bullet}\, \hat{e}_i) = \sum_{i=1}^{n} x_i \cdot \delta_{ij} = x_j. \tag{3.30}$$

QED.•

Corollary 3.5. Given $n \in \omega_1$, given an n-dimensional Euclidean vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ with a normal orthogonal basis (3.26) subject to (3.27), for any two vectors $\hat{x}$ and $\hat{y}$ given relative to the basis by (2.52) and (2.58) respectively:

$$\hat{x} \,\hat{\bullet}\, \hat{y} = \sum_{i=1}^{n} x_i \cdot y_i. \tag{3.31}$$

Proof: By (2.52), (2.58), and (3.27), it follows from Definition 3.1 that (cf. (3.30))

$$\hat{x}\,\hat{\bullet}\,\hat{y} = \left(\hat{\sum_{i=1}^{n}} x_i \,\hat{\cdot}\, \hat{e}_i\right)\hat{\bullet}\left(\hat{\sum_{j=1}^{n}} y_j \,\hat{\cdot}\, \hat{e}_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i \cdot y_j\right)\hat{\cdot}\left(\hat{e}_i \,\hat{\bullet}\, \hat{e}_j\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i \cdot y_j\right)\cdot \delta_{ij} = \sum_{i=1}^{n} x_i \cdot y_i. \tag{3.32}$$

QED.•

3.3. Projective real arithmetical vector spaces

Definition 3.7.

$$\overline{E}_1 \doteqdot R^{1\times} \doteqdot \left\{\left\langle x_1\right\rangle \middle| x_1 \in R\right\} = \left\{\left\{x_1\right\} \middle| x_1 \in R\right\} \neq R, \tag{3.33}$$

$$\overline{E}_n \doteqdot R^{n\times} \doteqdot \underbrace{R \times R \times \ldots \times R}_{n \text{ times } R} \doteqdot R^{(n-1)\times} \times R$$
$$\doteqdot \underbrace{[[\ldots}_{n-2}[R \times R] \times R] \times \ldots] \times R] \times R \text{ for each } n \in \omega_2. \tag{3.34}$$

Definition 3.8. For each $n \in \omega_1$, for each $\overline{x}_{[1,n]} \in \overline{E}_n$, for each $\overline{y}_{[1,n]} \in \overline{E}_n$, for each $a \in R$:

$$\overline{x}_{[1,n]} \,\bar{+}\, \overline{y}_{[1,n]} = \overline{y}_{[1,n]} \,\bar{+}\, \overline{x}_{[1,n]} \doteqdot \left\langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\right\rangle, \tag{3.35}$$

$$\bar{-}\, \overline{x}_{[1,n]} \doteqdot \left\langle -x_1, -x_2, \ldots, -x_n\right\rangle, \tag{3.36}$$

$$a \,\bar{\cdot}\, \overline{x}_{[1,n]} \doteqdot \left\langle a \cdot x_1, a \cdot x_2, \ldots, a \cdot x_n\right\rangle = \left\langle x_1 \cdot a, x_2 \cdot a, \ldots, x_n \cdot a\right\rangle \doteqdot \overline{x}_{[1,n]} \cdot a, \tag{3.37}$$

where

$$\overline{x}_{[1,n]} \doteqdot \left\langle x_1, x_2, \ldots, x_n\right\rangle, \; \overline{y}_{[1,n]} \doteqdot \left\langle y_1, y_2, \ldots, y_n\right\rangle, \tag{3.38}$$

in accordance with (2.20). Also,

$$\overline{0}_{[1,1]} \doteqdot \left\langle 0\right\rangle = \left\{0\right\}, \tag{3.39}$$

$$\overline{0}_{[1,n]} \doteqdot \langle \underbrace{0, \ldots, 0}_{n \text{ zeros}} \rangle \doteqdot \langle \overline{0}_{[1,n-1]}, 0\rangle \doteqdot \underbrace{\langle\langle \ldots}_{n-1}\langle 0,0\rangle,0\rangle,\ldots\rangle,0\rangle, \tag{3.40}$$

which are instances of (2.24) and (2.20) respectively.•

Comment 3.2. The operations

$$\bar{+} : \overline{E}_n \times \overline{E}_n \to \overline{E}_n, \bar{-} : \overline{E}_n \to \overline{E}_n, \tag{3.41}$$

$$\bar{\cdot} : \left(R \times \overline{E}_n\right) \cup \left(\overline{E}_n \times R\right) \to \overline{E}_n, \tag{3.42}$$

as defined by (3.35)-(3.37) subject to (3.38), are distinct from the operations +, -, and $\cdot$ for scalars of R indicated in Definition 3.5 and they are also distinct from the operations $\hat{+}$, $\hat{-}$, and $\hat{\cdot}$ defined in Definitions 2.4 and 2.6. Most mathematician denote the opetations $\bar{+}$, $\bar{-}$, and $\bar{\cdot}$ equivocally (homonymously, homographically) with the operations +, -, and ·, in terms of which the former are defined. But I do not follow this practice for avoidance of confusion.•

Corollary 3.6. 1) For each $n \in \omega_1$, for each $\bar{x}_{[1,n]} \in \bar{E}_n$:

$$\bar{x}_{[1,n]} \bar{+} \bar{0}_{[1,n]} = \bar{0}_{[1,n]} \bar{+} \bar{x}_{[1,n]} = \langle x_1 + 0,\ldots,x_n + 0\rangle = \langle x_1,\ldots,x_n\rangle = \bar{x}_{[1,n]}, \tag{3.43}$$

$$\bar{x}_{[1,n]} \bar{+} \left(\bar{-}\,\bar{x}_{[1,n]}\right) = \langle x_1 + (-x_1),\ldots,x_n + (-x_n)\rangle = \langle\underbrace{0,\ldots,0}_{n}\rangle = \bar{0}_{[1,n]}. \tag{3.44}$$

2) For each $a \in R$:

$$a \,\bar{\cdot}\, \bar{0}_{[1,n]} = \langle\underbrace{a\cdot 0,\ldots,a\cdot 0}_{n}\rangle = \langle\underbrace{0,\ldots,0}_{n}\rangle = \bar{0}_{[1,n]}. \tag{3.45}$$

Hence, the ordered n-tuple $\bar{0}_{[1,n]}$ is *the zero element of* $\bar{E}_n$, whereas the element $\bar{-}\,\bar{x}_{[1,n]} \in \bar{E}_{\{n\}}$ is *the additive inverse* of the element $\bar{x}_{[1,n]} \in \bar{E}_n$.

Proof: The corollary follows from Definition 3.8.•

Definition 3.9. For each $n \in \omega_1$, the algebraic system that, along with the field ***R***, includes the set $\bar{E}_n$, defined by Definition 3.7, and the operations on $\bar{E}_n$, defined by Definition 3.8 and by Corollary 3.6, is a *specific instance* (*concrete interpretation*) *of the n-dimensional projective abstract linear* (*vector*) *space* $\hat{\boldsymbol{E}}_n^{\mathrm{p}}(\boldsymbol{R})$ or briefly $\hat{\boldsymbol{E}}_n^{\mathrm{p}}$, which has been defined by Definitions 2.6 and 2.7. This instance will denoted by '$\bar{\boldsymbol{E}}_n^{\mathrm{p}}(\boldsymbol{R})$' or briefly by '$\bar{\boldsymbol{E}}_n^{\mathrm{p}}$' and be called an *n-dimensional projective [real] arithmetical linear*, or *vector*, *space over the field **R** of real numbers*¸ the understanding being that the prepositive adjectival qualifier "real" and the postpositive qualifier "over the field ***R*** of real numbers" are concurrent. Accordingly, an ordered *n*-tuple being an element of $\bar{E}_{\{n\}}$ is called an *n-dimensional real arithmetical vector* or a *real arithmetical n-vector*. Some further distinguished attributes of $\hat{\boldsymbol{E}}_n^{\mathrm{p}}(\boldsymbol{R})$ as an *n*-dimensional vector space are explicated below.•

Definition 3.10. Given $n \in \omega_1$, in accordance with (3.35) and (3.36), it follows that for each $\langle \bar{x}_{[1,n]}, \bar{y}_{[1,n]}\rangle \in \bar{E}_{\{} \times \bar{E}_n$:

$$\begin{aligned}\bar{x}_{[1,n]} = \left(\bar{-}\,\bar{y}_{[1,n]}\right) \doteq \bar{x}_{[1,n]} \bar{+} \left(\bar{-}\,\bar{y}_{[1,n]}\right) &= \langle x_1 - y_1,\ldots,x_n - y_n\rangle \\ &= \langle x_1 + (-y_1),\ldots,x_n + (-y_n)\rangle.\end{aligned} \tag{3.46}$$

Theorem 3.7. Given $n \in \omega_1$, for each $\bar{x}_{[1,n]} \in \bar{E}_n$:

$$\bar{x}_{[1,n]} = \mathop{\bar{+}}_{i=1}^{n} x_i \,\bar{\cdot}\, \bar{e}_{i[1,n]}, \tag{3.47}$$

where

$$\bar{e}_{1[1,n]} \doteq \langle 1,0,0,...,0 \rangle, \bar{e}_{2[1,n]} \doteq \langle 0,1,0,...,0 \rangle, ..., \bar{e}_{n[1,n]} \doteq \langle 0,0,...,0,1 \rangle \qquad (3.48)$$

or equivalently

$$\bar{e}_{i[1,n]} \doteq \langle \delta_{i1},...,\delta_{in} \rangle \text{ for each } i \in \omega_{1,n}; \qquad (3.49)$$

'δ_{ij}' is the Kronecker delta-symbol. In accordance with Definition 2.12 and Comment 2.3, the symbol '$\bar{\sum}$' can be used interchangeably with '$\bar{+}$'.

Proof: Making use of the instance of (3.37) with 'x_i' in place of '*a*' and with '$\bar{e}_{i[1,n]}$' in place of '$\bar{x}_{[1,n]}$', and then making use of *n*–1 pertinent instances of (3.35), one can develop equation (3.47) thus:

$$\begin{aligned} \bar{x}_{[1,n]} &= \bar{+}_{i=1}^{n} x_i \,\bar{\cdot}\, \bar{e}_{i[1,n]} = \bar{+}_{i=1}^{n} x_i \,\bar{\cdot}\, \langle \delta_{i1},...,\delta_{in} \rangle = \bar{+}_{i=1}^{n} \langle x_i \cdot \delta_{i1},...,x_i \cdot \delta_{in} \rangle \\ &= \langle x_1,0,0,...,0,0 \rangle \,\bar{+}\, \langle 0,x_2,0,..,0,0 \rangle \,\bar{+}\, ... \,\bar{+}\, \langle 0,0,...,0,x_n \rangle = \langle x_1,x_2,...,x_n \rangle. \end{aligned} \qquad (3.47_1)$$

QED.•

Comment 3.3. The equation (3.47) can be rewritten as

$$\bar{x}_{[1,n]} \,\bar{-}\, \bar{+}_{i=1}^{n} x_i \,\bar{\cdot}\, \bar{e}_{i[1,n]} = \bar{0}_{[1,n]}, \qquad (3.47_2)$$

which means that the $n+1$ vectors $\bar{x}_{[1,n]}$, $\bar{e}_{1[1,n]}$, ..., $\bar{e}_{n[1,n]}$ are linearly dependent.•

Definition 3.11. 1) Given $n \in \omega_1$, the ordered *n*-tuple $\bar{\bar{e}}_{[1,n][1,n]}$ defined as

$$\bar{\bar{e}}_{[1,n][1,n]} \doteq \langle \bar{e}_{1[1,n]},...,\bar{e}_{n[1,n]} \rangle \in \bar{E}_{\{n\}}^{n\times} \qquad (3.50)$$

subject to (3.48) or (3.49) is a *basis of* $\bar{\boldsymbol{E}}_n^{\mathrm{p}}(\boldsymbol{R})$, which is specifically called a *unit basis*.•

3.4. Euclidean real arithmetical vector spaces

Definition 3.12. Given $n \in \omega_1$, a vector space $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$ or briefly $\bar{\boldsymbol{E}}_n$ over the field $\boldsymbol{R}$ of real numbers is called an *n-dimensional Euclidean real arithmetical linear*, or *vector*, *space* if and only if it is an *n-dimensional projective real arithmetical linear*, or *vector*, *space* $\bar{\boldsymbol{E}}_n^{\mathrm{p}}(\boldsymbol{R})$ *together* with is a real-valued binary function $\bar{\bullet} : \bar{E}_n \times \bar{E}_n \to R$ such that for each $\langle \bar{x}_{[1,n]}, \bar{y}_{[1,n]} \rangle \in \bar{E}_n \times \bar{E}_n$:

$$\bar{x}_{[1,n]} \,\bar{\bullet}\, \bar{y}_{[1,n]} \doteq \bar{+}_{i=1}^{n} x_i \cdot y_i, \qquad (3.51)$$

in compliance with (3.32). The operation $\bar{\bullet}$ so defined is is called the *inner*, or *scalar*, *multiplication function on* $\bar{E}_{\{n\}}$, the understanding being that it satisfies the variants of IMA1–IMA5 with '$\bar{\bullet}$', '$\bar{E}_n$', '$\bar{x}_{[1,n]}$', '$\bar{y}_{[1,n]}$', and '$\bar{z}_{[1,n]}$' in place of '$\hat{\bullet}$', '$\hat{E}$', '$\hat{x}$', '$\hat{y}$', and '$\hat{z}$', respectively.•

Corollary 3.8. Given $n \in \omega_1$, the ordered n-tuple $\bar{\bar{e}}_{[1,n][1,n]}$ defined by (3.50) subject to (3.48) or (3.49) is a *normal orthogonal*, or *orthonormal*, *basis of* $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$, i.e.

$$\bar{e}_{i[1,n]} \bar{\bullet} \bar{e}_{j[1,n]} = \delta_{ij} \text{ for each } i \in \omega_{1,n} \text{ and each } j \in \omega_{1,n}. \tag{3.52}$$

Proof: Given $i \in \omega_{1,n}$, given $j \in \omega_{1,n}$, the instance of equation (3.51) with $\bar{x}_{[1,n]} = \bar{e}_{i[1,n]}$ and $\bar{y}_{[1,n]} = \bar{e}_{j[1,n]}$, subject to (3.49), yields

$$\bar{e}_{i[1,n]} \bar{\bullet} \bar{e}_{j[1,n]} = \mathop{+}_{k=1}^{n} \delta_{ik} \cdot \delta_{jk} = \delta_{ij}. \tag{3.53}$$

QED.•

Comment 3.4. Making use of (3.47) and of the similar relation:

$$\bar{y}_{[1,n]} = \mathop{\bar{+}}_{j=1}^{n} y_j \,\bar{\cdot}\, \bar{e}_{j[1,n]},$$

the inner product $\bar{x}_{[1,n]} \bar{\bullet} \bar{y}_{[1,n]}$ can be developed with the help of (3.52) thus:

$$\begin{gathered} \bar{x}_{[1,n]} \bar{\bullet} \bar{y}_{[1,n]} = \left(\mathop{\bar{+}}_{i=1}^{n} x_i \,\bar{\cdot}\, \bar{e}_{i[1,n]} \right) \bar{\bullet} \left(\mathop{\bar{+}}_{j=1}^{n} y_j \,\bar{\cdot}\, \bar{e}_{j[1,n]} \right) \\ = \mathop{+}_{i=1}^{n} \mathop{+}_{j=1}^{n} \left(x_i \cdot y_j \right) \cdot \left(\bar{e}_{i[1,n]} \bar{\bullet} \bar{e}_{j[1,n]} \right) = \mathop{+}_{i=1}^{n} \mathop{+}_{j=1}^{n} \left(x_i \cdot y_j \right) \cdot \delta_{ij} = \mathop{+}_{i=1}^{n} x_i \cdot y_i, \end{gathered} \tag{3.54}$$

which is in agreement both with (3.51), as expected.•

Corollary 3.9. Given $n \in \omega_1$, for each $\bar{x}_{[1,n]}$ as given by (3.38) or (3.47):

$$x_i = \bar{x}_{[1,n]} \bar{\bullet} \bar{e}_{i[1,n]} \text{ for each } i \in \omega_{1,n}. \tag{3.55}$$

Proof (*mutatis mutandis*, the same as that of Corollary 3.4**):** Given $j \in \omega_{1,n}$, it follows from (3.47) by Definition 3.12 and Corollary 3.8 that

$$\begin{gathered} \bar{e}_{j[1,n]} \bar{\bullet} \bar{x}_{[1,n]} = \bar{e}_{j[1,n]} \bar{\bullet} \left(\mathop{\bar{+}}_{i=1}^{n} x_i \cdot \bar{e}_{i[1,n]} \right) = \mathop{\bar{+}}_{i=1}^{n} x_i \cdot \left(\bar{e}_{j[1,n]} \bar{\bullet} \bar{e}_{i[1,n]} \right) \\ = \mathop{+}_{i=1}^{n} x_i \cdot \delta_{ij} = x_j. \end{gathered} \tag{3.56}$$

QED.•

Definition 3.13. Given $n \in \omega_1$, given an n-dimensional Euclidean real abstract vector space $\hat{\boldsymbol{E}}$ defined by (2.18), given an orthonormal basis $\bar{\hat{e}}_{[1,n]}$, defined by (3.26) subject to (3.27), $\hat{\boldsymbol{E}}_n$ and the *respective* n-dimensional Euclidean real arithmetical vector space $\bar{\boldsymbol{E}}_n$ can formally be defined in analogy with (2.18) respectively thus:

$$\hat{\boldsymbol{E}}_n \doteqdot \hat{\boldsymbol{E}}_n(\boldsymbol{R}) \doteqdot \hat{\boldsymbol{E}}^{\mathrm{g}} \cup \boldsymbol{R} \cup \hat{\cdot} \cup \bar{\hat{e}}_{[1,n]} \cup \hat{\bullet}, \tag{3.57}$$

$$\bar{\boldsymbol{E}}_{\}} \doteqdot \bar{\boldsymbol{E}}_n(\boldsymbol{R}) \doteqdot \bar{\boldsymbol{E}}_n^{\mathrm{g}} \cup \boldsymbol{R} \cup \bar{\cdot} \cup \bar{\bullet}, \tag{3.58}$$

where $\bar{\boldsymbol{E}}_n^{\mathrm{g}}$ is a commutative additive (Abelian) group of ordered *n*-tuples of real numbers constituting the underlying set $\bar{E}_n$, which is formally defined in analogy with (2.12) as:

$$\bar{\boldsymbol{E}}_n^{\mathrm{g}} \doteqdot \bar{E}_{\{n\backslash} \cup \bar{+} \cup \bar{=}. \tag{3.59}$$

3.5. A *coordinatization* of $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ and a *vectorization* of $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$

Theorem 3.8. Given $n \in \omega_1$, given an n-dimensional Euclidean real abstract vector space $\hat{\boldsymbol{E}}_n$, given an orthonormal basis $\bar{\hat{e}}_{[1,n]}$ in $\hat{\boldsymbol{E}}_n$, defined by (3.26) subject to (3.27), let

$$\bar{\mu}_{\bar{\hat{e}}_{[1,n]}} \doteqdot \bar{C}_{\bar{\hat{e}}_{[1,n]}} \cup \bar{C}^{+}_{\bar{\hat{e}}_{[1,n]}}, \bar{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1} \doteqdot \bar{C}_{\bar{\hat{e}}_{[1,n]}}^{-1} \cup \bar{C}^{+\,-1}_{\bar{\hat{e}}_{[1,n]}}, \tag{3.60}$$

where

$$\begin{aligned} \bar{C}_{\bar{\hat{e}}_{[1,n]}} \doteqdot \Big\{ \big\langle \hat{x} \hat{+} \hat{y}, \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]} \big\rangle \Big| \hat{x} \hat{+} \hat{y} = \hat{\sum_{i=1}^{n}} (x_i + y_i) \hat{\cdot} \hat{e}_i \\ \text{and } \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]} = \bar{\sum_{i=1}^{n}} (x_i + y_i) \bar{\cdot} \bar{e}_{i[1,n]} \\ \text{and } \langle x_j, y_j \rangle \in R \times R \text{ for each } j \in \omega_{1,n} \Big\}, \end{aligned} \tag{3.61}$$

$$\begin{aligned} \bar{C}_{\bar{\hat{e}}_{[1,n]}}^{-1} \doteqdot \Big\{ \big\langle \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]}, \hat{x} \hat{+} \hat{y} \big\rangle \Big| \hat{x} \hat{+} \hat{y} = \hat{\sum_{i=1}^{n}} (x_i + y_i) \hat{\cdot} \hat{e}_i \\ \text{and } \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]} = \bar{\sum_{i=1}^{n}} (x_i + y_i) \bar{\cdot} \bar{e}_{i[1,n]} \\ \text{and } \langle x_j, y_j \rangle \in R \times R \text{ for each } j \in \omega_{1,n} \Big\}, \end{aligned} \tag{3.62}$$

subject to (3.48) or (3.49), and

$$\bar{C}^{+}_{\bar{\hat{e}}_{[1,n]}} \doteqdot \{\langle \hat{\bullet}, \bar{\bullet} \rangle\}, \bar{C}^{+\,-1}_{\bar{\hat{e}}_{[1,n]}} \doteqdot \{\langle \bar{\bullet}, \hat{\bullet} \rangle\}. \tag{3.63}$$

The function $\bar{\mu}_{\bar{\hat{e}}_{[1,n]}} : \hat{\boldsymbol{E}}_n(\boldsymbol{R}) \to \bar{\boldsymbol{E}}_n(\boldsymbol{R})$ is the *isomorphism from* $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ *to* $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$, whereas the inverse function $\bar{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1} : \bar{\boldsymbol{E}}_n(\boldsymbol{R}) \to \hat{\boldsymbol{E}}_n(\boldsymbol{R})$ is the *isomorphism from* $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$ *to* $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$. That is to say, the definitions (3.61) and (3.62) imply the following bijective (one-to-one) correspondences:

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{0}) = \overline{0}_{[1,n]},\ \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{0}_{[1,n]}) = \hat{0}, \tag{3.64}$$

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{e}_j) = \overline{e}_{j[1,n]} \text{ and } \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{e}_{j[1,n]}) = \hat{e}_j \text{ for each } j \in \omega_{1,n}, \tag{3.65}$$

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{x}) = \overline{x}_{[1,n]},\ \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{x}_{[1,n]}) = \hat{x}, \tag{3.66}$$

$$\begin{gathered}\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{-}\,\hat{x}) = \bar{-}\,\overline{x}_{[1,n]} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{-})\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{x}),\\ \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\bar{-}\,\overline{x}_{[1,n]}) = \hat{-}\hat{x} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\bar{-})\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{x}_{[1,n]}),\end{gathered} \tag{3.67}$$

$$\begin{gathered}\overline{C}_{\bar{\hat{e}}_{[1,n]}}(a \,\hat{\cdot}\, \hat{x}) = a \,\bar{\cdot}\, \overline{x}_{[1,n]} = a \,\bar{\cdot}\, \overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{x}),\\ \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(a \,\bar{\cdot}\, \overline{x}_{[1,n]}) = a \,\hat{\cdot}\, \hat{x} = a \,\hat{\cdot}\, \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{x}_{[1,n]}),\end{gathered} \tag{3.68}$$

$$\begin{gathered}\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{x} \,\hat{+}\, \hat{y}) = \overline{x}_{[1,n]} \,\bar{+}\, \overline{y}_{[1,n]} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{x})\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{+})\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{y}),\\ \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{x}_{[1,n]} + \overline{y}_{[1,n]}) = \hat{x} \,\hat{+}\, \hat{y} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{x}_{[1,n]})\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\bar{+})\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{y}_{[1,n]}),\end{gathered} \tag{3.69}$$

subject to (3.48) or (3.49) and also subject to

$$\hat{x} \,\hat{+}\, \hat{y} = \hat{\sum_{i=1}^{n}}(x_i + y_i) \,\hat{\cdot}\, \hat{e}_i \text{ and } \overline{x}_{[1,n]} \,\bar{+}\, \overline{y}_{[1,n]} = \bar{\sum_{i=1}^{n}}(x_i + y_i) \,\bar{\cdot}\, \overline{e}_{i[1,n]}. \tag{3.70}$$

The relations (3.64)–(3.69) hold with ‘μ’ in place of ‘C’, and in addition

$$\begin{gathered}\overline{\mu}_{\bar{\hat{e}}_{[1,n]}}(\hat{x} \,\hat{\bullet}\, \hat{y}) = \overline{x}_{[1,n]} \,\bar{\bullet}\, \overline{y}_{[1,n]} = \overline{\mu}_{\bar{\hat{e}}_{[1,n]}}(\hat{x})\overline{\mu}_{\bar{\hat{e}}_{[1,n]}}(\hat{\bullet})\overline{\mu}_{\bar{\hat{e}}_{[1,n]}}(\hat{y})\\ = \overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{x})\overline{C}^{+}_{\bar{\hat{e}}_{[1,n]}}(\hat{\bullet})\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{y}) = \sum_{i=1}^{n}(x_i \cdot y_i),\\ \overline{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{x}_{[1,n]} \,\bar{\bullet}\, \overline{y}_{[1,n]}) = \hat{x} \,\hat{\bullet}\, \hat{y} = \overline{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{x}_{[1,n]})\overline{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\bar{\bullet})\overline{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{y}_{[1,n]})\\ = \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{x}_{[1,n]})\overline{C}^{+\,-1}_{\bar{\hat{e}}_{[1,n]}}(\bar{\bullet})\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{y}_{[1,n]}) = \sum_{i=1}^{n}(x_i \cdot y_i),\end{gathered} \tag{3.71}$$

whence

$$\overline{\mu}_{\bar{\hat{e}}_{[1,n]}}(\hat{\bullet}) = \overline{C}^{+}_{\bar{\hat{e}}_{[1,n]}}(\hat{\bullet}) = \bar{\bullet},\ \overline{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\bar{\bullet}) = \overline{C}^{+\,-1}_{\bar{\hat{e}}_{[1,n]}}(\bar{\bullet}) = \hat{\bullet}. \tag{3.72}$$

Accordingly, the functions $\overline{C}_{\bar{\hat{e}}_{[1,n]}}$ and $\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}$ are called *projective isomorphisms*, whereas the functions $\overline{C}^{+}_{\bar{\hat{e}}_{[1,n]}}$ and $\overline{C}^{+\,-1}_{\bar{\hat{e}}_{[1,n]}}$ are called *metric*, or *inner product*, *isomorphisms*.

Proof: 1) Equations (3.70) reduce to:

$$\begin{gathered}\hat{x} \,\hat{+}\, \hat{y} = \hat{0} \text{ and } \overline{x}_{[1,n]} \,\bar{+}\, \overline{y}_{[1,n]} = \overline{0}_{[1,n]}\\ \text{if } x_i = y_i = 0 \text{ for each } i \in \omega_{1,n},\end{gathered} \tag{3.62$_1$}$$

$$\hat{x} \hat{+} \hat{y} = \hat{e}_j \text{ and } \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]} = \bar{e}_{j[1,n]} \quad \text{if } x_i = \delta_{ij} \text{ and } y_i = 0 \text{ for each } i \in \omega_{1,n}, \tag{3.63$_1$}$$

$$\hat{x} \hat{+} \hat{y} = \hat{x} \text{ and } \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]} = \bar{x}_{[1,n]} \quad \text{if } y_i = 0 \text{ for each } i \in \omega_{1,n}, \tag{3.64$_1$}$$

$$\hat{x} \hat{+} \hat{y} = \hat{-}\hat{z} = \hat{\sum_{i=1}^{n}}(-z_i)\hat{\cdot}\hat{e}_i \text{ and } \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]} = \bar{-}\bar{z} = \bar{\sum_{i=1}^{n}}(-z_i)\bar{\cdot}\bar{e}_{i[1,n]} \quad x_i = -z_i \text{ and } y_i = 0 \text{ for each } i \in \omega_{1,n}, \tag{3.65$_1$}$$

$$\hat{x} \hat{+} \hat{y} = a \hat{\cdot} \hat{z} = a \hat{\cdot} \hat{\sum_{i=1}^{n}} z_i \hat{\cdot} \hat{e}_i \text{ and } \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]} = a \bar{\cdot} \bar{z} = a \bar{\cdot} \bar{\sum_{i=1}^{n}} z_i \bar{\cdot} \bar{e}_{i[1,n]} \quad \text{if } x_i = a \cdot z_i \text{ and } y_i = 0 \text{ for each } i \in \omega_{1,n}. \tag{3.66$_1$}$$

Hence, the ordered pairs $\langle \hat{0}, \bar{0}_{[1,n]} \rangle$, $\langle \hat{e}_j, \bar{e}_{j[1,n]} \rangle$, $\langle \hat{x}, \bar{x}_{[1,n]} \rangle$, $\langle \hat{-}\hat{z}, \bar{-}\bar{z} \rangle$, and $\langle a \hat{\cdot} \hat{z}, a \bar{\cdot} \bar{z} \rangle$, defined above, belong to $\bar{C}_{\bar{\hat{e}}_{[1,n]}}$, while the reverse-ordered pairs belong to $\bar{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}$. At the same time, it immediately follows from definitions (3.61) and (3.62) that the ordered pair $\langle \hat{x} \hat{+} \hat{y}, \bar{x}_{[1,n]} \bar{+} \bar{y}_{[1,n]} \rangle$, defined by (3.70), belongs to $\bar{C}_{\bar{\hat{e}}_{[1,n]}}$ and that the reverse-ordered pair belongs to $\bar{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}$. In developing the trains (3.69), use of equations (3.66) and of their variants with 'y' in place of 'x' has been made.

2) The metric isomorphism $\bar{C}^{+}_{\bar{\hat{e}}_{[1,n]}}$ affects neither any element of $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ nor any function of $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ other than $\hat{\bullet}$. Likewise, the inverse metric isomorphism $\bar{C}^{+}_{\bar{\hat{e}}_{[1,n]}}{}^{-1}$ affects neither any element of $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$ nor any function of $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$ other than $\bar{\bullet}$. Therefore, replacement of 'C' with 'μ' in the relations (3.64)–(3.69) does not alter those relations semantically.

3) It follows from (3.31) and (3.51) that

$$\hat{x} \hat{\bullet} \hat{y} = \bar{x}_{[1,n]} \bar{\bullet} \bar{y}_{[1,n]} = \sum_{i=1}^{n}(x_i \cdot y_i) \in R. \tag{3.73}$$

This equality is in agreement with the fact that the isomorphisms $\bar{\mu}_{\bar{\hat{e}}_{[1,n]}}$ and $\bar{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1}$, and hence $\bar{C}_{\bar{\hat{e}}_{[1,n]}}$ and $\bar{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}$, do not affect either scalars or functions of the field $\boldsymbol{R}$, which is common to both Euclidean spaces $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ and $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$. At the same time, application of '$\bar{\mu}_{\bar{\hat{e}}_{[1,n]}}$' to '$\hat{x} \hat{\bullet} \hat{y}$'or of '$\bar{\mu}_{\bar{\hat{e}}_{[1,n]}}^{-1}$' to '$\bar{x}_{[1,n]} \bar{\bullet} \bar{y}_{[1,n]}$' yields (3.71). QED.•

Comment 3.5. By (3.29) and (3.55), definitions (3.61) and (3.62) can be restated thus:

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}} \doteq \left\{ \left\langle \hat{x} \,\hat{+}\, \hat{y}, \bar{x}_{[1,n]} \,\bar{+}\, \bar{y}_{[1,n]} \right\rangle \middle| \bar{x}_{[1,n]} \,\bar{+}\, \bar{y}_{[1,n]} = \overset{n}{\underset{i=1}{\bar{+}}} \left[\left(\hat{x} \,\hat{+}\, \hat{y} \right) \hat{\bullet}\, \hat{e}_i \right] \bar{\cdot}\, \bar{e}_{i[1,n]} \right. \\ \left. \text{and } \left\langle \hat{x}, \hat{y} \right\rangle \in \hat{E}_n \times \hat{E}_n \right\}, \tag{3.61$_1$}$$

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{\ -1} \doteq \left\{ \left\langle \bar{x}_{[1,n]} \,\bar{+}\, \bar{y}_{[1,n]}, \hat{x} \,\hat{+}\, \hat{y} \right\rangle \middle| \hat{x} \,\hat{+}\, \hat{y} = \overset{n}{\underset{i=1}{\hat{+}}} \left[\left(\bar{x}_{[1,n]} \,\bar{+}\, \bar{y}_{[1,n]} \right) \bar{\bullet}\, \bar{e}_{i[1,n]} \right] \hat{\cdot}\, \hat{e}_i \right. \\ \left. \text{and } \left\langle \bar{x}_{[1,n]}, \bar{y}_{[1,n]} \right\rangle \in \overline{E}_n \times \overline{E}_n \right\}. \tag{3.62$_1$}$$

Definition 3.14. Given $n \in \omega_1$, given a Euclidean real abstract vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, given a basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_{\{n\}}^{\ n\times}$ in $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, defined by (3.26) subject to (3.27), the projective isomorphism $\overline{C}_{\bar{\hat{e}}_{[1,n]}}$ is called *the coordinatization of $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ relative to the basis $\bar{\hat{e}}_{[1,n]}$* and the inverse projective isomorphism $\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{\ -1}$ is called an *abstract vectorization of the Euclidean real arithmetical vector space $\overline{\boldsymbol{E}}_n(\boldsymbol{R})$ relative to the same basis $\bar{\hat{e}}_{[1,n]}$*.•

Definition 3.15: *Extensions of* $\overline{C}_{\bar{\hat{e}}_{[1,n]}}$ and $\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{\ -1}$ ***to the power sets*** $\mathsf{P}(\hat{E}_n)$ ***and*** $\mathsf{P}(\overline{E}_n)$. Given $n \in \omega_1$, given a Euclidean real abstract vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, given a basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_{\{n\}}^{\ n\times}$ in $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$:

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}\left(\hat{X}\right) \doteq \left\{ \bar{x}_{[1,n]} \middle| \bar{x}_{[1,n]} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}\left(\hat{x}\right) \text{and } \hat{x} \in \hat{X} \right\} \subseteq \overline{E}_n \\ \text{for each } \hat{X} \subseteq \hat{E}_n, \tag{3.74}$$

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{\ -1}\left(\overline{X}\right) \doteq \left\{ \hat{x} \middle| \hat{x} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{\ -1}\left(\bar{x}_{[1,n]}\right) \text{and } \bar{x}_{[1,n]} \in \overline{X} \right\} \subseteq \hat{E}_n \\ \text{for each } \overline{X} \subseteq \overline{E}_n. \tag{3.75}$$

Hence particularly,

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}\left(\hat{E}_n\right) = \overline{E}_n, \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{\ -1}\left(\overline{E}_n\right) = \hat{E}_n. \tag{3.76}$$

Comment 3.6. By the conventional definition of a power set as (see, e.g., Halmos [1960, p. 19]), the power sets $\mathsf{P}(\hat{E}_n)$ and $\mathsf{P}(\overline{E}_n)$ can contextually be defined as:

$$\hat{X} \subseteq \hat{E}_n \text{ if and only if } \hat{X} \in \mathsf{P}\left(\hat{E}_n\right), \tag{3.77}$$

$$\overline{X} \subseteq \overline{E}_n \text{ if and only if } \overline{X} \in \mathsf{P}\left(\overline{E}_n\right). \tag{3.78}$$

Therefore, the functions $\overline{C}_{\bar{\hat{e}}_{[1,n]}}$ and $\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{\ -1}$, which have originally been defined by (3.61) and (3.62) or (3.61$_1$) and (3.62$_2$), are extended by Definition 3.15 from the sets $\hat{E}_n$ and $\overline{E}_n$ to the power sets $\mathsf{P}(\hat{E}_n)$ and $\mathsf{P}(\overline{E}_n)$ respectively. In accordance with the presently common practice, the extensions are denoted by the same symbols as the original functions. By (3.70), *the arithmetical vector set* $\overline{C}_{\bar{\hat{e}}_{[1,n]}}(\hat{X}) \subseteq \overline{E}_n$ *is the image of the abstract vector set* $\hat{X}_{\{n\}} \subseteq \hat{E}_n$ in $\overline{E}_n$ under the mapping $\overline{C}_{\bar{\hat{e}}_{[1,n]}}$. Similarly, by (3.71), *the abstract vector set* $\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}(\overline{X}_{\{n\}}) \subseteq \hat{E}_n$ *is the image of the arithmetical vector set* $\overline{X}_{\{n\}} \subseteq \overline{E}_n$ in $\hat{E}_n$ under the mapping $\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}$.•

3.6. Isomorphisms of Euclidean vector spaces

Theorem 3.9. Given $n \in \omega_1$, given two n-dimensional Euclidean abstract vector spaces $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ and $\hat{\boldsymbol{E}}'_n(\boldsymbol{R})$, given an orthonormal basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_n^{n\times}$ in $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, defined by (3.26) subject to (3.27), given an orthonormal basis $\bar{\hat{e}}'_{[1,n]} \in \hat{E}'^{n\times}_n$ in $\hat{\boldsymbol{E}}'_n(\boldsymbol{R})$, defined by the variants (3.26) and (3.27) with '*e*´' in place of '*e*', let in analogy with (3.60)–(3.63): $C_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}}$ and $C_{\bar{\hat{e}}'_{[1,n]} \to \bar{\hat{e}}_{[1,n]}}$

$$\begin{gathered} \mu_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}} \doteqdot C_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}} \cup C^{+}_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}}, \\ \mu_{\bar{\hat{e}}'_{[1,n]} \to \bar{\hat{e}}_{[1,n]}} \doteqdot C_{\bar{\hat{e}}'_{[1,n]} \to \bar{\hat{e}}_{[1,n]}} \cup C^{+}_{\bar{\hat{e}}'_{[1,n]} \to \bar{\hat{e}}_{[1,n]}} \\ \doteq C_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}}^{-1} \cup C^{+\,-1}_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}} \doteq \mu_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}}^{-1} \end{gathered} \tag{3.79}$$

where

$$\begin{gathered} C_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}} \doteqdot \Big\{ \langle \hat{x} \hat{+} \hat{y}, \hat{x}' \hat{+}' \hat{y}' \rangle \Big| \hat{x} \hat{+} \hat{y} = \hat{\sum_{i=1}^{n}} (x_i + y_i) \hat{\cdot} \hat{e}_i \\ \text{and } \hat{x}' \hat{+}' \hat{y}' = \hat{\sum_{i=1}^{n}}{}' (x_i + y_i) \hat{\cdot}' \hat{e}'_i \\ \text{and } \langle x_j, y_j \rangle \in R \times R \text{ for each } j \in \omega_{1,n} \Big\}, \end{gathered} \tag{3.80}$$

$$\begin{gathered} C_{\bar{\hat{e}}'_{[1,n]} \to \bar{\hat{e}}_{[1,n]}} \doteqdot \Big\{ \langle \hat{x}' \hat{+}' \hat{y}', \hat{x} \hat{+} \hat{y}, \rangle \Big| \hat{x} \hat{+} \hat{y} = \hat{\sum_{i=1}^{n}} (x_i + y_i) \hat{\cdot} \hat{e}_i \\ \text{and } \hat{x}' \hat{+}' \hat{y}' = \hat{\sum_{i=1}^{n}}{}' (x_i + y_i) \hat{\cdot}' \hat{e}'_i \\ \text{and } \langle x_j, y_j \rangle \in R \times R \text{ for each } j \in \omega_{1,n} \Big\} \doteq C_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}}^{-1}, \end{gathered} \tag{3.81}$$

$$C^{+}_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}} \doteqdot \{\langle \hat{\bullet}, \hat{\bullet}' \rangle\}, C^{+}_{\bar{\hat{e}}'_{[1,n]} \to \bar{\hat{e}}_{[1,n]}} \doteqdot \{\langle \hat{\bullet}', \hat{\bullet} \rangle\} \doteq C^{+\,-1}_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}}. \tag{3.82}$$

The functions $\mu_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}} : \hat{\boldsymbol{E}}_n(\boldsymbol{R}) \to \hat{\boldsymbol{E}}'_n(\boldsymbol{R})$ and $\mu_{\bar{\hat{e}}'_{[1,n]} \to \bar{\hat{e}}_{[1,n]}} : \hat{\boldsymbol{E}}'_n(\boldsymbol{R}) \to \hat{\boldsymbol{E}}_n(\boldsymbol{R})$ are is the *isomorphism from* $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ *to* $\hat{\boldsymbol{E}}'_n(\boldsymbol{R})$ and *from* $\hat{\boldsymbol{E}}'_n(\boldsymbol{R})$ *to* $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ respectively. That is to say, the definitions (3.61) and (3.62) imply the bijective (one-to-one) correspondences, which are expressed by the pertinent variants of relations (3.64)–(3.72), namely by ones with

$$\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}},\ C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}},\ C^{+}_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}},\ \hat{0}',\ \hat{e}'_j,\ \hat{x}',\ \hat{y}',\ \hat{z}',\ \hat{\dot{-}}',\ \hat{\dotplus}',\ \hat{\bullet}'$$

in place of (3.83)

$$\bar{\mu}_{\bar{\hat{e}}_{[1,n]}},\ \bar{C}_{\bar{\hat{e}}_{[1,n]}},\ \bar{C}^{+}_{\bar{\hat{e}}_{[1,n]}},\ \bar{0}_{[1,n]},\ \bar{e}_{j[1,n]},\ \bar{x}_{[1,n]},\ \bar{y}_{[1,n]},\ \bar{z}_{[1,n]},\ \bar{\dot{-}},\ \bar{\dotplus},\ \bar{\bullet},$$

respectively. For instance, in this case, the relations

$$\begin{aligned}
&\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{x}\,\hat{\bullet}\,\hat{y}) = \hat{x}'\,\hat{\bullet}'\,\hat{y}' = \mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{x})\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{\bullet})\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{y})\\
&\quad = C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{x})C^{+}_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{\bullet})C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{y}) = \overset{n}{\underset{i=1}{+}}(x_i\cdot y_i),\\
&\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{x}'\,\hat{\bullet}'\,\hat{y}') = \hat{x}\,\hat{\bullet}\,\hat{y} = \mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{x}')\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{\bullet}')\bar{\mu}_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{y}')\\
&\quad = C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{x}')C^{+}_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{\bullet}')C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{y}') = \overset{n}{\underset{i=1}{+}}(x_i\cdot y_i)
\end{aligned} \tag{3.84}$$

and the relations

$$\begin{aligned}
&\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{\bullet}) = C^{+}_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{\bullet}) = \hat{\bullet}',\\
&\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{\bullet}') = C^{+}_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}(\hat{\bullet}') = \hat{\bullet},
\end{aligned} \tag{3.85}$$

which follow from them, come in this case instead of relations (3.71) and (3.72) respectively.

Proof: The proof of the theorem is, mutatis *mutandis*, word for word the same as that of Theorem 3.8. Particularly, one should make substitutions (3.83) and to use the pertinent primed variants of $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ instead of using relations of $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$. For instance, it follows from (3.31) and from the primed variant of (3.31) that

$$\hat{x}\,\hat{\bullet}\,\hat{y} = \hat{x}'\,\hat{\bullet}'\,\hat{y}' = \overset{n}{\underset{i=1}{+}}(x_i\cdot y_i)\in R, \tag{3.86}$$

instead of (3.73). Application of '$\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}$' to '$\hat{x}\,\hat{\bullet}\,\hat{y}$' or of '$\mu_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}$' to '$\hat{x}'\,\hat{\bullet}'\,\hat{y}'$' yields (3.84).•

Comment 3.7 (Analogous to Comment 3.5**).** By (3.29) and by the variant of (3.29) with $\hat{x}'$, $\hat{\bullet}'$, and $\hat{e}'_i$ in place of $\hat{x}$, $\hat{\bullet}$, and $\hat{e}_i$, definitions (3.61) and (3.62) can be restated thus:

$$\begin{aligned}
&C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}} \doteqdot \left\{\langle \hat{x}\,\hat{\dotplus}\,\hat{y}, \hat{x}'\,\hat{\dotplus}'\,\hat{y}'\rangle \,\middle|\, \hat{x}'\,\hat{\dotplus}'\,\hat{y}' = \overset{n}{\underset{i=1}{\hat{\dotplus}'}}\left[(\hat{x}\,\hat{\dotplus}\,\hat{y})\,\hat{\bullet}\,\hat{e}_i\right]\hat{\cdot}'\,\hat{e}'_i\right.\\
&\qquad\left.\text{and } \langle \hat{x},\hat{y}\rangle\in\hat{E}_{\{n\}}\times\hat{E}_{\{n\}}\right\} \doteqdot \hat{C}_{\bar{\hat{e}}'_{[1,n]}\to\bar{\hat{e}}_{[1,n]}}{}^{-1},
\end{aligned} \tag{3.80$_1$}$$

$$C_{\bar{\hat{e}}'_{[1,n]}\to\bar{\hat{e}}_{[1,n]}} \doteqdot \left\{\langle \hat{x}'\,\hat{+}'\,\hat{y}', \hat{x}\,\hat{+}\,\hat{y},\rangle \middle| \hat{x}\,\hat{+}\,\hat{y} = \hat{\sum_{i=1}^{n}}\left[(\hat{x}'\,\hat{+}'\,\hat{y}')\,\hat{\bullet}'\,\hat{e}'_i\right]\hat{\cdot}\,\hat{e}_i \right.$$
$$\left. \text{and } \langle \hat{x}', \hat{y}'\rangle \in \hat{E}'_{\{n\}} \times \hat{E}'_{\{n\}} \right\} \doteqdot C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}{}^{-1}. \tag{3.81$_1$}$$

Definition 3.16: ***Extensions of*** $C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}$ and $C_{\bar{\hat{e}}'_{[1,n]}\to\bar{\hat{e}}_{[1,n]}}$ ***to the power sets*** $\mathsf{P}(\hat{E}_n)$ ***and*** $\mathsf{P}(\overline{E}_n)$ **(**Analogous to Definition 3.15 subject to Comments 3.6**).** Given $n \in \omega_1$, given a Euclidean real abstract vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, given a basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_{\{n\}}{}^{n\times}$ in $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$:

$$C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{X}) \doteqdot \left\{\hat{x}' \middle| \hat{x}' = C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{x}) \text{ and } \hat{x} \in \hat{X}\right\} \subseteq \hat{E}'_n$$
$$\text{for each } \hat{X} \subseteq \hat{E}_n, \tag{3.87}$$

$$C_{\bar{\hat{e}}'_{[1,n]}\to\bar{\hat{e}}_{[1,n]}}(\hat{X}') \doteqdot \left\{\hat{x} \middle| \hat{x} = C_{\bar{\hat{e}}'_{[1,n]}\to\bar{\hat{e}}_{[1,n]}}(\hat{x}') \text{ and } \hat{x}' \in \hat{X}'\right\} \subseteq \hat{E}_n$$
$$\text{for each } \hat{X}' \subseteq \hat{E}'_n. \tag{3.88}$$

Hence particularly,

$$C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}}(\hat{E}_{\{}) = \hat{E}'_n,\; C_{\bar{\hat{e}}'_{[1,n]}\to\bar{\hat{e}}_{[1,n]}}(\hat{E}'_n) = \hat{E}_n\,. \tag{3.89}$$

Comment 3.8. 1) The nomenclature of functions defined by (3.60)–(3.63) is incorporated into the nomenclature of functions defined by (3.79)–(3.82) thus:

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}} = C_{\bar{\hat{e}}_{[1,n]}\to\bar{\bar{e}}_{[1,n][1,n]}},\; \overline{C}_{\bar{\hat{e}}_{[1,n]}}{}^{-1} = C_{\bar{\hat{e}}_{[1,n]}\to\bar{\bar{e}}_{[1,n][1,n]}}{}^{-1} = C_{\bar{\bar{e}}_{[1,n][1,n]}\to\bar{\hat{e}}_{[1,n]}}\,, \tag{3.90}$$

where $\bar{\bar{e}}_{[1,n][1,n]} \in \overline{E}_n{}^{n\times}$ is the orthonormal basis of $\overline{\boldsymbol{E}}_n(\boldsymbol{R})$, which is given by (3.50) subject to (3.48) or (3.49). Hence,

$$\overline{C}_{\bar{\hat{e}}'_{[1,n]}} = C_{\bar{\hat{e}}'_{[1,n]}\to\bar{\bar{e}}_{[1,n][1,n]}},\; \overline{C}_{\bar{\hat{e}}'_{[1,n]}}{}^{-1} = C_{\bar{\hat{e}}'_{[1,n]}\to\bar{\bar{e}}_{[1,n][1,n]}}{}^{-1} = C_{\bar{\bar{e}}_{[1,n][1,n]}\to\bar{\hat{e}}'_{[1,n]}}\,. \tag{3.91}$$

2) Given $n \in \omega_1$, given two n-dimensional vector spaces $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ and $\hat{\boldsymbol{E}}'_n(\boldsymbol{R})$, given bases $\bar{\hat{e}}_{[1,n]} \in \hat{E}_n{}^{n\times}$ and $\bar{\hat{e}}'_{[1,n]} \in \hat{E}'_n{}^{n\times}$, it is evident that

$$C_{\bar{\hat{e}}_{[1,n]}\to\bar{\hat{e}}'_{[1,n]}} = C_{\bar{\hat{e}}_{[1,n]}\to\bar{\bar{e}}_{[1,n][1,n]}} \circ C_{\bar{\bar{e}}_{[1,n][1,n]}\to\bar{\hat{e}}'_{[1,n]}} = C_{\bar{\hat{e}}_{[1,n]}} \circ C_{\bar{\hat{e}}'_{[1,n]}}{}^{-1},$$
$$\hat{C}_{\bar{\hat{e}}'_{[1,n]}\to\bar{\hat{e}}_{[1,n]}} \doteqdot C_{\bar{\hat{e}}'_{[1,n]}\to\bar{\bar{e}}_{[1,n][1,n]}} \circ C_{\bar{\bar{e}}_{[1,n][1,n]}\to\bar{\hat{e}}_{[1,n]}} = \overline{C}_{\bar{\hat{e}}'_{[1,n]}} \circ \overline{C}_{\bar{\hat{e}}_{[1,n]}}{}^{-1}. \tag{3.92}$$

These relations imply that if

$$\hat{x} = \hat{\sum_{i=1}^{n}} x_i\,\hat{\cdot}\,\hat{e}_i \text{ and } \hat{x}' = \hat{\sum_{i=1}^{n}}{}' x_i\,\hat{\cdot}'\,\hat{e}'_i$$
$$\text{for each } \bar{x}_{[1,n]} \doteqdot \langle x_1,...,x_n\rangle = \bar{\sum_{i=1}^{n}} x_i\,\bar{\cdot}\,\bar{e}_{i[1,n]} \in R^{n\times} \tag{3.93}$$

subject to (3.48) or (3.49) then

$$
\begin{aligned}
\hat{x}' &= C_{\bar{\hat{e}}_{[1,n]} \to \bar{\hat{e}}'_{[1,n]}}(\hat{x}) = C_{\bar{\hat{e}}_{[1,n]} \to \bar{\bar{e}}_{[1,n][1,n]}}\left(C_{\bar{\bar{e}}_{[1,n][1,n]} \to \bar{\hat{e}}'_{[1,n]}}(\hat{x})\right) \\
&= \left(C_{\bar{\hat{e}}_{[1,n]} \to \bar{\bar{e}}_{[1,n][1,n]}} \circ C_{\bar{\bar{e}}_{[1,n][1,n]} \to \bar{\hat{e}}'_{[1,n]}}\right)(\hat{x}) = \left(\overline{C}_{\bar{\hat{e}}_{[1,n]}} \circ \overline{C}_{\bar{\hat{e}}'_{[1,n]}}{}^{-1}\right)(\hat{x}), \\
\hat{x} &= C_{\bar{\hat{e}}'_{[1,n]} \to \bar{\hat{e}}_{[1,n]}}(\hat{x}') = C_{\bar{\hat{e}}'_{[1,n]} \to \bar{\bar{e}}_{[1,n][1,n]}}\left(C_{\bar{\bar{e}}_{[1,n][1,n]} \to \bar{\hat{e}}_{[1,n]}}(\hat{x}')\right) \\
&= \left(C_{\bar{\hat{e}}'_{[1,n]} \to \bar{\bar{e}}_{[1,n][1,n]}} \circ C_{\bar{\bar{e}}_{[1,n][1,n]} \to \bar{\hat{e}}_{[1,n]}}\right)(\hat{x}') = \left(\overline{C}_{\bar{\hat{e}}'_{[1,n]}} \circ \overline{C}_{\bar{\hat{e}}_{[1,n]}}{}^{-1}\right)(\hat{x}'),
\end{aligned} \tag{3.94}
$$

These relations illustrate the *symmetric* and *transitive* properties of isomorphisms of Euclidean real vector spaces of the same dimension.•

3.7. The del-operator in a Euclidean vector space

Definition 3.17. For each $n \in \omega_1$, given n-dimensional *Euclidean* space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, given an orthonormal basis (3.26) subject to (3.27) or (3.28) in $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$:

$$
\hat{\nabla} \stackrel{\rightarrow}{=} \hat{\sum_{i=1}^{n}} \hat{e}_i \,\hat{\cdot}\, \nabla_i\,, \tag{3.95}
$$

where

$$
\nabla_i \stackrel{\rightarrow}{=} \frac{\partial}{\partial x_i} \text{ for each } i \in \omega_{1,n}\,. \tag{3.96}
$$

In analogy with (3.29), it immediately follows from (3.95) that

$$
\nabla_i = \hat{e}_i \,\hat{\bullet}\, \hat{\nabla} \text{ for each } i \in \omega_{1,n}\,. \tag{3.97}
$$

The differential operator $\hat{\nabla}$ is called *the del-operator in* $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$.•

Definition 3.18. For each $n \in \omega_1$:

$$
\overline{\nabla}_{[1,n]} \stackrel{\rightarrow}{\equiv} \left\langle \nabla_n, \ldots, \nabla_n \right\rangle = \sum_{i=1}^{n} \bar{e}_{i[1,n]} \cdot \nabla_i \tag{3.98}
$$

subject to (3.48) or (3.49). In analogy with (3.55), it immediately follows from (3.98) that

$$
\nabla_i = \bar{e}_{i[1,n]} \,\bar{\bullet}\, \overline{\nabla}_{[1,n]} \text{ for each } i \in \omega_{1,n}\,. \tag{3.99}
$$

The differential operator $\overline{\nabla}_{[1,n]}$ is called the *del-operator in* $\overline{\boldsymbol{E}}_n(\boldsymbol{R})$.•

4. Affine additive groups

4.1. Affine group point manifolds

Definition 4.1. 1) Let $\hat{\boldsymbol{E}}^{\mathrm{g}}$ be a vector (linear) group defined by Definition 2.4 and let $\hat{E}$ be as before the underlying set of its elements called vectors; $\hat{E}$ may sometimes be identified with $\hat{\boldsymbol{E}}^{\mathrm{g}}$. In accordance with Definition 2.4, the binary composition operation of addition and the singulary operation of additive inversion operation *in* $\hat{\boldsymbol{E}}$ (or respectively *on* $\hat{E} \times \hat{E}$ and *on* $\hat{E}$) are denoted by '$\hat{+}$' and '$\hat{-}$' respectively. The latter operation is defined relative the *additive identity element of* $\hat{\boldsymbol{E}}^{g}$ (or *of*, and also *in*, $\hat{E}$) that is denoted by '$\hat{0}$' and is called the *null vector*. Elements (vectors) of $\hat{E}$ are denoted by the *variables* '$\hat{x}$', '$\hat{y}$', and '$\hat{z}$', which can be furnished with some appropriate labels as Arabic numeral subscripts '$_1$', '$_2$', etc or as primes.

2) An *affine additive group* (*AAG*) $\dot{\boldsymbol{E}}^{g}$ is an algebraic system that consists of a certain *underlying set of points* $\dot{E}$, called its *affine additive group manifold* (*AAGM*), and of a certain *vector group* $\hat{\boldsymbol{E}}^{g}$ whose underlying set $\hat{E}$ of elements, called *vectors*, is related to $\dot{E}$ by a *binary surjection*

$$\hat{V} : \dot{E} \times \dot{E} \to \hat{E}\,, \tag{4.1}$$

which satisfies the following two *AAGM axioms* (*AAGMA's*).

AAGMA1: The law of composition of vectors from ordered pairs of points – The set of bijections between $\dot{E}$ *and* $\hat{E}$. For each $\langle \dot{x}, \dot{y} \rangle \in \dot{E} \times \dot{E}$, there is exactly one $\hat{z} \in \hat{E}$ such that

$$\hat{z} = \hat{V}_{\dot{x}}(\dot{y}) \doteqdot \hat{V}(\dot{x}, \dot{y}) \tag{4.2}$$

and conversely for each $\langle \hat{z}, \dot{x} \rangle \in \hat{E} \times \dot{E}$, there is exactly one $\dot{y} \in \dot{E}$ such that (4.2) holds, i.e.

$$\dot{y} = \hat{V}_{\dot{x}}^{-1}(\hat{z})\,. \tag{4.3}$$

That is to say, given $\dot{x} \in \dot{E}$, the singulary functions $\hat{V}_{\dot{x}}\colon \dot{E} \to \hat{E}$ and $\hat{V}_{\dot{x}}^{-1}\colon \hat{E} \to \dot{E}$, as defined in terms of the binary function (4.1) by (4.2) and (4.3), are two mutually inverse *bijections.*

AAGMA2: The Chasle, or *triangle*, *law*. For each $\langle \dot{x}, \dot{y}, \dot{z} \rangle \in \dot{E}^{3\times}$,

$$\hat{V}(\dot{x}, \dot{y}) \hat{+} \hat{V}(\dot{y}, \dot{z}) \hat{+} \hat{V}(\dot{z}, \dot{x}) = \hat{0}\,. \tag{4.4}$$

3) The *commutative* [*abstract*] *additive group* (*CAG*) $\dot{\boldsymbol{E}}^{g}$ and its underlying vector set $\hat{E}$ are said to be *adjoint* of the AAG group $\dot{\boldsymbol{E}}$ and of its underlying point set (AAGM) $\dot{E}$ respectively. «*Togetherness*» of $\dot{E}$, $\hat{\boldsymbol{E}}^{g}$, and $\hat{V}$ as constituent parts forming a single whole algebraic system $\dot{\boldsymbol{E}}^{g}$ can be expressed by the following formal definition of the latter:

$$\dot{\boldsymbol{E}}^{g} \doteqdot \dot{E} \cup \hat{\boldsymbol{E}}^{g} \cup \hat{V} = \dot{E} \cup \left(\hat{E} \cup \hat{+} \cup \hat{-}\right) \cup \hat{V} \tag{4.5}$$

subject to (2.12).•

Comment 4.1: Definition 4.1 has been made with the purpose to introduce specifically the notions of an affine *additive* group and of an affine *additive* group manifold for convenience in further references. At the same time, Definition 4.1 can obviously be altered to introduce the like notions with "*multiplicative*" instead of "*additive*" or in general without either qualifier. With the help of the appropriate substitutions, all corollaries that are deduced below from Definition 4.1 can be restated so as to become corollaries of the respective modified definition.•

Corollary 4.1: ***The identity law for*** $\hat{V}$**.** For each $\dot{x} \in \dot{E}$:

$$\hat{V}(\dot{x}, \dot{x}) = \hat{0} \tag{4.6}$$

and hence

$$\hat{V}_{\dot{x}}(\dot{x}) = \hat{0}\,, \tag{4.7}$$

$$\hat{V}_{\dot{x}}^{-1}(\hat{0}) = \dot{x}\,. \tag{4.8}$$

Proof: (4.6) follows from (4.4) at $\dot{z} \doteq \dot{y} \doteq \dot{x}$. (4.7) follows from (4.2) at $\dot{y} \doteq \dot{x}$, by (4.6). (4.8) follows from (4.3) at $\hat{z} \doteq \hat{0}$, by (4.7).•

Corollary 4.2: ***The basic inversion law for*** $\hat{V}$**.** For each $\langle \dot{x}, \dot{y} \rangle \in \dot{E} \times \dot{E}$:

$$\hat{V}(\dot{y}, \dot{x}) = \hat{-}\hat{V}(\dot{x}, \dot{y}), \tag{4.9}$$

where $\hat{-}\hat{V}(\dot{x}, \dot{y})$ is *the additive inverse of* $\hat{V}(\dot{x}, \dot{y})$. That is to say, $\hat{V}(\dot{y}, \dot{x})$ and $\hat{V}(\dot{x}, \dot{y})$ are *the additive inverse of each other*.

Proof: By the variant of (4.6) with '$\dot{y}$' or '$\dot{z}$' in place of '$\dot{x}$', it follows from (4.4) at $\dot{z} = \dot{y}$ that

$$\hat{V}(\dot{x}, \dot{y}) \hat{+} \hat{V}(\dot{y}, \dot{x}) \hat{=} \hat{0} \text{ for each } \langle \dot{x}, \dot{y} \rangle \in \dot{E} \times \dot{E}\,. \tag{4.10}$$

The corollary immediately follows from (4.10) by the item CAGA4 of Definition 2.4.•

Corollary 4.3: *A modified triangle law*. For each $\langle \dot{x}, \dot{y}, \dot{z} \rangle \in \dot{E}^{3\times}$:

$$\hat{V}(\dot{x}, \dot{y}) \hat{+} \hat{V}(\dot{y}, \dot{z}) = \hat{V}(\dot{x}, \dot{z}). \tag{4.11}$$

Proof: By the equation $\hat{V}(\dot{z}, \dot{x}) = \hat{-}\hat{V}(\dot{x}, \dot{z})$, which is the variant of (4.9) with '$\dot{z}$' in place of '$\dot{y}$', and also by the item CAGA4 of Definition 2.4, equation (4.11) is equivalent to (4.4).•

Corollary 4.4. The binary surjection $\hat{V}$, (4.1), has the property that for each $\langle \dot{x}, \dot{y} \rangle \in \dot{E} \times \dot{E}$, there is exactly one $\hat{z} \in \hat{E}$ such that

$$\hat{z} = \hat{V}_{\dot{x}}(\dot{y}) \doteq \hat{V}(\dot{x}, \dot{y}) = \hat{-}\hat{V}(\dot{y}, \dot{x}) \doteq \hat{-}\hat{V}_{\dot{y}}(\dot{x}) \tag{4.12}$$

and conversely for each $\langle \hat{z}, \dot{y} \rangle \in \hat{E} \times \dot{E}$, there is exactly one $\dot{x} \in \dot{E}$ such that both (4.12) and hence (4.3) hold and in addition

$$\dot{x} = \hat{V}_{\dot{y}}^{-1}(\hat{-}\,\hat{z}). \tag{4.13}$$

That is to say, in accordance with AAGMA1, relation (4.3) is the inverse of relation (4.12) at $\dot{x}$ held constant, whereas relation (4.13) is the inverse of relation (4.12) at $\dot{y}$ held constant. At the same time, relations (4.3) and (4.13) are mutually inverses at $\hat{z}$ held constant.

Proof: The train of equations (4.12) is the train (4.1), which is developed by supplementing it by equation (4.9) and also by the variant with '$\dot{x}$' and '$\dot{y}$' exchanged of the definition occurring in (4.1). The train (4.12) is equivalent to this one:

$$\hat{-}\,\hat{z} = \hat{V}_{\dot{y}}(\dot{x}) \stackrel{\rightarrow}{\equiv} \hat{V}(\dot{y}, \dot{x}) = \hat{-}\hat{V}(\dot{x}, \dot{y}) \stackrel{\leftarrow}{\equiv} \hat{-}\hat{V}_{\dot{x}}(\dot{y}), \tag{4.12$_1$}$$

while (4.13) is equivalent to the first equation in (4.12$_1$).QED.•

Comment 4.2. By Corollary 4.1, at $\hat{z} \stackrel{\rightarrow}{\equiv} \hat{0}$ and $\dot{x} \stackrel{\rightarrow}{\equiv} \dot{y}$, the conjunction of equations (4.12) and (4.13) reduces to the conjunction of the variants of equations (4.6)–(4.8) with '$\dot{y}$' in place of '$\dot{x}$'.•

Theorem 4.1. There is a binary composition *surjection*

$$\dot{P}\colon \dot{E} \times \hat{E} \to \dot{E}\ , \tag{4.14}$$

such that for each $\hat{z} \in \hat{E}$: (a) for each $\dot{x} \in \dot{E}$: there is exactly one $\dot{y} \in \dot{E}$ such that

$$\dot{y} = \dot{P}_{\hat{z}}(\dot{x}) \stackrel{\rightarrow}{\equiv} \dot{P}(\dot{x}, \hat{z}) \stackrel{\rightarrow}{\equiv} \hat{V}_{\dot{x}}^{-1}(\hat{z}), \tag{4.15}$$

and conversely (b) for each $\dot{y} \in \dot{E}$: there is exactly one $\dot{x} \in \dot{E}$ such that

$$\dot{x} = \dot{P}_{\hat{z}}^{-1}(\dot{y}) = \dot{P}_{\hat{-}\hat{z}}(\dot{y}) \stackrel{\rightarrow}{\equiv} \hat{V}_{\dot{y}}^{-1}(\hat{-}\,\hat{z}). \tag{4.16}$$

By (4.16), for each $\hat{z} \in \hat{E}$:

$$\dot{P}_{\hat{z}}^{-1} = \dot{P}_{\hat{-}\hat{z}}, \tag{4.17}$$

the understanding being that the singular functions

$$\dot{P}_{\hat{z}}\colon \dot{E} \to \dot{E} \text{ and } \dot{P}_{\hat{z}}^{-1}\colon \dot{E} \to \dot{E}, \tag{4.18}$$

which are defined in terms of the binary function (4.1) by (4.15) and (4.16), are two mutually inverse *bijections*.

Proof: The final definientia of the trains of definitions (4.15) and (4.16) are given by equations (4.3) and (4.13) respectively, which are, by Corollary 4.4, mutually inverses at $\hat{z}$ held constant. At the same time, the relation '$\dot{x} = \dot{P}_{\hat{z}}^{-1}(\dot{y})$', occurring in (4.16), is the inverse of the relation '$\dot{y} = \dot{P}_{\hat{z}}(\dot{x})$', occurring in (4.15), while the definition $\dot{P}_{\hat{-}\hat{z}}(\dot{y}) \stackrel{\rightarrow}{\equiv} \hat{V}_{\dot{y}}^{-1}(\hat{-}\,\hat{z})$, occurring in (4.16), is the variant of the definition $\dot{P}(\dot{x},\hat{z}) \dot{\equiv} \hat{V}_{\dot{x}}^{-1}(\hat{z})$, occurring in (4.15), with ‘ $\dot{y}$ ’ in place of ‘ $\dot{x}$ ’ and ‘ $\dot{-}\hat{z}$ ’ in place of ‘ $\hat{z}$ ’.•

Comment 4.3. It should be recalled that the function $\hat{V}_{\dot{x}}^{-1}$, e.g., is the inverse of $\hat{V}_{\dot{x}}$ at $\dot{x}$ held constant. At the same time, the function $\dot{P}_{\hat{z}}^{-1}$ is the inverse of $\dot{P}_{\hat{z}}$ at $\hat{z}$ held constant. Therefore, the equations ‘ $\dot{P}_{\hat{z}}(\dot{x}) = \hat{V}_{\dot{x}}^{-1}(\hat{z})$ ’ and $\dot{P}_{\dot{-}\hat{z}}(\dot{y}) \dot{\equiv} \hat{V}_{\dot{y}}^{-1}(\dot{-}\hat{z})$, e.g., which occur in (4.14) and (4.15), cannot be rewritten as ‘ $\dot{P}_{\hat{z}}^{-1}(\dot{x}) = \hat{V}_{\dot{x}}(\hat{z})$ ’ and ‘ $\dot{P}_{\dot{-}\hat{z}}^{-1}(\dot{y}) \dot{\equiv} \hat{V}_{\dot{y}}(\dot{-}\hat{z})$’ respectively. The former two equations are true by definition, whereas the latter two are false.•

Definition 4.2. 1) The surjection (4.1) is called *the first*, or *basic, surjection of the affine additive group manifold* $\dot{E}$ and also *the vectorization of the set* $\dot{E} \times \dot{E}$.

2) The surjection (4.13) is called *the second surjection of the affine additive group manifold* $\dot{E}$ and also *the pointillage of the set* $\dot{E} \times \hat{E}$.

3) Given $\dot{x} \in \dot{E}$, the bijection $\hat{V}_{\dot{x}}$ as defined by (4.2) is called *the vectorization of the point set* $\dot{E}$ *relative to the point* $\dot{x}$, whereas the inverse bijection $\hat{V}_{\dot{x}}^{-1}$ is called *the pointillage of the vector set* $\hat{E}$ *relative to the point* $\dot{x}$.

4) Given $\hat{z} \in \hat{E}$, the bijection $\dot{P}_{\hat{z}}$ as defined by (4.14) and having the property (4.16) is called *the translation of the affine additive group manifold* $\dot{E}$ *over the vector* $\hat{z}$. In this case, the inverse bijection $\dot{P}_{\hat{z}}^{-1}$ is, by (4.16), *the translation of the affine additive group manifold* $\dot{E}$ *over the vector* $\dot{-}\hat{z}$.•

Corollary 4.5.

$$\dot{P}_{\hat{0}}^{-1}(\dot{x}) = \dot{P}_{\hat{0}}(\dot{x}) \dot{\equiv} \dot{P}(\hat{0},\dot{x}) \dot{\equiv} \hat{V}_{\dot{x}}^{-1}(\hat{0}) = \dot{x} \text{ for each } \dot{x} \in \dot{E}, \tag{4.19}$$

whence

$$\dot{P}_{\hat{0}}^{-1} = \dot{P}_{\hat{0}} \dot{\equiv} \dot{P}(\hat{0}) = I_{\dot{E}}, \tag{4.20}$$

where $I_{\dot{E}}$ is the identity function from $\dot{E}$ onto $\dot{E}$.

Proof: The corollary follows from (4.15)–(4.17) by (4.8).•

Definition 4.3. 1) For each $\langle \dot{x},\dot{y}\rangle \in \dot{E}\times\dot{E}$: the ordered pair $\langle \dot{x},\dot{y}\rangle$, or $(\dot{x},\dot{y})$, is called *the position group-vector of the point* $\dot{y}$ *relative to the point* $\dot{x}$. The point $\dot{x}$ is called *the base*, or *tail, of the position group-vector* $\langle \dot{x},\dot{y}\rangle$, whereas the point $\dot{y}$ is called *the head*, or *terminal*, *of the position group-vector* $\langle \dot{x},\dot{y}\rangle$.

2) In contrast to a position group-vector, which belongs to the set $\dot{E}\times\dot{E}$, a group-vector, which belongs to the set $\hat{E}$ is called a *free group-vector*.•

Comment 4.4. The term "*position group-vector*" ("*of a point relative to a point*") as specified in Definition 4.3 should not be confused with the term '*group-vector*' without the qualifier '*position*'. By AAGMA1, to each ordered pair of points $\dot{x}$ and $\dot{y}$ in $\dot{E}$, different or not, there corresponds a unique group-vector $\hat{z}=\hat{V}(\dot{x},\dot{y})$ in $\hat{E}$. Since $\hat{V}$ is a surjection, therefore any group-vector $\hat{z}\in\hat{E}$ is *a class of equivalence of ordered pairs* $\langle\dot{x},\dot{y}\rangle\in\dot{E}\times\dot{E}$ *of points relative to the surjection* $\hat{V}$. In this case, this class is a regular one, i.e. a set, so that

$$\hat{z}\doteq\left\{\langle\dot{x},\dot{y}\rangle\middle|\langle\dot{x},\dot{y}\rangle\in\dot{E}\times\dot{E}\text{ and }\hat{V}(\dot{x},\dot{y})=\hat{z}\right\}\text{ for each }\hat{z}\in\hat{E} \tag{4.21}$$

and particularly

$$\hat{0}=\left\{\langle\dot{x},\dot{x}\rangle\middle|\dot{x}\in\dot{E}\text{ and }\hat{V}(\dot{x},\dot{x})=\hat{0}\right\}\in\hat{E}\,. \tag{4.22}$$

These relations are of course *tautologies*, but they demonstrate that any attempt to treat the vector as an *arrow* that has certain end points, i.e. a certain tail (base) point and a certain head (terminal) point, is inconsistent. Therefore, the term "position *group-vector*" should not mislead the reader. Either of these terms is just a synonym of the term "ordered pair of points".

2) Incidentally, if a vector group $\hat{\boldsymbol{E}}^{\mathrm{g}}$ is treated as an *autonomous* algebraic system in no connection with any affine group $\dot{\boldsymbol{E}}^{\mathrm{g}}$ then a group-vector in $\hat{\boldsymbol{E}}^{\mathrm{g}}$ can be regarded as an insensible *nonempty individual*. A point of $\dot{\boldsymbol{E}}^{\mathrm{g}}$ is also an insensible *nonempty individual*. If, however, $\hat{\boldsymbol{E}}^{\mathrm{g}}$ is treated as the adjoint vector group of a certain affine group $\dot{\boldsymbol{E}}^{\mathrm{g}}$ then, a group-vector of $\hat{\boldsymbol{E}}^{\mathrm{g}}$ including the null group-vector becomes, as explicated in the previous item, a *set* (*regular class*, *small class*) *of equivalence* of ordered pairs of points of $\dot{\boldsymbol{E}}^{\mathrm{g}}$ and therefore it ceases to be a nonempty individual. At the same time, a separate ordered pair $\langle\dot{x},\dot{y}\rangle\in\dot{E}\times\dot{E}$, i.e. a separate position group-vector, is a set, namely $\langle\dot{x},\dot{y}\rangle=\{\dot{x},\{\dot{x},\dot{y}\}\}$, and therefore it is not a nonempty individual either.

3) In the general case, a single point in $\dot{E}$ is not a group-vector in $\hat{E}$, except a certain special case to be explicated by Theorem 4.2 below in subsection 4.3.•

Definition 4.4. For each $n\in\omega_1$:

$$\bar{\dot{x}}_{[1,n]}\doteq\langle\dot{x}_1,\ldots,\dot{x}_n\rangle\in\dot{E}^{n\times}, \tag{4.23}$$

the understanding being that

$$\bar{\dot{x}}_{[1,1]} \rightleftharpoons \langle \dot{x}_1 \rangle = \{\dot{x}_1\} \in \dot{E}^{1\times}. \tag{4.24}$$

Corollary 4.6: ***The general contraction law for $\hat{V}$ with respect to $\hat{\dotplus}$.*** For each $n \in \omega_3$, for each $\bar{\dot{x}}_{[1,n]}$ satisfying (4.23):

$$\hat{V}(\dot{x}_1, \dot{x}_2) \hat{\dotplus} \hat{V}(\dot{x}_2, \dot{x}_3) \hat{\dotplus} \ldots \hat{\dotplus} \hat{V}(\dot{x}_{n-2}, \dot{x}_{n-1}) \hat{\dotplus} \hat{V}(\dot{x}_{n-1}, \dot{x}_n) = \hat{V}(\dot{x}_1, \dot{x}_n). \tag{4.25}$$

Proof: The proof of the corollary is one by induction on 'n'. Equation (4.25) at $n = 3$ is true by (4.11). Let us, therefore, assume that equation (4.25) is true. In this case, by the variant of (4.11) with '$\dot{x}_1$', '$\dot{x}_n$', and '$\dot{x}_{n+1}$' in place of '$\dot{x}$', '$\dot{y}$', and '$\dot{z}$', respectively, it follows from (4.25) with n+1 in place of n that

$$\begin{aligned} &\left[\hat{V}(\dot{x}_1, \dot{x}_2) \hat{\dotplus} \hat{V}(\dot{x}_2, \dot{x}_3) \hat{\dotplus} \ldots \hat{\dotplus} \hat{V}(\dot{x}_{n-2}, \dot{x}_{n-1}) \hat{\dotplus} \hat{V}(\dot{x}_{n-1}, \dot{x}_n)\right] \hat{\dotplus} \hat{V}(\dot{x}_n, \dot{x}_{n+1}) \\ &\quad = \hat{V}(\dot{x}_1, \dot{x}_n) \hat{\dotplus} \hat{V}(\dot{x}_n, \dot{x}_{n+1}) = \hat{V}(\dot{x}_1, \dot{x}_{n+1}), \end{aligned} \tag{4.26}$$

where '$\dot{x}_{n+1}$' is, besides '$\dot{x}_1$', ..., '$\dot{x}_n$', another variable with values in $\dot{E}$. The pair of square parenthesis on the left-hand side of expression (4.26) indicates the way in which the corresponding parts of that expression should be associated with respect to the vary last occurrence of '$\hat{\dotplus}$'. By the item CAGA2 of Definition 2.4, the pair of square parenthesis can be omitted, which proves that the variant of (4.25) with '$n+1$' in place of 'n' is true. QED.•

Corollary 4.7: ***A polygon law or a generalized Chasle law.*** For each $n \in \omega_2$, for each $\bar{\dot{x}}_{[1,n]} \in \dot{E}^{n\times}$:

$$\hat{V}(\dot{x}_1, \dot{x}_2) \hat{\dotplus} \hat{V}(\dot{x}_2, \dot{x}_3) \hat{\dotplus} \ldots \hat{\dotplus} \hat{V}(\dot{x}_{n-2}, \dot{x}_{n-1}) \hat{\dotplus} \hat{V}(\dot{x}_{n-1}, \dot{x}_1) = \hat{0}. \tag{4.27}$$

Proof: (4.27) immediately follows from (4.25) at $\dot{x}_{n+1} = \dot{x}_1$, by (4.6).•

Corollary 4.8: ***The general inversion law for $\hat{V}$ with respect to $\hat{\dotplus}$.*** For each $n \in \omega_3$: for each $\bar{\dot{x}}_{[1,n]} \in \dot{E}^{n\times}$:

$$\begin{aligned} \hat{\dot{-}}\hat{V}(\dot{x}_1, \dot{x}_n) &= \hat{\dot{-}}\left[\hat{V}(\dot{x}_1, \dot{x}_2) \hat{\dotplus} \hat{V}(\dot{x}_2, \dot{x}_3) \hat{\dotplus} \ldots \hat{\dotplus} \hat{V}(\dot{x}_{n-2}, \dot{x}_{n-1}) \hat{\dotplus} \hat{V}(\dot{x}_{n-1}, \dot{x}_n)\right] \\ &= \left[\hat{\dot{-}}\hat{V}(\dot{x}_{n-1}, \dot{x}_n)\right] \hat{\dotplus} \left[\hat{\dot{-}}\hat{V}(\dot{x}_{n-2}, \dot{x}_{n-1})\right] \hat{\dotplus} \ldots \hat{\dotplus} \left[\hat{\dot{-}}\hat{V}(\dot{x}_2, \dot{x}_3)\right] \hat{\dotplus} \left[\hat{\dot{-}}\hat{V}(\dot{x}_1, \dot{x}_2)\right] \\ &= \hat{V}(\dot{x}_n, \dot{x}_{n-1}) \hat{\dotplus} \hat{V}(\dot{x}_{n-1}, \dot{x}_{n-2}) \hat{\dotplus} \ldots \hat{\dotplus} \hat{V}(\dot{x}_3, \dot{x}_2) \hat{\dotplus} \hat{V}(\dot{x}_2, \dot{x}_1) = \hat{V}(\dot{x}_n, \dot{x}_1). \end{aligned} \tag{4.28}$$

Proof: By the variant of (4.10) with '$\dot{x}_1$' and '$\dot{x}_n$' in place of '$\dot{x}$' and '$\dot{y}$', respectively, it immediately follows from (4.25) that

$$\begin{aligned} &\hat{\dot{-}}\left[\hat{V}(\dot{x}_1, \dot{x}_2) \hat{\dotplus} \hat{V}(\dot{x}_2, \dot{x}_3) \hat{\dotplus} \ldots \hat{\dotplus} \hat{V}(\dot{x}_{n-2}, \dot{x}_{n-1}) \hat{\dotplus} \hat{V}(\dot{x}_{n-1}, \dot{x}_n)\right] \\ &\quad = \hat{\dot{-}}\hat{V}(\dot{x}_1, \dot{x}_n) = \hat{V}(\dot{x}_n, \dot{x}_1) \end{aligned} \tag{4.29}$$

At the same time, it is evident that

$$\hat{V}\left(\dot{x}_n,\dot{x}_1\right)=\hat{V}\left(\dot{x}_n,\dot{x}_{n-1}\right)\hat{+}\hat{V}\left(\dot{x}_{n-1},\dot{x}_n\right)\hat{+}\ldots\hat{+}\hat{V}\left(\dot{x}_3,\dot{x}_2\right)\hat{+}\hat{V}\left(\dot{x}_2,\dot{x}_1\right), \tag{4.30}$$

because this is the variant of (4.25) with ‘ $n-i+1$ ’ in place of ‘ i ’ for each $i\in\omega_{1,n}$. In this case, by (4.10),

$$\hat{V}\left(\dot{x}_{i+1},\dot{x}_i\right)=\dot{-}\hat{V}\left(\dot{x}_i,\dot{x}_{i+1}\right) \text{ for each } i\in\omega_{1,n-1}. \tag{4.31}$$

The conjunction of equations (4.29)-(4.31) is equivalent to (4.28). In the above proof, use of the item CAGA2 of Definition 2.4 has tacitly been made. At the same time, the item CAGA5 of Definition 2.4 has not been. QED.•

4.2. Extensions of the surjection $\hat{V}$ and of the bijections $\dot{P}_{\hat{z}}$ and $\dot{P}_{\hat{z}}^{-1}$ to the power sets of their domains of definition

Definition 4.5. 1) For each $\dot{x}\in\dot{E}$, for each $\dot{Y}\subseteq\dot{E}$:

$$\begin{gathered}\hat{V}\left(\{\dot{x}\},\dot{Y}\right)=\hat{V}_{\{\dot{x}\}}\left(\dot{Y}\right)\\ \doteqdot\left\{\hat{z}\middle|\hat{z}=\hat{V}\left(\dot{x},\dot{y}\right)=\hat{V}_{\dot{x}}\left(\dot{y}\right)\text{and}\left\langle\dot{x},\dot{y}\right\rangle\in\{\dot{x}\}\times\dot{Y}\right\}\subseteq\hat{E}.\end{gathered} \tag{4.32}$$

2) For each $\hat{z}\in\hat{E}$:

$$\begin{gathered}\dot{P}_{\hat{z}}\left(\dot{X}\right)=\dot{P}\left(\dot{X},\hat{z}\right)=\hat{V}_{\dot{X}}^{-1}\left(\hat{z}\right)\\ \doteqdot\left\{\dot{y}\middle|\dot{y}=\dot{P}_{\hat{z}}\left(\dot{x}\right)=\dot{P}\left(\dot{x},\hat{z}\right)=\hat{V}_{\dot{x}}^{-1}\left(\hat{z}\right)\text{and}\,\dot{x}\in\dot{X}\right\}\subseteq\dot{E}\\ \text{for each }\dot{X}\subseteq\dot{E},\end{gathered} \tag{4.33}$$

$$\begin{gathered}\dot{P}_{\hat{z}}^{-1}\left(\dot{Y}\right)=\dot{P}_{\dot{-}\hat{z}}\left(\dot{Y}\right)=\hat{V}_{\dot{Y}}^{-1}\left(\dot{-}\hat{z}\right)\\ \doteqdot\left\{\dot{x}\middle|\dot{x}=\dot{P}_{\hat{z}}^{-1}\left(\dot{y}\right)=\dot{P}_{\dot{-}\hat{z}}\left(\dot{y}\right)=\hat{V}_{\dot{y}}^{-1}\left(\dot{-}\hat{z}\right)\text{and}\,\dot{y}\in\dot{Y}\right\}\subseteq\dot{E}\\ \text{for each }\dot{Y}\subseteq\dot{E}.\end{gathered} \tag{4.34}$$

Hence particularly,

$$\begin{gathered}\hat{V}\left(\{\dot{x}\},\dot{E}\right)=\hat{V}_{\{\dot{x}\}}\left(\dot{E}\right)=\hat{E},\\ \dot{P}_{\hat{z}}\left(\dot{E}\right)=\dot{P}\left(\dot{E},\hat{z}\right)=\hat{V}_{\dot{E}}^{-1}\left(\hat{z}\right)=\dot{E},\ .\\ \dot{P}_{\hat{z}}^{-1}\left(\dot{E}\right)=\dot{P}_{\dot{-}\hat{z}}\left(\dot{E}\right)\doteqdot\hat{V}_{\dot{E}}^{-1}\left(\dot{-}\hat{z}\right)=\dot{E}.\end{gathered} \tag{4.35}$$

Comment 4.5 (analogous to Comment 3.6**):** By the conventional definition of a power set (see, e.g., Halmos [1960, p. 19]), the power sets $\mathsf{P}\left(\dot{E}\times\dot{E}\right)$ and $\mathsf{P}\left(\dot{E}\right)$ can contextually be defined as:

$$\dot{X}\subseteq\dot{E}\text{ if and only if }\dot{X}\in\mathsf{P}\left(\dot{E}\right) \tag{4.36}$$

and similarly with ‘ $\dot{Y}$ ’ in place of ‘ $\dot{X}$ ’. Therefore, definitions (4.32)–(4.34) extend the relations (4.2), (4.15), and (4.16) and the functions involved in them to the pertinent power sets. In accordance with the presently common practice, the extension of each one of the functions is denoted by the same symbol as that denoting the original function. By (4.32), the vector set $\hat{V}(\{\dot{x}\},\dot{Y}) \subseteq \hat{E}$ is the image of the set $\{\dot{x}\}\times\dot{Y} \subseteq \dot{E}\times\dot{E}$ in the adjoint vector set $\hat{E}$ under the mapping $\hat{V}$. Similarly, the point set $\dot{P}_{\hat{z}}(\dot{X}) \subseteq \dot{E}$ is, by (4.33), the image (injection) of a certain subset $\dot{X}$ of the underlying set $\dot{E}$ in that same $\dot{E}$ under the mapping $\dot{P}_{\hat{z}}$, whereas the point set $\dot{P}_{\hat{z}}^{-1}(\dot{Y}) \subseteq \dot{E}$ is, by (4.34), the image (injection) of a certain subset $\dot{Y}$ of the underlying set $\dot{E}$ in that same $\dot{E}$ under the mapping $\dot{P}_{\hat{z}}^{-1}$. At the same time, the point set $\dot{P}_{\hat{z}}(\dot{X}) \subseteq \dot{E}$ can alternatively be treated as the translation of $\dot{X}$ in the result of translation of $\dot{E}$ over the vector $\hat{z} \in \hat{E}$, whereas the point set $\dot{P}_{\hat{z}}^{-1}(\dot{Y}) \subseteq \dot{E}$ can alternatively be treated as the translation of $\dot{Y}$ in the result of translation of $\dot{E}$ over the vector $\hat{-}\hat{z} \in \hat{E}$.•

4.3. Self-adjoint affine additive group manifolds

Theorem 4.2. A commutative additive group $\hat{\boldsymbol{E}}^{\mathrm{g}}$ and its underlying vector set $\hat{E}$ are *self-adjoint affine ones*, denoted also by '$\ddot{\boldsymbol{E}}^{\mathrm{g}}$' and '$\ddot{E}$' respectively, i.e.

$$\ddot{\boldsymbol{E}}^{\mathrm{g}} \doteqdot \hat{\boldsymbol{E}}^{\mathrm{g}} \text{ and } \ddot{E} \doteqdot \hat{E}, \tag{4.37}$$

if the surjection $\hat{V}$, (4.1), is defined as:

$$\hat{V}(\hat{x},\hat{y}) \doteqdot \hat{y} \hat{-} \hat{x} \doteqdot \hat{y} \hat{+} (\hat{-}\hat{x}) \in \hat{E} \text{ for each } \langle \hat{x},\hat{y}\rangle \in \hat{E}\times\hat{E}. \tag{4.38}$$

Proof: In accordance with (4.37) and (4.38), let $\dot{x} \doteqdot \hat{x} \in \hat{E}$ and similarly for all other point-valued variables with an overdot. By (4.38), equation (4.2) becomes

$$\hat{z} = \hat{V}_{\hat{x}}(\hat{y}) = \hat{V}(\hat{x},\hat{y}) \doteqdot \hat{y} \hat{-} \hat{x}. \tag{4.39}$$

Solving equation (4.39) with respect to '$\hat{y}$' yields the pertinent specification of equation (4.3) in the form

$$\hat{y} = \hat{V}_{\hat{x}}^{-1}(\hat{z}) \doteqdot \hat{z} \hat{+} \hat{x}. \tag{4.40}$$

By (4.38), equation (4.4) becomes

$$\hat{y} \hat{-} \hat{x} \hat{+} \hat{z} \hat{-} \hat{y} \hat{+} \hat{x} \hat{-} \hat{z} = \hat{0}, \tag{4.41}$$

which is a tautology. Thus, under definitions (4.37) and (4.38), both axioms AAGMA1and AAGMA2 of Definition 4.1 are satisfied. QED.•

Comment 4.6. 1) By (4.38) or (4.39), equation (4.6) becomes

$$\hat{x} \hat{-} \hat{x} = \hat{0}. \tag{4.42}$$

This equation is a tautology, i.e. it is always true, although it cannot be deduced from (4.41) as was done in the general case. Likewise, one can readily check the validity of all corollaries that have been deduced from Definition 4.1.•

5. Affine spaces

5.1. An affine space as an affine additive group

Preliminary Remark 5.1. When $\hat{\boldsymbol{E}}^{\mathrm{g}}$ is successively supplemented by the appropriate additional attributes to become first a vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ and ultimately an *n*-dimensional Euclidean vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, $\dot{\boldsymbol{E}}^{\mathrm{g}}$ is automatically self-adjusted to all current metamorphoses of its adjoint CAG $\hat{\boldsymbol{E}}^{\mathrm{g}}$ to become first an affine space $\dot{\boldsymbol{E}}(\boldsymbol{R})$ and ultimately an *n*-dimensional Euclidean affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$. The following definitions of this subsection are subject to the above way of development of $\dot{\boldsymbol{E}}^{\mathrm{g}}$.•

Definition 5.1. 1) An affine additive group (AAG) $\dot{\boldsymbol{E}}^{\mathrm{g}}$ is called an *affine space over the field* $\boldsymbol{R}$ *of real numbers* and also a *real affine space* (*RAS*) and it is denoted by '$\dot{\boldsymbol{E}}$' or '$\dot{\boldsymbol{E}}(\boldsymbol{R})$' if and only if the commutative additive group $\hat{\boldsymbol{E}}^{\mathrm{g}}$ adjoint of $\dot{\boldsymbol{E}}^{\mathrm{g}}$ is a *vector* (*linear*) *space over* $\boldsymbol{R}$, i.e. a *real vector* (*linear*) *space*, denoted also by '$\hat{\boldsymbol{E}}$' or '$\hat{\boldsymbol{E}}(\boldsymbol{R})$'.

2) A real vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ is a *self-ajoint real affine space*, which will alternatively be denoted by '$\ddot{\boldsymbol{E}}(\boldsymbol{R})$' or briefly by '$\ddot{\boldsymbol{E}}$' if it is a *self-ajoint real affine additive group* as defined by Theorem 4.2.•

Definition 5.2. Given $n \in \omega_1$, an affine space $\dot{\boldsymbol{E}}$ over $\boldsymbol{R}$ is said to be *n-dimensional* and it will be denoted by '$\dot{\boldsymbol{E}}_n$' and also by '$\dot{\boldsymbol{E}}_n(\boldsymbol{R})$' if and only if the vector space $\hat{\boldsymbol{E}}$ adjoint of $\dot{\boldsymbol{E}}$ is an *n*-dimensional vector space over $\boldsymbol{R}$, denoted by '$\hat{\boldsymbol{E}}_n$' or '$\hat{\boldsymbol{E}}_n(\boldsymbol{R})$'. Accordingly, besides the variables such as '$\dot{x}$' and '$\hat{x}$', which denote elements of the respective underlying sets $\dot{E}$ and $\hat{E}$ in general, the variables such as '$\dot{x}_{\{n\}}$' and '$\hat{x}_{\{n\}}$' will often be used to denote elements of the above underlying sets once they are specified as $\dot{E}_n$ and $\hat{E}_n$ respectively.•

Corollary 5.1. Given $n \in \omega_1$, the *n*-dimensional real arithmetical vector space $\overline{\boldsymbol{E}}_n(\boldsymbol{R})$ is an *n*-dimensional self-adjoint real affine space, provided that

$$\hat{V}(\hat{x},\hat{y}) = \overline{V}(\overline{x}_{[1,n]},\overline{y}_{[1,n]}) \doteqdot \overline{y}_{[1,n]} = \overline{x}_{[1,n]} \doteqdot \langle y_1 - x_1,\ldots,y_n - x_n \rangle$$
$$= \langle -x_1 + y_1,\ldots,-x_n + y_n \rangle = -\overline{x}_{[1,n]} + \overline{y}_{[1,n]} \in \overline{E}_{\{n\}}, \tag{5.1}$$

in accordance with (3.16) and (4.39).

Proof: The corollary is a specification of Theorem 4.2 in the particular case where

$$\dot{E} = \hat{E} = \overline{E}_n,\ \dot{x} = \hat{x} = \overline{x}_{[1,n]} \in \overline{E}_n. \tag{5.2}$$

and similarly with any other pertinent letter, as '*y*' or '*z*', in place of '*x*'.•

Definition 5.3. An affine space $\dot{\boldsymbol{E}}(\boldsymbol{R})$ is called a *Euclidean affine space over* ***R*** and also a *Euclidean real affine space* if the vector space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ adjoint of $\dot{\boldsymbol{E}}(\boldsymbol{R})$ is a Euclidean real one.•

Definition 5.4. Given $n \in \omega_1$, an *n*-dimensional real affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ is called an *n-dimensional Euclidean real affine space* if the vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ adjoint of $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ is an *n*-dimensional Euclidean real vector space.•

5.2. A combined rectilinear coordinate system in an *n*-dimensional real affine space and a combined normal orthogonal rectilinear coordinate system in an *n*-dimensional Euclidean real affine space

Convention 5.1. For more clarity, the variables and constants that denote subsets (parts) of $\dot{E}_n$ or $\hat{E}_n$ will hereafter be provided with a subscript '$_n$', whereas the variables and constants that denote points in $\dot{E}_n$ or vectors in $\hat{E}_n$, will be provided with a subscript '$_{\{n\}}$'. Thus, for instance, '$\dot{X}_n$' or '$\dot{x}_{\{n\}}$' will be used instead of or interchangeably with '$\dot{X}$' or '$\dot{x}$' for mentioning an arbitrary subset of $\dot{E}_n$ or an arbitrary point in $\dot{E}_n$ respectively. Likewise, '$\hat{X}_n$' or '$\hat{x}_{\{n\}}$' will be used instead of or interchangeably with '$\hat{X}$' or '$\hat{x}$' for mentioning an arbitrary subset of $\hat{E}_n$ or an arbitrary vector in $\hat{E}_n$ respectively. The letters '*X*' and '*x*' in the above examples can be replaced by any appropriate letters (as '*Y*' and '*y*' or '*Z*' and '*z*') of the same fonts of the Latin or Greek alphabet. Also, 'x' can be replaced by the digit '0' with the understanding that $\dot{0}$ or $\dot{0}_{\{n\}}$ is an arbitrary given reference (origin) point in $\dot{E}_n$, whereas $\hat{0}_{\{n\}}$ is the null vector in $\hat{E}_n$, which has previously been denoted by '$\hat{0}$'.•

Definition 5.5. 1) Given $n \in \omega_1$, given an *n*-dimensional real affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, given a point $\dot{0}_{\{n\}} \in \dot{E}_n$, given a basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_n^{\;n\times}$ as defined by (2.50): the ordered pair $c_{\{n\}}$, defined as:

$$c_{\{n\}} \rightleftharpoons \left\langle \dot{0}_{\{n\}}, \bar{\hat{e}}_{[1,n]} \right\rangle \in \dot{E}_n \times \hat{E}_n^{\;n\times} \tag{5.3}$$

is called a *combined rectilinear*, or *Cartesian, coordinate system in* $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ *with the origin* $\dot{0}_{\{n\}}$ *in* $\dot{E}_n$ *and basis* $\bar{\hat{e}}_{[1,n]}$ *in* $\hat{E}_n$.

2) If the space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ is Euclidean and if its basis $\bar{\hat{e}}_{[1,n]}$ is a *normal orthogonal basis* (*NOB*), defined by (3.26) subject to (3.27), then the coordinate system $c_{\{n\}}$ is called a *combined normal orthogonal* (*rectangular*), or *orthonormal, rectilinear coordinate system* (*CNORCS* or *CONRCS*) *in* $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$.•

Convention 5.2. Henceforth, the spaces $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$¸ and $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$ are assumed to be Euclidean and the coordinate system $c_{\{n\}}$, which is defined by (5.3), is assumed to be a *CNORCS* (*CONRCS*).•

Comment 5.1. 1) By the "only-if" part of the theorem of ordered pairs for sets, (2.22), the fact that a coordinate system $c_{\{n\}}$ as defined by (5.3) is given means that both the origin $\dot{0}_{\{n\}} \in \dot{E}_n$ of $c_{\{n\}}$ and the basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_n^{\;n\times}$ of $c_{\{n\}}$ are given. At the same time, by Theorem 3.8 and Definition 3.14, given a basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_n^{\;n\times}$, there are two mutually inverse isomorphisms $\bar{C}_{\bar{\hat{e}}_{[1,n]}}$ and $\bar{C}_{\bar{\hat{e}}_{[1,n]}}^{\;-1}$ between the *n*-dimensional vector spaces $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ and $\bar{\boldsymbol{E}}_n(\boldsymbol{R})$ with the property that *for each* $\hat{x}_{\{n\}} \in \hat{E}_n$ *there is exactly one* $\bar{x}_{[1,n]} \in \bar{E}_n$ *such that*

$$\bar{x}_{[1,n]} = \bar{C}_{\bar{\hat{e}}_{[1,n]}}\left(\hat{x}_{\{n\}}\right), \tag{5.4}$$

and conversely, for each $\bar{x}_{[1,n]} \in \bar{E}_n$*: there is exactly one* $\hat{x}_{\{n\}} \in \hat{E}_n$ *such that*

$$\hat{x}_{\{n\}} = \bar{C}_{\bar{\hat{e}}_{[1,n]}}^{\;-1}\left(\bar{x}_{[1,n]}\right), \tag{5.5}$$

Also, by the item AAGMA1 of Definition 5.1, given a point $\dot{0}_{\{n\}} \in \dot{E}_n$, there are two mutually inverse bijections $\hat{V}_{\dot{0}_{\{n\}}}$ and $\hat{V}_{\dot{0}_{\{n\}}}^{\;-1}$ between the *n*-dimensional affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ and its adjoint n-dimensional vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ with the property that *for each* $\dot{x}_{\{n\}} \in \dot{E}_n$*, there is exactly one* $\hat{x}_{\{n\}} \in \hat{E}_n$ *such that*

$$\hat{x}_{\{n\}} = \hat{V}_{\dot{0}_{\{n\}}}\left(\dot{x}_{\{n\}}\right) = \hat{V}_{\dot{0}_{\{n\}}}\left(\dot{0}_{\{n\}}, \dot{x}_{\{n\}}\right), \tag{5.6}$$

and conversely, for each $\hat{x}_{\{n\}} \in \hat{E}_n$ *: there is exactly one* $\dot{x}_{\{n\}} \in \dot{E}_n$ *such that*

$$\dot{x}_{\{n\}} = \hat{V}_{\dot{0}_{\{n\}}}{}^{-1}\left(\hat{x}_{\{n\}}\right). \tag{5.7}$$

2) In connection with the above said, it is worthy to recall that besides the isomorphisms $\overline{C}_{\bar{\hat{e}}_{[1,n]}}$ and $\overline{C}_{\bar{\hat{e}}_{[1,n]}}{}^{-1}$, which depend on the choice of the basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_n{}^{n\times}$, and also besides the bijections $\hat{V}_{\dot{0}_{\{n\}}}$ and $\hat{V}_{\dot{0}_{\{n\}}}{}^{-1}$, which depend on the choice of the origin $\dot{0}_{\{n\}} \in \dot{E}_n$, there are in the affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ two mutually inverse bijections, which do not depend on the choice of a coordinate system $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, but which do depend on the choice of a vector in $\hat{E}_n$, and hence in $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, as a parameter. Namely, according to Theorem 4.1, *given* $\hat{z}_{\{n\}} \in \hat{E}_n$, *for each* $\dot{x}_{\{n\}} \in \dot{E}_n$ *there is exactly one* $\dot{y}_{\{n\}} \in \dot{E}_n$ *such that*

$$\dot{y}_{\{n\}} = \dot{P}_{\hat{z}_{\{n\}}}\left(\dot{x}_{\{n\}}\right) \overset{\rightarrow}{\equiv} \dot{P}\left(\dot{x}_{\{n\}}, \hat{z}_{\{n\}}\right) \overset{\rightarrow}{\equiv} \hat{V}_{\dot{x}_{\{n\}}}{}^{-1}\left(\hat{z}_{\{n\}}\right), \tag{5.8}$$

and conversely, *for each* $\dot{y}_{\{n\}} \in \dot{E}_n$ *there is exactly one* $\dot{x}_{\{n\}} \in \dot{E}_n$ *such that*

$$\dot{x}_{\{n\}} = \dot{P}_{\hat{z}_{\{n\}}}{}^{-1}\left(\dot{y}_{\{n\}}\right) \overset{\rightarrow}{\equiv} \dot{P}_{\hat{-}\hat{z}_{\{n\}}}\left(\dot{y}_{\{n\}}\right) \overset{\rightarrow}{\equiv} \hat{V}_{\dot{x}_{\{n\}}}{}^{-1}\left(\hat{-}\,\hat{z}_{\{n\}}\right). \tag{5.9}$$

In this case, according to the item 4 of Definition 4.2, the bijection $\dot{P}_{\hat{z}_{\{n\}}}$ is *the translation of the affine space* $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ *over the vector* $\hat{z}_{\{n\}}$, *whereas the inverse bijection* $\dot{P}_{\hat{z}_{\{n\}}}{}^{-1}$ *is the translation of* $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ *over the vector* $\hat{-}\,\hat{z}_{\{n\}}$.•

Corollary 5.2. Given $n \in \omega_1$, given a CNORCS $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ as defined by (5.3), there is a *composite bijection*

$$\overline{K}_{c_{\{n\}}} \overset{\rightarrow}{\equiv} \overline{C}_{\bar{\hat{e}}_{[1,n]}} \circ \hat{V}_{\dot{0}_{\{n\}}} : \dot{E}_n \to \overline{E}_n, \tag{5.10}$$

with the property that for each $\dot{x}_{\{n\}} \in \dot{E}_n$ there is exactly one $\bar{x}_{[1,n]} \in \overline{E}_n$ such that

$$\bar{x}_{[1,n]} = \overline{K}_{c_{\{n\}}}\left(\dot{x}_{\{n\}}\right) \tag{5.11}$$

and there is also the *inverse composite bijection*

$$\overline{K}_{c_{\{n\}}}{}^{-1} \overset{\rightarrow}{\equiv} \hat{V}_{\dot{0}_{\{n\}}}{}^{-1} \circ \overline{C}_{\bar{\hat{e}}_{[1,n]}}{}^{-1} : \overline{E}_n \to \dot{E}_n \tag{5.12}$$

with the property that for each $\bar{x}_{[1,n]} \in \overline{E}_n$ there is exactly one $\dot{x}_{\{n\}} \in \dot{E}_n$ such that

$$\dot{x}_{\{n\}} = \overline{K}_{c_{\{n\}}}{}^{-1}\left(\bar{x}_{[1,n]}\right). \tag{5.13}$$

Proof: The corollary follows from Comment 5.1.•

Definition 5.6. Given $n \in \omega_1$, given a CNORCS $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ as defined by (5.3):

1) For each $\dot{x}_{\{n\}} \in \dot{E}_{\{n\}}$, the arithmetical vector (point) $\overline{x}_{[1,n]} \in \overline{E}_n$ as given by (5.11) (see also (3.47) subject to (3.48) or (3.49)) is called *the ordered n-tuple of coordinates of the point* $\dot{x}_{\{n\}} \in \dot{E}_n$ *relative to the coordinate system* $c_{\{n\}}$.

2) For each $\overline{x}_{[1,n]} \in \overline{E}_n$, the point $\dot{x}_{\{n\}} \in \dot{E}_n$ as given by (5.13) is called *the point with the n-tuple* $\overline{x}_{[1,n]} \in \overline{E}_n$ *of coordinates relative to the coordinate system* $c_{\{n\}}$.

3) The bijection (5.10) is called *the coordinatization of the [abstract] affine space* $\dot{\boldsymbol{E}}_n(\mathbf{R})$ *relative to the coordinate system* $c_{\{n\}}$, whereas the inverse bijection (5.12) is called *the [abstract] pointillage of the arithmetical vector space* $\overline{\boldsymbol{E}}_n(\mathbf{R})$ *relative to* $c_{\{n\}}$ (cf. Definition 3.14).

Corollary 5.3. Given $n \in \omega_1$, given a CNORCS $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ as defined by (5.3):

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}} = \overline{K}_{c_{\{n\}}} \circ \hat{V}_{\dot{0}_{\{n\}}}^{-1} : \hat{E}_n \to \overline{E}_n , \tag{5.14}$$

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1} = \hat{V}_{\dot{0}_{\{n\}}} \circ \overline{K}_{c_{\{n\}}}^{-1} : \overline{E}_n \to \hat{E}_n , \tag{5.15}$$

$$\hat{V}_{\dot{0}_{\{n\}}} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1} \circ \overline{K}_{c_{\{n\}}} : \dot{E}_n \to \hat{E}_n , \tag{5.16}$$

$$\hat{V}_{\dot{0}_{\{n\}}}^{-1} = \overline{K}_{c_{\{n\}}}^{-1} \circ \overline{C}_{\bar{\hat{e}}_{[1,n]}} : \hat{E}_n \to \dot{E}_n . \tag{5.17}$$

Proof: Multiplying (5.10) by '$\hat{V}_{\dot{0}_{\{n\}}}^{-1}$' from the right or by '$\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1}$' from the left yields (5.14) and (5.16) respectively. Equations (5.15) and (5.17) are deduced from (5.12) in the similar way. Alternatively, (5.15) and (5.17) can be obtained by forming the inverses of the expressions on both sides of each one of equations (5.14) and (5.16).•

Corollary 5.4. Given $n \in \omega_1$, given a CNORCS $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ as defined by (5.3), the ordered n-tuple $\overline{x}_{[1,n]} \in \overline{E}_n$ of coordinates of a given point $\dot{x}_{\{n\}} \in \dot{E}_n$ relative to the coordinate system $c_{\{n\}}$ coincides with the ordered n-tuple of coordinates of the vector $\hat{x}_{\{n\}} \in \hat{E}_n$, as defined by (5.6), relative to basis $\bar{\hat{e}}_{[1,n]} \in \hat{E}_n^{\,n\times}$.

Proof: Substituting '$\hat{x}_{\{n\}}$' as given by (5.6) into the expression on the right-hand side of equation (5.4), and then making use of (5.10) in the result, one obtains

$$\overline{x}_{[1,n]} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}\left(\hat{V}_{\dot{0}_{\{n\}}}\left(\dot{x}_{\{n\}}\right)\right) = \left(\overline{C}_{\bar{\hat{e}}_{[1,n]}} \circ \hat{V}_{\dot{0}_{\{n\}}}\right)\left(\dot{x}_{\{n\}}\right) = \overline{K}_{c_{\{n\}}}\left(\dot{x}_{\{n\}}\right), \tag{5.18}$$

in agreement with (5.11).•

Corollary 5.5. Given $n \in \omega_1$, given a CNORCS $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ as defined by (5.3):

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}\left(\hat{0}_{\{n\}}\right) = \overline{0}_{[1,n]},\ \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{\ -1}\left(\overline{0}_{[1,n]}\right) = \hat{0}_{\{n\}}, \tag{5.19}$$

$$\hat{V}_{\dot{0}_{\{n\}}}\left(\dot{0}_{\{n\}}\right) = \hat{0}_{\{n\}}, \hat{V}_{\dot{0}_{\{n\}}}^{\ -1}\left(\hat{0}_{\{n\}}\right) = \dot{0}_{\{n\}}, \tag{5.20}$$

$$\overline{K}_{c_{\{n\}}}\left(\dot{0}_{\{n\}}\right) = \overline{0}_{[1,n]},\ \overline{K}_{c_{\{n\}}}^{\ -1}\left(\overline{0}_{[1,n]}\right) = \dot{0}_{\{n\}}, \tag{5.21}$$

where $\hat{0}_{\{n\}}$ is the zero vector in $\hat{E}_n$ and $\overline{0}_{[1,n]}$ is the zero vector in $\overline{E}_n$ as defined by (3.39) and (3.40).

Proof: The corollary follows from equation (3.64) (Theorem 3.8), Corollary 4.1, and Corollary 5.2, by Convention 5.1.•

5.3. Extensions of the bijections $\hat{V}_{\dot{0}}$, $\hat{V}_{\dot{0}}^{-1}$, $\dot{P}_{\hat{z}}$, $\dot{P}_{\hat{z}}^{-1}$, $K_{c_{\{n\}}}$, and $K_{c_{\{n\}}}^{\ -1}$ to the power sets of their domains of definition

Corollary 5.6. 1) Given $n \in \omega_1$, given $\dot{0}_{\{n\}} \in \dot{E}_n$, for each $\dot{Y}_n \subseteq \dot{E}_n$:

$$\begin{gathered}\hat{V}\left(\left\{\dot{0}_{\{n\}}\right\}, \dot{Y}_n\right) = \hat{V}_{\left\{\dot{0}_{\{n\}}\right\}}\left(\dot{Y}_n\right) \\ \doteqdot \left\{\hat{z} \middle| \hat{z} = \hat{V}(\dot{x}, \dot{y}) = \hat{V}_{\dot{x}}(\dot{y}) \text{and} \langle \dot{x}, \dot{y}\rangle \in \left\{\dot{0}_{\{n\}}\right\} \times \dot{Y}_n\right\} \subseteq \hat{E}_n.\end{gathered} \tag{5.22}$$

2) Given $n \in \omega_1$, for each $\hat{z}_{\{n\}} \in \hat{E}_n$:

$$\begin{gathered}\dot{P}_{\hat{z}_{\{n\}}}\left(\dot{X}_n\right) = \dot{P}\left(\dot{X}_n, \hat{z}_{\{n\}}\right) = \hat{V}_{\dot{X}_{\{n\}}}^{\ -1}\left(\hat{z}_{\{n\}}\right) \\ \doteqdot \left\{\dot{y}_{\{n\}} \middle| \dot{y}_{\{n\}} = \dot{P}_{\hat{z}_{\{n\}}}\left(\dot{x}_{\{n\}}\right) = \dot{P}\left(\dot{x}_{\{n\}}, \hat{z}_{\{n\}}\right) = \hat{V}_{\dot{x}}^{-1}\left(\hat{z}_{\{n\}}\right) \text{and}\ \dot{x}_{\{n\}} \in \dot{X}_n\right\} \subseteq \dot{E}_n \\ \text{for each } \dot{X}_n \subseteq \dot{E}_n,\end{gathered} \tag{5.23}$$

$$\begin{gathered}\dot{P}_{\hat{z}_{\{n\}}}^{\ -1}\left(\dot{Y}_n\right) = \dot{P}_{\hat{-}\hat{z}_{\{n\}}}\left(\dot{Y}_n\right) = \hat{V}_{\dot{Y}_n}^{\ -1}\left(\hat{-}\,\hat{z}_{\{n\}}\right) \\ \doteqdot \left\{\dot{x}_{\{n\}} \middle| \dot{x}_{\{n\}} = \dot{P}_{\hat{z}_{\{n\}}}^{\ -1}\left(\dot{y}_{\{n\}}\right) = \dot{P}_{\hat{-}\hat{z}_{\{n\}}}\left(\dot{y}_{\{n\}}\right) = \hat{V}_{\dot{y}_{\{n\}}}^{\ -1}\left(\hat{-}\,\hat{z}_{\{n\}}\right) \text{and}\ \dot{y}_{\{n\}} \in \dot{Y}_n\right\} \subseteq \dot{E}_n \\ \text{for each } \dot{Y}_n \subseteq \dot{E}_n.\end{gathered} \tag{5.24}$$

Hence particularly,

$$\begin{gathered}\hat{V}\left(\left\{\dot{0}_{\{n\}}\right\}, \dot{E}_n\right) = \hat{V}_{\left\{\dot{0}_{\{n\}}\right\}}\left(\dot{E}_n\right) = \hat{E}_n, \\ \dot{P}_{\hat{z}_{\{n\}}}\left(\dot{E}_n\right) = \dot{P}\left(\dot{E}_n, \hat{z}_{\{n\}}\right) = \hat{V}_{\dot{E}_n}^{\ -1}\left(\hat{z}_{\{n\}}\right) = \dot{E}_n, \\ \dot{P}_{\hat{z}_{\{n\}}}^{\ -1}\left(\dot{E}_n\right) = \dot{P}_{\hat{-}\hat{z}_{\{n\}}}\left(\dot{E}_n\right) \doteqdot \hat{V}_{\dot{E}_n}^{\ -1}\left(\hat{-}\,\hat{z}_{\{n\}}\right) = \dot{E}_n.\end{gathered} \tag{5.25}$$

Proof: The corollary is a specification of Definition 4.5 subject to Convention 5.1 and particularly subject to equations (5.6), (5.8). and (5.9).•

Definition 5.7. Given $n \in \omega_1$, given a CNORCS $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ as defined by (5.3):

$$\overline{K}_{c_{\{n\}}}\left(\dot{X}_n\right) \doteqdot \left\{\overline{x}_{[1,n]} \middle| \overline{x}_{[1,n]} = \overline{K}_{c_{\{n\}}}\left(\dot{x}_{\{n\}}\right) \text{and } \dot{x}_{\{n\}} \in \dot{X}_n\right\} \subseteq \overline{E}_n \quad \text{for each } \dot{X}_{\{n\}} \subseteq \dot{E}_n, \tag{5.26}$$

$$\overline{K}_{c_{\{n\}}}^{-1}\left(\overline{X}_n\right) \doteqdot \left\{\dot{x}_{\{n\}} \middle| \dot{x}_{\{n\}} = \overline{K}_{c_{\{n\}}}^{-1}\left(\overline{x}_{[1,n]}\right) \text{and } \overline{x}_{[1,n]} \in \overline{X}_n\right\} \subseteq \dot{E}_n \quad \text{for each } \overline{X}_{\{n\}} \subseteq \overline{E}_n, \tag{5.27}$$

whence

$$\overline{K}_{c_{\{n\}}}\left(\dot{E}_n\right) = \overline{E}_n, \overline{K}_{c_{\{n\}}}^{-1}\left(\overline{E}_n\right) = \dot{E}_n. \tag{5.28}$$

Comment 5.2 (analogous to Comments 3.6 and 4.5**).** By (4.36) and (3.78), definitions (5.26) and (5.27) extend the functions $K_{c_{\{n\}}}$ and $K_{c_{\{n\}}}^{-1}$, as defined originally by (5.10) and (5.12), from the sets $\dot{E}_n$ and $\overline{E}_n$ to the power sets $\mathsf{P}\left(\dot{E}_n\right)$ and $\mathsf{P}\left(\overline{E}_n\right)$ respectively. In accordance with the presently common practice, the extensions have been denoted by the same symbols as the original functions, – just as in Definitions 3.15 and 4.5. By (5.26), the arithmetical vector set $\overline{K}_{c_{\{n\}}}\left(\dot{X}_n\right) \subseteq \overline{E}_n$ is the image of the point set $\dot{X}_n \subseteq \dot{E}_n$ in the arithmetical vector space $\overline{\boldsymbol{E}}_n$ under the mapping $\overline{K}_{c_{\{n\}}}$. Similarly, by (5.27), the point set $\overline{K}_{c_{\{n\}}}^{-1}\left(\overline{X}_n\right) \subseteq \dot{E}_n$ is the image of the arithmetical vector set $\overline{X}_{\{n\}} \subseteq \overline{E}_n$ in the affine space $\dot{\boldsymbol{E}}_{\{n\}}$ under the mapping $\overline{K}_{c_{\{n\}}}^{-1}$.•

Corollary 5.7. Given $n \in \omega_1$, given a CNORCS $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ as defined by (5.3): the [syntactical] variants of relations (5.10), (5.12), and (5.14)-(5.17) with '$\mathsf{P}\left(\dot{E}_n\right)$', '$\mathsf{P}\left(\overline{E}_n\right)$', and '$\mathsf{P}\left(\hat{E}_n\right)$' in place of '$\dot{E}_n$', '$\overline{E}_n$', and '$\hat{E}_n$' respectively are semantically sound; that is,

$$\overline{K}_{c_{\{n\}}} \doteqdot \overline{C}_{\bar{\hat{e}}_{[1,n]}} \circ \hat{V}_{\dot{0}_{\{n\}}} : \mathsf{P}\left(\dot{E}_n\right) \to \mathsf{P}\left(\overline{E}_n\right), \tag{5.29}$$

$$\overline{K}_{c_{\{n\}}}^{-1} \doteqdot \hat{V}_{\dot{0}_{\{n\}}}^{-1} \circ \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1} : \mathsf{P}\left(\overline{E}_n\right) \to \mathsf{P}\left(\dot{E}_n\right), \tag{5.30}$$

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}} = \overline{K}_{c_{\{n\}}} \circ \hat{V}_{\dot{0}_{\{n\}}}^{-1} : \mathsf{P}\left(\hat{E}_n\right) \to \mathsf{P}\left(\overline{E}_n\right), \tag{5.31}$$

$$\overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1} = \hat{V}_{\dot{0}_{\{n\}}} \circ \overline{K}_{c_{\{n\}}}^{-1} : \mathsf{P}\left(\overline{E}_n\right) \to \mathsf{P}\left(\hat{E}_n\right), \tag{5.32}$$

$$\hat{V}_{\dot{0}_{\{n\}}} = \overline{C}_{\bar{\hat{e}}_{[1,n]}}^{-1} \circ \overline{K}_{c_{\{n\}}} : \mathsf{P}\left(\dot{E}_n\right) \to \mathsf{P}\left(\hat{E}_n\right), \tag{5.33}$$

$$\hat{V}_{\dot{0}_{\{n\}}}^{-1} = \overline{K}_{c_{\{n\}}}^{-1} \circ \overline{C}_{\bar{\hat{e}}_{[1,n]}} : \mathsf{P}\left(\hat{E}_n\right) \to \mathsf{P}\left(\dot{E}_n\right). \tag{5.34}$$

Proof: By (5.10) subject to (5.4) and (5.6), it follows from (5.26) that

$$\overline{K}_{c_{\{n\}}}\left(\dot{X}_n\right) \doteqdot \left\{\overline{x}_{[1,n]} \middle| \overline{x}_{[1,n]} = \overline{C}_{\bar{\hat{e}}_{[1,n}}\left(\hat{V}_{\dot{0}_{\{n\}}}\left(\dot{x}_{\{n\}}\right)\right) \text{and } \dot{x}_{\{n\}} \in \dot{X}_n\right\}$$
$$= \left\{\overline{x}_{[1,n]} \middle| \overline{x}_{[1,n]} = \left(\overline{C}_{\bar{\hat{e}}_{[1,n}} \circ \hat{V}_{\dot{0}_{\{n\}}}\right)\left(\dot{x}_{\{n\}}\right) \text{and } \dot{x}_{\{n\}} \in \dot{X}_n\right\} \tag{5.35}$$
$$= \left(\overline{C}_{\bar{\hat{e}}_{[1,n}} \circ \hat{V}_{\dot{0}_{\{n\}}}\right)\left(\dot{X}_n\right) \text{for each } \dot{X}_n \subseteq \dot{E}_n,$$

which proves (5.29). Equation (5.30) is proved analogously from (5.27) by (5.12) subject to (5.5) and (5.7). Equations (5.30)–(5.34) are proved from (5.14)-(5.17) after the same manner by making use of (5.4)–(5.7), (5.11), and (5.13).•

5.4. Simplest figures in an *n*-dimensional Euclidean real affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$

Definition 5.8. 1) In the following definitions, it is assumed that $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ is a given *n*-dimensional Euclidean real affine space, in which a CNORCS $c_{\{n\}}$, defined by (5.3). It is also assumed that $\dot{a} \doteqdot \dot{a}_{\{n\}} \in \dot{E}_n$ and $\dot{b} \doteqdot \dot{b}_{\{n\}} \in \dot{E}_n$ are arbitrary given points and that $\dot{x} \doteqdot \dot{x}_{\{n\}} \in \dot{E}_n$ is a current point – three points in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, the coordinates of which relative to $c_{\{n\}}$ are coordinates of the respective arithmetical vectors

$$a_{[1,n]} \doteqdot \langle a_1, a_2, ..., a_n \rangle \in \overline{E}_n, b_{[1,n]} \doteqdot \langle b_1, b_2, ..., b_n \rangle \in \overline{E}_n,$$
$$x_{[1,n]} \doteqdot \langle x_1, x_2, ..., x_n \rangle \in \overline{E}_n, \tag{5.36}$$

so that

$$\hat{V}\left(\dot{0}_{\{n\}}, \dot{a}_{\{n\}}\right) = \hat{\underset{i=1}{\overset{n}{+}}} a_i \,\hat{\cdot}\, \hat{e}_i, \hat{V}\left(\dot{0}_{\{n\}}, \dot{b}_{\{n\}}\right) = \hat{\underset{i=1}{\overset{n}{+}}} b_i \,\hat{\cdot}\, \hat{e}_i, \hat{V}\left(\dot{0}_{\{n\}}, \dot{x}_{\{n\}}\right) = \hat{\underset{i=1}{\overset{n}{+}}} x_i \,\hat{\cdot}\, \hat{e}_i. \tag{5.37}$$

2) The point set $\vec{p}\left(\dot{a}, \dot{b}\right) \subset \dot{E}_n$, defined as:

$$\vec{p}\left(\dot{a}, \dot{b}\right) \to \left\{\dot{x} \middle| \hat{V}(\dot{a}, \dot{x}) = \hat{\underset{i=1}{\overset{n}{+}}} (x_i - a_i) \,\hat{\cdot}\, \hat{e}_i \text{ and } x_j \in \left[a_j, b_j\right] \text{for each } j \in \omega_{1,n}\right\}, \tag{5.38}$$

is called the [*space*] *position vector*, or [*space*] *radius-vector*, *of the point* $\dot{b}$ *relative to the point* $\dot{a}$.

3) Particularly,

$$\vec{p}\left(\dot{0}, \dot{a}\right) \to \hat{\underset{i=1}{\overset{n}{+}}} a_i \,\vec{\cdot}\, \vec{e}_i$$
$$\to \left\{\dot{x} \middle| \hat{V}\left(\dot{0}, \dot{x}\right) = \hat{\underset{i=1}{\overset{n}{+}}} x_i \,\hat{\cdot}\, \hat{e}_i \text{ and } x_j \in \left[0, a_j\right] \text{for each } j \in \omega_{1,n}\right\}, \tag{5.39}$$

$$\vec{p}\left(\dot{0},\dot{b}\right)\to \hat{\sum_{i=1}^{n}} b_i \,\hat{\cdot}\, \vec{e}_i$$

$$\to \left\{ \dot{x} \middle| \hat{V}\left(\dot{0},\dot{x}\right) = \hat{\sum_{i=1}^{n}} x_i \,\hat{\cdot}\, \hat{e}_i \text{ and } x_j \in \left[0, b_j\right] \text{for each } j \in \omega_{1,n} \right\}, \tag{5.40}$$

the understanding being that

$$\vec{e}_i \to \left\{ \dot{x} \middle| \hat{V}\left(\dot{0},\dot{x}\right) = x \,\hat{\cdot}\, \hat{e}_i \text{ and } x \in [0,1] \right\} \text{for each } i \in \omega_{1,n}\,. \tag{5.41}$$

4) According to the above item 2, $\vec{p}\left(\dot{a},\dot{b}\right) \in \mathsf{P}\left(\dot{E}_n\right)$, where $\mathsf{P}\left(\dot{E}_n\right)$ is the power set of the set $\dot{E}_n$. Let $\vec{P}_{c_{\{n\}}} \subset \mathsf{P}\left(\dot{E}_n\right)$ be the set of all position vectors of $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ relative to the origin $\dot{0}_{\{n\}}$ of the given $c_{\{n\}}$. There is a real-valued binary function $\vec{\bullet} : \vec{P}_{c_{\{n\}}} \times \vec{P}_{c_{\{n\}}} \to R$, which is called the *inner*, or *scalar*, *multiplication function on* $\vec{P}_{c_{\{n\}}}$ and which is contextually defines as:

$$\vec{e}_i \,\vec{\bullet}\, \vec{e}_j \to \hat{e}_i \,\hat{\bullet}\, \hat{e}_j = \delta_{ij} \text{ for each } i \in \omega_{1,n} \text{ and each } j \in \omega_{1,n} \tag{5.42}$$

subject to (3.27). Therefore, for each $\vec{p}\left(\dot{0},\dot{a}\right) \in \vec{P}_{c_{\{n\}}}$ and each $\vec{p}\left(\dot{0},\dot{b}\right) \in \vec{P}_{c_{\{n\}}}$,

$$\begin{aligned} \vec{p}\left(\dot{0},\dot{a}\right) \vec{\bullet}\, \vec{p}\left(\dot{0},\dot{b}\right) &= \left(\hat{\sum_{i=1}^{n}} a_i \,\hat{\cdot}\, \vec{e}_i \right) \vec{\bullet} \left(\hat{\sum_{j=1}^{n}} b_j \,\hat{\cdot}\, \vec{e}_j \right) = \hat{\sum_{i=1}^{n}} \hat{\sum_{j=1}^{n}} \left(a_i \cdot b_j\right) \cdot \left(\vec{e}_i \,\vec{\bullet}\, \vec{e}_j\right) \\ &= \hat{\sum_{i=1}^{n}} \hat{\sum_{j=1}^{n}} \left(a_i \cdot b_j\right) \cdot \delta_{ij} = \hat{\sum_{i=1}^{n}} \left(a_i \cdot b_i\right) \\ &= \hat{\sum_{i=1}^{n}} \hat{\sum_{j=1}^{n}} \left(a_i \cdot b_j\right) \cdot \left(\hat{e}_i \,\hat{\bullet}\, \hat{e}_j\right) = \left(\hat{\sum_{i=1}^{n}} a_i \,\hat{\cdot}\, \hat{e}_i \right) \hat{\bullet} \left(\hat{\sum_{j=1}^{n}} b_j \,\hat{\cdot}\, \hat{e}_j \right) = \hat{V}\left(\dot{0},\dot{a}\right) \hat{\bullet}\, \hat{V}\left(\dot{0},\dot{b}\right). \end{aligned} \tag{5.43}$$

Consequently, in (5.39) and (5.40), the new binary function $\vec{\cdot} : \left(R \times \vec{P}_{c_{\{n\}}}\right) \cup \left(\vec{P}_{c_{\{n\}}} \times R\right) \to \vec{P}_{c_{\{n\}}}$ of multiplication of each one of the orthonormal position vectors $\vec{e}_1$, $\vec{e}_2$, , $\vec{e}_n$, and hence of each one of the position vectors of the set by a scalar or *R* in either order has been defined contextually as a by-side product of explicit definitions of $\vec{p}\left(\dot{0},\dot{a}\right)$ and $\vec{p}\left(\dot{0},\dot{b}\right)$.•

5) The ordered pair $\dot{c}_{\{n\}}$, defined as:

$$\dot{c}_{\{n\}} \equiv \left\langle \dot{0}_{\{n\}}, \bar{\vec{e}}_{[1,n]} \right\rangle \in \dot{E}_n \times \vec{P}_{c_{\{n\}}}{}^{n\times} \tag{5.44}$$

subject to

$$\bar{\vec{e}}_{[1,n]} \to \left\langle \vec{e}_1, \vec{e}_1, \ldots, \vec{e}_n \right\rangle \in \vec{P}_{c_{\{n\}}}{}^{n\times} \tag{5.45}$$

is called a *uniform normal orthogonal* (*rectangular*), or *orthonormal, rectilinear coordinate system* (*UNORCS* or *UONRCS*) *in* $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$. It is understood that the coordinates of any point of

$\dot{x}_{\{n\}} \in \dot{E}_n$ relative to $\dot{c}_{\{n\}}$ are the same as the coordinates of that point relative to $c_{\{n\}}$. However, a vector $\hat{x}_{\{n\}} \in \hat{E}_n$ cannot be expanded into the basis (5.45). Therefore, $\dot{c}_{\{n\}}$ is unusable.•

Definition 5.9. 1) Given $n \in \omega_1$, given $\overline{x}_{*[1,n]} \in \overline{E}_n$, given $r \in (0,+\infty)$,

$$\begin{aligned}
\overline{B}_n^{\mathrm{o}}\left(r, \overline{x}_{*[1,n]}\right) &\doteq \left\{\overline{x}_{[1,n]} \middle| \overline{x}_{[1,n]} \in \overline{E}_n \text{ and } \left|\overline{x}_{[1,n]} = \overline{x}_{*[1,n]}\right| < r\right\}, \\
\overline{B}_n^{\mathrm{c}}\left(r, \overline{x}_{*[1,n]}\right) &\doteq \left\{\overline{x}_{[1,n]} \middle| \overline{x}_{[1,n]} \in \overline{E}_n \text{ and } \left|\overline{x}_{[1,n]} = \overline{x}_{*[1,n]}\right| \le r\right\}, \\
\overline{B}_n^{\mathrm{b}}\left(r, \overline{x}_{*[1,n]}\right) &\doteq \left\{\overline{x}_{[1,n]} \middle| \overline{x}_{[1,n]} \in \overline{E}_n \text{ and } \left|\overline{x}_{[1,n]} = \overline{x}_{*[1,n]}\right| = r\right\}
\end{aligned} \tag{5.46}$$

subject to

$$\left|\overline{x}_{[1,n]} = \overline{x}_{*[1,n]}\right| \doteq \sqrt{\sum_{i=1}^{n} \left(x_i - x_{*i}\right)^2} \ge 0\,. \tag{5.47}$$

The sets $\overline{B}_n^{\mathrm{o}}\left(r, \overline{x}_{*[1,n]}\right)$, $\overline{B}_n^{\mathrm{c}}\left(r, \overline{x}_{*[1,n]}\right)$, and $\overline{B}_n^{\mathrm{b}}\left(r, \overline{x}_{*[1,n]}\right)$ are called respectively the *open sphere* (or the *spherical neighborhood*), the *closed sphere*, and the *spherical surface, in* $\overline{E}_n$ *of radius* r *centered at the point* $\overline{x}_{*[1,n]}$. The word "*ball*" can be used interchangeably with the word "*sphere*".

2) Given a CNORCS $c_{\{n\}}$ in $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$, relative to which the coordinates of a certain point $\dot{x}_{*\{n\}} \in \dot{E}_n$ are coordinates of the arithmetical vector

$$\overline{x}_{*[1,n]} = \left\langle x_{*1}, x_{*2}, ..., x_{*n} \right\rangle \in \overline{E}_n\,, \tag{5.48}$$

so that

$$\hat{V}\left(\dot{0}_{\{n\}}, \dot{x}_{*\{n\}}\right) = \hat{\sum_{i=1}^{n}} x_{*i} \,\hat{\cdot}\, \hat{e}_i\,, \tag{5.49}$$

the sets $\dot{B}_n^{\mathrm{o}}\left(r, \dot{x}_{*\{n\}}\right)$, $\dot{B}_n^{\mathrm{c}}\left(r, \dot{x}_{*\{n\}}\right)$, and $\dot{B}_n^{\mathrm{b}}\left(r, \dot{x}_{*\{n\}}\right)$, defined as:

$$\begin{aligned}
\dot{B}_n^{\mathrm{o}}\left(r, \dot{x}_{*\{n\}}\right) &\doteq \left\{\dot{x} \middle| \overline{x}_{[1,n]} \in \overline{B}_n^{\mathrm{o}}\left(r, \overline{x}_{*[1,n]}\right) \text{and}\, \hat{V}\left(\dot{x}_{*\{n\}}, \dot{x}\right) = \hat{\sum_{i=1}^{n}} \left(x_i - x_{*i}\right) \hat{\cdot}\, \hat{e}_i\right\}, \\
\dot{B}_n^{\mathrm{c}}\left(r, \dot{x}_{*\{n\}}\right) &\doteq \left\{\dot{x} \middle| \overline{x}_{[1,n]} \in \overline{B}_n^{\mathrm{c}}\left(r, \overline{x}_{*[1,n]}\right) \text{and}\, \hat{V}\left(\dot{x}_{*\{n\}}, \dot{x}\right) = \hat{\sum_{i=1}^{n}} \left(x_i - x_{*i}\right) \hat{\cdot}\, \hat{e}_i\right\}, \\
\dot{B}_n^{\mathrm{b}}\left(r, \dot{x}_{*\{n\}}\right) &\doteq \left\{\dot{x} \middle| \overline{x}_{[1,n]} \in \overline{B}_n^{\mathrm{b}}\left(r, \overline{x}_{*[1,n]}\right) \text{and}\, \hat{V}\left(\dot{x}_{*\{n\}}, \dot{x}\right) = \hat{\sum_{i=1}^{n}} \left(x_i - x_{*i}\right) \hat{\cdot}\, \hat{e}_i\right\},
\end{aligned} \tag{5.50}$$

are called respectively the *open sphere* (or the *spherical neighborhood*), the *closed sphere*, and the *spherical surface, in* $\dot{E}_n$ *of radius* r *centered at the point* $\dot{x}_{*\{n\}}$.•

Comment 5.3. 1) At $n = 1$, definitions (5.44) become

$$\overline{B}_1^{\mathrm{o}}\left(r,\overline{x}_{*[1,1]}\right)=\left(x_{*1}-r,x_{*1}+r\right),\ \overline{B}_1^{\mathrm{c}}\left(r,\overline{x}_{*[1,1]}\right)=\left[x_{*1}-r,x_{*1}+r\right],$$
$$\overline{B}_1^{\mathrm{b}}\left(r,\overline{x}_{*[1,1]}\right)=\left\{x_{*1}-r,x_{*1}+r\right\}, \tag{5.51}$$

the understanding being that

$$\overline{x}_{*[1,1]}\rightleftharpoons\left\langle x_{*1}\right\rangle\rightleftharpoons\left\{x_{*1}\right\}. \tag{5.52}$$

2) Definitions (5.50) are evidently equivalent to

$$B_n^{\mathrm{o}}\left(r,\,\dot{x}_{*\{n\}}\right)$$
$$\rightleftharpoons\left\{\dot{x}\middle|\overline{x}_{[1,n]}\in\overline{E}_n\text{ and }\left|\overline{x}_{[1,n]}=\overline{x}_{*[1,n]}\right|<r\text{ and }\hat{V}\left(\dot{x}_{*\{n\}},\dot{x}\right)=\hat{\sum_{i=1}^{n}}\left(x_i-x_{*i}\right)\hat{\cdot}\,\hat{e}_i\right\},$$
$$B_n^{\mathrm{c}}\left(r,\,\dot{x}_{*\{n\}}\right)$$
$$\rightleftharpoons\left\{\dot{x}\middle|\overline{x}_{[1,n]}\in\overline{E}_n\text{ and }\left|\overline{x}_{[1,n]}=\overline{x}_{*[1,n]}\right|\le r\text{ and }\hat{V}\left(\dot{x}_{*\{n\}},\dot{x}\right)=\hat{\sum_{i=1}^{n}}\left(x_i-x_{*i}\right)\hat{\cdot}\,\hat{e}_i\right\}, \tag{5.50a}$$
$$B_n^{\mathrm{b}}\left(r,\,\dot{x}_{*\{n\}}\right)$$
$$\rightleftharpoons\left\{\dot{x}\middle|\overline{x}_{[1,n]}\in\overline{E}_n\text{ and }\left|\overline{x}_{[1,n]}=\overline{x}_{*[1,n]}\right|=r\text{ and }\hat{V}\left(\dot{x}_{*\{n\}},\dot{x}\right)=\hat{\sum_{i=1}^{n}}\left(x_i-x_{*i}\right)\hat{\cdot}\,\hat{e}_i\right\}.$$

5.5. A real-valued function defined in $\dot{E}_n$ versus a real-valued function defined in $\overline{E}_n$

Preliminary Remark 5.1. In what follows, it is shown that given a coordinate $c_{\{n\}}$ in $\dot{E}_n$, a function $\varPhi_{\dot{E}_n}$, defined on a certain set $\dot{X}_n\subseteq\dot{E}_n$ can be reduced to the respective real-valued function $\varPhi_{\overline{E}_n}$ defined on the respective set $\overline{X}_n\subseteq\overline{E}_n$.•

Definition 5.10. Given a Euclidean real affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ of a given dimension $n\in\omega_1$, let $\varPhi_{\dot{E}_n}$ be a *real-valued* function from $\dot{X}_n\subseteq\dot{E}_n$, to $Y\subseteq R=(-\infty,\infty)$, i.e. a function whose *domain of departure* (D_{dp}), *domain of arrival* (D_{a}), *domain of definition* (D_{df}), and *domain of variation* (D_{v}) are $\dot{E}_n$, R, $\dot{X}_n$, and Y respectively. Thus, symbolically,

$$\varPhi_{\dot{E}_n}:\dot{X}_n\to Y, \tag{5.53}$$

so that

$$D_{\mathrm{dp}}\left(\varPhi_{\dot{E}_n}\right)=\dot{E}_n,\ D_{\mathrm{a}}\left(\varPhi_{\dot{E}_n}\right)=R,\ D_{\mathrm{df}}\left(\varPhi_{\dot{E}_n}\right)=\dot{X}_n\subseteq\dot{E}_n,\ D_{\mathrm{v}}\left(\varPhi_{\dot{E}_n}\right)=Y\subseteq R. \tag{5.54}$$

Consequently, for each $\dot{x}_{\{n\}}\in\dot{X}_n$ there is exactly one $y\in Y$ such that

$$y=\varPhi_{\dot{E}_n}\left(\dot{x}_{\{n\}}\right). \tag{5.55}$$•

Definition 5.11. By (5.13) and (5.27), it follows from Definition 5.10 that there is a composite function

$$\Phi_{\bar{E}_n} : \bar{X}_n \to Y \tag{5.56}$$

subject to

$$\Phi_{\bar{E}_n} \stackrel{\rightarrow}{=} \Phi_{\dot{E}_n} \circ \bar{K}_{c_{\{n\}}}{}^{-1}, \tag{5.57}$$

so that

$$D_{\mathrm{dp}}\left(\Phi_{\bar{E}_n}\right) = \bar{E}_n, D_{\mathrm{a}}\left(\Phi_{\bar{E}_n}\right) = R, D_{\mathrm{df}}\left(\Phi_{\bar{E}_n}\right) = \bar{X}_n \subseteq \bar{E}_n, D_{\mathrm{v}}\left(\Phi_{\bar{E}_n}\right) = Y \subseteq R. \tag{5.58}$$

Consequently, for each $\bar{x}_{[1,n]} \in \bar{X}_n$ there is exactly one $y \in Y$ such that

$$y = \Phi_{\bar{E}_n}\left(\bar{x}_{[1,n]}\right) = \Phi_{\dot{E}_n}\left(\bar{K}_{c_{\{n\}}}{}^{-1}\left(\bar{x}_{[1,n]}\right)\right) \tag{5.59}$$ •

5.6. A real-valued function defined in $\dot{T} \times \dot{E}_n$ versus a real-valued function defined in $\bar{T} \times \bar{E}_n$ or in $R \times \bar{E}_n$

Preliminary Remark 5.2. If exists, a hypothetical measurable time-dependent physical field occurring in an *n*-dimensional Euclidean real affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ should be described by one or more real-valued functions such as $\Psi_{\dot{T}\times\dot{E}_n}$, which is defined on the direct product $\dot{T} \times \dot{X}_n$, where $\dot{X}_n$ is a certain connected subset of $\dot{E}_n$. In what follows, it is shown that given coordinate systems ω in $\dot{T}$ and $c_{\{n\}}$ in $\dot{E}_n$, a function $\Psi_{\dot{T}\times\dot{E}_n}$ can be reduced to a certain real-valued function $\Psi_{\bar{T}\times\bar{E}_n}$ defined on the direct product $\bar{T} \times \bar{X}_n$, where $\bar{X}_n$ is the pertinent subset of $\bar{E}_n$.•

Definition 5.12. Given a Euclidean real affine space $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$ of a given dimension $n \in \omega_1$, let $\Psi_{\dot{T}\times\dot{E}_n}$ be a *real-valued* function from $\dot{T} \times \dot{X}_n \subseteq \dot{T} \times \dot{E}_n$ to $Y \subseteq R = (-\infty, \infty)$, i.e. a function whose *domain of departure* (D_{dp}), *domain of arrival* (D_{a}), *domain of definition* (D_{df}), and *domain of variation* (D_{v}) are $\dot{T} \times \dot{E}_n$, *R*, $\dot{T} \times \dot{X}_n$, and Y respectively. Thus, symbolically,

$$\Psi_{\dot{T}\times\dot{E}_n} : \dot{T} \times \dot{X}_n \to Y, \tag{5.60}$$

so that

$$\begin{gathered} D_{\mathrm{dp}}\left(\Psi_{\dot{T}\times\dot{E}_n}\right) = \dot{T} \times \dot{E}_n, D_{\mathrm{a}}\left(\Psi_{\dot{T}\times\dot{E}_n}\right) = R, \\ D_{\mathrm{df}}\left(\Psi_{\dot{T}\times\dot{E}_n}\right) = \dot{T} \times \dot{X}_n \subseteq \dot{T} \times \dot{E}_n, D_{\mathrm{v}}\left(\Psi_{\dot{T}\times\dot{E}_n}\right) = Y \subseteq R. \end{gathered} \tag{5.61}$$

Consequently, for each $\left\langle \dot{\xi}, \dot{x}_{\{n\}} \right\rangle \in \dot{T} \times \dot{X}_n$ there is exactly one $y \in Y$ such that

$$y = \Psi_{\dot{T}\times\dot{E}_n}\left(\dot{\xi}, \dot{x}_{\{n\}}\right). \tag{5.62}$$

Definition 5.13. By (5.13) and (5.27), it follows from Definition 5.12 that there is a composite function

$$\Psi_{\overline{T}\times\overline{E}_n} : \overline{T} \times \overline{X}_n \to Y \tag{5.63}$$

subject to

$$\Psi_{\overline{T}\times\overline{E}_n} \doteqdot \Psi_{\dot{T}\times\dot{E}_n} \circ \left(\overline{K}_\omega^{\;-1} \times \overline{K}_{c_{\{n\}}}^{\;-1}\right), \tag{5.64}$$

so that

$$\begin{gathered} D_{\mathrm{dp}}\left(\Psi_{\overline{T}\times\overline{E}_n}\right) = \overline{T} \times \overline{E}_n, D_{\mathrm{a}}\left(\Psi_{\overline{T}\times\overline{E}_n}\right) = R, \\ D_{\mathrm{df}}\left(\Psi_{\overline{T}\times\overline{E}_n}\right) = \overline{T} \times \overline{X}_n \subseteq \overline{T} \times \overline{E}_n, D_{\mathrm{v}}\left(\Psi_{\overline{T}\times\overline{E}_n}\right) = Y \subseteq R. \end{gathered} \tag{5.65}$$

Consequently, for each $\left\langle \overline{\xi}, \overline{x}_{[1,n]} \right\rangle \in \overline{T} \times \overline{X}_n$ there is exactly one $y \in Y$ such that

$$\begin{aligned} y = \Psi_{\overline{T}\times\overline{E}_n}\left(\overline{\xi}, \overline{x}_{[1,n]}\right) &= \Psi_{\dot{T}\times\dot{E}_n}\left(\left(\overline{K}_\omega^{\;-1} \times \overline{K}_{c_{\{n\}}}^{\;-1}\right)\left(\overline{\xi}, \overline{x}_{[1,n]}\right)\right) \\ &= \Psi_{\dot{T}\times\dot{E}_n}\left(\overline{K}_\omega^{\;-1}\left(\overline{\xi}\right), \overline{K}_{c_{\{n\}}}^{\;-1}\left(\overline{x}_{[1,n]}\right)\right). \end{aligned} \tag{5.66}$$ •

Definition 5.14. The mapping $\chi : \overline{T} \to R$, under which, e.g., $\chi\left(\left\langle x_0 \right\rangle\right) = x_0$ for each $\left\langle x_0 \right\rangle \in \overline{T}$, is a bijection. Hence, the mapping $\chi^{-1} : R \to \overline{T}$, under which $\chi^{-1}\left(x_0\right) = \left\langle x_0 \right\rangle$ for each $x_0 \in R$, is the inverse of that bijection. Consequently, for each $\left\langle x_0, \overline{x}_{[1,n]} \right\rangle \in R \times \overline{X}_n$, i.e. for each $x_0 \in R$ and each $\overline{x}_{[1,n]} \in \overline{X}_n$,

$$\psi^{\langle 1,n \rangle}\left(x_0, \overline{x}_{[1,n]}\right) \doteqdot \psi_{R\times\overline{E}_n}\left(x_0, \overline{x}_{[1,n]}\right) \doteqdot \Psi_{\overline{T}\times\overline{E}_n}\left(\chi\left(x_0\right), \overline{x}_{[1,n]}\right) \doteqdot \Psi_{\overline{T}\times\overline{E}_n}\left(\left\langle x_0 \right\rangle, \overline{x}_{[1,n]}\right), \tag{5.67}$$

so that

$$\begin{gathered} D_{\mathrm{dp}}\left(\psi^{\langle 1,n \rangle}\right) = R \times \overline{E}_n, D_{\mathrm{a}}\left(\psi^{\langle 1,n \rangle}\right) = R, \\ D_{\mathrm{df}}\left(\psi^{\langle 1,n \rangle}\right) = R \times \overline{X}_n \subseteq R \times \overline{E}_n, D_{\mathrm{v}}\left(\psi^{\langle 1,n \rangle}\right) = Y \subseteq R; \end{gathered} \tag{5.68}$$

the superscript '$\langle 1,n \rangle$' on 'ψ' stands for the *combined weight of the function* $\psi^{\langle 1,n \rangle}$, i.e. it indicates that there is 1 independent real-valued time-like variable as 'x_0' and *n* independent real-valued spatial variables as 'x_1', …, 'x_n', to which the functional variable (operator) '$\psi^{\langle 1,n \rangle}$' can apply with the understanding that the last *n* variables are components of an arithmetical vector. Consequently, for each $\left\langle x_0, \overline{x}_{[1,n]} \right\rangle \in R \times \overline{X}_n$ there is exactly one $y \in Y \subseteq R$ such that $y = \psi^{\langle 1,n \rangle}\left(x_0, \overline{x}_{[1,n]}\right)$.•

5.7. An *n*-dimensional primitive (Bravais) affine lattice

Preliminary Remark 5.3. This short subsection is a minor digression, which is designed to generalize the notion of a *three-dimensional primitive* (*Bravais*) *crystal lattice* (see, e.g., Landau & Lifshitz [1980, chapter XIII, §129, pp. 403–404]) in the *three-dimensional Euclidean real affine space* $\dot{\boldsymbol{E}}_3(\boldsymbol{R})$ to the case of any dimension $n \in \omega_1$ and to demonstrate that an *n*-dimensional primitive lattice can formally be defined as an *n-dimensional affine additive group* $\dot{\boldsymbol{E}}_n^{\mathrm{g}}$ in an *n-dimensional Euclidean real affine space* $\dot{\boldsymbol{E}}_n(\boldsymbol{R})$.•

Theorem 5.1. Given $n \in \omega_1$, given an *n*-dimensional Euclidean vector space $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$, let $\hat{a}_1$, ..., $\hat{a}_n$ be *n* linearly independent vectors in $\hat{E}_n$. The set $\hat{\Lambda}_n$, defined as

$$\hat{\Lambda}_n \doteqdot \left\{ \hat{z} \middle| \hat{z} = \hat{\sum_{i=1}^{n}} m_i \hat{\cdot}\, \hat{a}_i \text{ and } m_j \in I_{-\infty,\infty} \text{ for each } j \in \omega_{1,n} \right\} \subseteq \hat{E}_n, \tag{5.69}$$

is the underlying vector set, which *together* with the restricted binary addition function $\hat{+} : \hat{\Lambda}_n \times \hat{\Lambda}_n \to \hat{\Lambda}_n$ and *together* with the restricted singulary additive inversion function $\hat{-} : \hat{\Lambda}_n \to \hat{\Lambda}_n$ relative to the null-vector $\hat{0}_{\{n\}}$ forms a *commutative additiver group in* $\hat{\boldsymbol{E}}_n(\boldsymbol{R})$ to be denoted by '$\hat{\boldsymbol{\Lambda}}_n$', so that formally

$$\hat{\boldsymbol{\Lambda}}_n \doteqdot \hat{\Lambda}_n \cup \hat{+} \cup \hat{-}. \tag{5.70}$$

The group $\hat{\boldsymbol{\Lambda}}_n$ will be called an *n-dimensional primitive* (or *Bravais*) *vector lattice*. Accordingly, the vectors $\hat{a}_1$, ..., $\hat{a}_n$ will be called *basis vectors of the vector lattice* $\hat{\boldsymbol{\Lambda}}_n$, whereas the ordered *n*-tuple

$$\bar{\hat{a}}_{[1,n]} \doteqdot \langle \hat{a}_1, ..., \hat{a}_n \rangle \in \hat{\Lambda}_n^{n\times} \subset \hat{E}_n^{n\times} \tag{5.71}$$

is called a *basis of the vector lattice* $\hat{\boldsymbol{\Lambda}}_n$. Elements of $\hat{\Lambda}_n$ are called *lattice vectors of* $\hat{\boldsymbol{\Lambda}}_n$ and therefore the underlying set $\hat{\Lambda}_n$ itself is called the *set of lattice vectors*.

Proof: By (5.48), a common (general) element $\hat{x}$ of the set $\hat{\Lambda}_n$ is given as

$$\hat{z} = \hat{\sum_{i=1}^{n}} m_i \hat{\cdot}\, \hat{a}_i, \text{ where } m_j \in I_{-\infty,\infty} \text{ for each } j \in \omega_{1,n}. \tag{5.72}$$

It can be verified by the corresponding straightforward computations that $\hat{\Lambda}_n$ satisfies all axioms of Definitions 2.4 and 4.1. QED.•

Definition 5.15. Given $n \in \omega_1$, an *n-dimensional primitive*, or *Bravais, affine lattice* $\dot{\boldsymbol{\Lambda}}_n$ is a certain *underlying set* $\dot{\Lambda}_n$ *of points*, which is called the *set of lattice points*, *together* with the *vector lattice* $\hat{\boldsymbol{\Lambda}}_n$ and also *together* with a *surjective binary function*

$$\hat{V} : \dot{\Lambda}_n \times \dot{\Lambda}_n \to \hat{\Lambda}_n, \tag{5.73}$$

which is the pertinent restriction of the function (4.1) and which satisfies two *lattice point axioms* (*LPA's*), being the pertinent restrictions of AAGMA1 and AAGMA2. LPA1 and LPA2 are, *mutatis mutandis*, word for word the instances of AAGMA1 and AAGMA2, in which '*E*' with an overdot or with a caret is replaced by 'Λ_n' with the same overscript. The «*togetherness*» as stated above can be expressed by the following formal definition of $\dot{\boldsymbol{\Lambda}}_n$:

$$\dot{\boldsymbol{\Lambda}}_n \doteqdot \dot{\Lambda}_n \cup \hat{\boldsymbol{\Lambda}}_n \cup \hat{V} = \dot{\Lambda}_n \cup \left(\hat{\Lambda}_n \cup \hat{+} \cup \hat{-}\right) \cup \hat{V} \tag{5.74}$$

subject to (5.49). The vector lattice $\hat{\boldsymbol{\Lambda}}_n$ and its underlying set $\hat{\Lambda}_n$ of lattice vectors are said to be *adjoint of the affine lattice* $\dot{\boldsymbol{\Lambda}}_n$ and of its *underlying set* $\hat{\Lambda}_n$ *of lattice points*, respectively.•

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