Optimisation of dynamical systems subject to meta-rules

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Outline

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- But suppose that it is not so simple. Suppose the shape of the system depends on the location in the system that we are currently at.
- So if the current state of the system is at the top of the torus, and we were to draw a trajectory from this point, we would expect suddenly the shape of the torus to change.

Dynamical systems with metarules

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- Because in reality the way a system changes depends on the direction a system is pushed from one state to another. The system is not static, but depends on the trajectories that are traced through it.
- In practice, this means that if we were to consider a system holistically, and consider a unique choice of initial tangent vector from each point - a vector field - in parameter space (ignoring situations where such is forbidden, since I am assuming Lorentzian geometry), then we would like to measure how a system would evolve / change in structure in a *natural* way, given that initial choice, or "push" in parameter space.

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- But suppose now that we wish to consider a set of transition matrices, and transition probabilities between these, which depend on the last state and the current state. In other words, a "meta-Markov" process. Then this is closer to the general idea I am trying to aim at.
- We are now ready to ask the central question.

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- Iterating this process a countably infinite number of times, we obtain the first jet bundle $\mathcal{J}M$, given by tuples (p, V), where V is an infinite matrix.
- In practice, however, V is of rank $dim(M)$.

Elements of the jet bundle associated to trajectories

Suppose now we have two points, p and q in our parameter space M.

- Consider the set of index preserving diffeomorphisms $Aut(M)$ on M. This will have a basis given by $\{f_{ij} : x_i \mapsto x_j\}$.

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- Then relative to any point $\gamma(t)$ we have a vector pointing in the direction of the perturbation of the point relative to the ijth element of $Aut(M)$ at $\gamma(t)$.
- This gives us a matrix of tangents (relative to these perturbations of γ), or an element of the first jet bundle, associated to each point of the path γ.

Meta-markov processes again

I claim that to specify the structure associated to the first jet bundle, we need a 6-tensor κ.

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- Then if T_{ij} is a unit transition probability, and U_{kl}, V_{mn} are unit tangent probabilities sitting in the tangent group $GL(n)$, we have that κ_{ijklmn} determines the result of acting on T_{ij} with U_{kl} "on the left" and V_{mn} "on the right". It is the "meta-rule transition to transition probability".

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- The analogy for left and right action is that a left action occurs subsequent to the state - it is where the trajectory is moving *to*, and a right action occurs prior - it is where the trajectory is moving *from*.

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The structural coefficients

- As in Riemannian geometry, we have structural coefficients given by

$$\Gamma^{pq}_{ijklmn} = \langle \partial_p E_{ij}, E_{kl}, \partial_q E_{mn} \rangle_\kappa$$

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- These can be computed as

$$\Gamma^{pq}_{ijklmn} = \kappa^{abc}_{ijk} (\Sigma_{g \in C_8 \otimes C_7} \{g \cdot \partial_p \partial_q \kappa_{abclmn}\})$$

  where summation is over the group product $C_8 \otimes C_7$ acting on the indices of $\partial_p \partial_q \kappa_{abclmn}$.

Geodesics

- γ is geodesic with respect to κ if

$$\nabla_{(X_{ij},\kappa)} X_{kl} = 0$$

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- γ is geodesic with respect to κ if
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where $X_{ij} : [0,1] \to \mathcal{J}M$ is the one parameter jet field associated to γ.
- $\nabla_{(X,\kappa)}$ is the affine connection with respect to κ, uniquely determined by
$$\partial_{ij}\langle \bar{\bar{X}}, \bar{\bar{Y}}, \bar{\bar{\bar{Z}}} \rangle_\kappa = \langle \partial_{ij}\bar{\bar{X}}, \bar{\bar{Y}}, \bar{\bar{\bar{Z}}} \rangle + \langle \bar{\bar{X}}, \partial_{ij}\bar{\bar{Y}}, \bar{\bar{\bar{Z}}} \rangle + \langle \bar{\bar{X}}, \bar{\bar{Y}}, \partial_{ij}\bar{\bar{\bar{Z}}} \rangle$$

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The cybernetic information functional

We wish to know what choice of κ is most natural, ie how a "physical" system will place constraints on allowable behaviour for κ.
Define $Cyb(M) := \{(\mathcal{J}M)^3 \to \mathcal{J}M\}$ as the space of left and right actions on the first jet bundle of M.

- We have an information functional given by

$$I := \int_M \int_{Cyb_m(M)} f(\partial_{ij}\partial_k logf)^3 dmdV$$

where $f = f(m, V) = \delta(\kappa(m) - V)$, with $m \in M$ a point in parameter space and $V \in Cyb_m(M)$ is a point in the space of meta-rules at m.

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  where $f = f(m, V) = \delta(\kappa(m) - V)$, with $m \in M$ a point in parameter space and $V \in Cyb_m(M)$ is a point in the space of meta-rules at m.
- ∂_{ij} is the derivative on function space. ∂_k is the derivative on normal space.

The key result

- I conjecture that, after some considerable work, it can be demonstrated that this simplifies to

$$\int_M Inv(\kappa) dm$$

where $Inv(\kappa)$ in a geometric invariant defined by

$$Inv(\kappa) := \kappa_{ijklmn}\Gamma_{ijabcdef}\Gamma_{klghpabc}\Gamma_{mndefghp}$$

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- This allows us to understand the geometric behaviour of a meta-dynamical system as $Inv(\kappa) = 0$, as a physical system will minimise the information associated to its relevant information functional.

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The Basic Problem

Jet bundles

Geometry

Optimisation

Concluding remarks

Final comments

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- Naturally a great deal of work remains to be done.

Questions