On some series related to Möbius function and Lambert W-function

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Abstract

We derive some new formulas, connecting some series with Möbius function with Sine Integral and Cosine Integral functions, give the formal proof for full version of Stirling's formula; investigate the values of new Dirichlet series function at natural numbers $\geqslant 2$ and it's behavior at the pole $s = 1$, connecting it with elementary constants.

Introduction

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}(\ln n! - n\ln n + n) = -\frac{1}{\pi}\int\limits_0^{2\pi} \frac{\sin x}{x} dx;$$

$$\sum_{n=1}^{\infty} \mu(n)(H_n - \gamma - \ln n) = 2\gamma + 2\ln 2\pi - 2\int\limits_0^{2\pi} \frac{1-\cos x}{x} dx;$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\left(\ln G(1+n) - \frac{n^2}{2}\ln n + \frac{3}{4}n^2\right) = \frac{1}{2} - \frac{1}{\pi}\int\limits_0^{2\pi} \frac{\sin x}{x} dx - \frac{1}{2\pi^2}\int\limits_0^{2\pi} \frac{1-\cos x}{x} dx,$$

where G is the Barnes G-function.

And some formulas of different type:

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}\left(1 + n + \frac{n^2}{2!} + ... + \frac{n^n}{n!}\right) = \frac{\pi^2}{6} - \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^3}\left(1 + n + \frac{n^2}{2!} + ... + \frac{n^n}{n!}\right) = \zeta(3) - \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4} \left(1 + n + \frac{n^2}{2!} + ... + \frac{n^n}{n!}\right) = \frac{\pi^4}{90} - \frac{7}{24}$$

There are well known expressions for $\ln x! = \ln \Gamma(1+x)$ and $H_x = \gamma + \psi(1+x)$:

$$\ln x! = x \ln x - x + \frac{1}{2} \ln 2\pi x + \int_0^{\infty} \frac{2 \arctan(\frac{t}{x})}{e^{2\pi t} - 1} dt$$

$$H_x = \ln x + \gamma + \frac{1}{2x} - \int_0^{\infty} \frac{2t}{t^2 + x^2} \frac{dt}{e^{2\pi t} - 1}$$

and each of them is easy to prove. But there is an interesting way to derive them. Let's start from the beginning, from simple formulas.

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!},$$

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Then

$$\frac{B_{2n}}{(2n)!} = \frac{2(-1)^{n+1}}{(2n-1)!} \int_0^{\infty} \frac{t^{2n-1}}{e^{2\pi t} - 1} dt$$

That's why

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n} = 1 - \frac{x}{2} + \int_0^{\infty} \frac{2x \sin xt}{e^{2\pi t} - 1} dt$$

Formal proof of full Stirling's formula

Formal proofs of different formulas are not the mathematical proofs in usual sense. They only could give a clue for the correct formula, which can be proved later by the strict mathematical reasoning. About the following identities we could only say that if they work for every polynomial, we could substitute them by other "natural" functions, to find a rigorous proof for the new identities later.

Let D be the differential operator $\frac{d}{dx}$. Applying Taylor's theorem, $e^{cD}f(x) = \sum\limits_{n=0}^{\infty}\dfrac{c^n f^{(n)}(x)}{n!} = f(x+c)$. Let's replace x in integral formula for $\dfrac{x}{e^x-1}$ by D and notice that formally

$$\frac{D}{e^D-1}\int\limits_x^{x+1} f(t)dt = f(x)$$

Then

$$f(x) = \left(1 - \frac{D}{2} + \int\limits_0^\infty \frac{2D\sin Dt}{e^{2\pi t}-1}dt\right)\int\limits_x^{x+1} f(z)dz =$$

$$= \int\limits_x^{x+1} f(z)dz - \frac{f(x+1)-f(x)}{2} + \frac{1}{i}\int\limits_0^\infty \frac{e^{iDt}-e^{-iDt}}{e^{2\pi t}-1}dt(f(x+1)-f(x))$$

And we obtain the full version of trapezoidal rule

$$\int\limits_x^{x+1} f(z)dz = \frac{f(x)+f(x+1)}{2} + i\int\limits_0^\infty \frac{f(1+x+it)-f(x+it)-f(1+x-it)+f(x-it)}{e^{2\pi t}-1}dt$$

Using this formula one can obtain Abel-Plana summation formula (which formally works for polynomials too, giving the values of Riemann zeta function at complex points with negative real part):

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int\limits_0^\infty f(x)dx + i\int\limits_0^\infty \frac{f(it)-f(-it)}{e^{2\pi t}-1}dt$$

But we are not interested in this formula now. Let's try to put $f(x) = \ln\Gamma(x)$, remembering Raabe's formula:

$$\int\limits_x^{x+1} \ln\Gamma(t)dt = x\ln x - x + \frac{1}{2}\ln 2\pi$$

So

$$\ln\Gamma(x) = x\ln x - x + \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln x - i\int\limits_0^\infty \frac{\ln(x+it)-\ln(x-it)}{e^{2\pi t}-1}dt$$

But $\ln(x+iy) = \dfrac{1}{2}\ln(x^2+y^2) + i\arctan\left(\dfrac{y}{x}\right)$, so we finally have

$$\ln x! = x\ln x - x + \frac{1}{2}\ln 2\pi x + \int\limits_0^\infty \frac{2\arctan(\frac{t}{x})}{e^{2\pi t}-1}dt$$

Strict proof of this formula can be derived, using the integral representations:

$$\arctan(x) = \int\limits_0^\infty \frac{\sin(xt)}{t}e^{-t}dt;$$

$$\ln x = \int\limits_0^\infty \frac{e^{-t}-e^{-xt}}{t}dt;$$

$$\ln x! = \int\limits_0^\infty \frac{e^{-xt}-xe^{-t}-1+x}{t(e^t-1)}dt,$$

all of which can be proved in the same manner as the formula for $\dfrac{x}{e^x-1}$, replacing Taylor coefficients by the Γ-function integral or $\Gamma\zeta$ integral multiplyed by the appropriate reciprocal factorials.

Derivation of stated results with μ-function

Differentiating the full Stirling's formula, we can now obtain integral representation for generalized Harmonic number H_x and Hurwitz zeta function $\zeta(k+1, x+1)$:

$$H_x = \ln x + \gamma + \frac{1}{2x} - \int\limits_0^\infty \frac{2t}{t^2+x^2}\frac{dt}{e^{2\pi t}-1}$$

$$\zeta(k+1,x+1)=\frac{1}{kx^k}-\frac{1}{2x^{k+1}}+i\int\limits_0^{\infty}\left(\frac{1}{(x+it)^{k+1}}-\frac{1}{(x-it)^{k+1}}\right)\frac{dt}{e^{2\pi t}-1}$$

Let's change the variable in all integrals $t=xny$ and use the Lambert series formula of Möbius function $\sum\limits_{n=1}^{\infty}\frac{\mu(n)}{e^{ny}-1}=e^{-y}\quad(y>0)$ with reciprocal zeta function formulas $\sum\limits_{n=1}^{\infty}\frac{\mu(n)}{n}=0,\ \sum\limits_{n=1}^{\infty}\frac{\mu(n)\ln n}{n}=-1$ to derive

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}(\ln\Gamma(1+nx)-(nx)\ln(nx)+nx)=-\frac{1}{2}+\frac{\cos 2\pi x}{\pi}\left(\frac{\pi}{2}-\int\limits_0^{2\pi x}\frac{\sin t}{t}dt\right)+$$

$$+\frac{\sin 2\pi x}{\pi}\left(\gamma+\ln 2\pi x-\int\limits_0^{2\pi x}\frac{1-\cos t}{t}dt\right);$$

$$\sum_{n=1}^{\infty}\mu(n)(H_{nx}-\gamma-\ln(nx))=2\cos 2\pi x\left(\gamma+\ln 2\pi x-\int\limits_0^{2\pi x}\frac{1-\cos t}{t}dt\right)-$$

$$-2\sin 2\pi x\left(\frac{\pi}{2}-\int\limits_0^{2\pi x}\frac{\sin t}{t}dt\right)$$

And not so pretty

$$\sum_{n=1}^{\infty}\mu(n)(nx)^{2k-1}\left(\zeta(2k,1+nx)-\frac{1}{(2k-1)(nx)^{2k-1}}\right)=\sum_{m=0}^{k-1}2(-1)^m(2\pi x)^{2m}\frac{(2k-2m-2)!}{(2k-1)!}+$$

$$+\frac{2(-1)^k(2\pi x)^{2k-1}}{(2k-1)!}\cos 2\pi x\left(\frac{\pi}{2}-\int\limits_0^{2\pi x}\frac{\sin t}{t}dt\right)+$$

$$+\frac{2(-1)^k(2\pi x)^{2k-1}}{(2k-1)!}\sin 2\pi x\left(\gamma+\ln 2\pi x-\int\limits_0^{2\pi x}\frac{1-\cos t}{t}dt\right);$$

$$\sum_{n=1}^{\infty}\mu(n)(nx)^{2k}\left(\zeta(2k+1,1+nx)-\frac{1}{2k(nx)^{2k}}\right)=\sum_{m=0}^{k-1}2(-1)^m(2\pi x)^{2m}\frac{(2k-2m-1)!}{(2k)!}+$$

$$+\frac{2(2\pi x)^{2k}(-1)^{k+1}}{(2k)!}\cos 2\pi x\left(\gamma+\ln 2\pi x-\int_0^{2\pi x}\frac{1-\cos t}{t}dt\right)+$$

$$+\frac{2(2\pi x)^{2k}(-1)^{k+1}}{(2k)!}\sin 2\pi x\left(\frac{\pi}{2}-\int_0^{2\pi x}\frac{\sin t}{t}dt\right)$$

Setting $x=1$ in all of these formulas we obtain the first two stated results. And also

$$\sum_{n=1}^{\infty}\mu(n)n\left(\frac{\pi^2}{6}-\left(1+\frac{1}{2^2}+...+\frac{1}{n^2}\right)-\frac{1}{n}\right)=2-2\pi^2+4\pi\int_0^{2\pi}\frac{\sin t}{t}dt;$$

$$\sum_{n=1}^{\infty}\mu(n)n^2\left(\zeta(3)-\left(1+\frac{1}{2^3}+...+\frac{1}{n^3}\right)-\frac{1}{2n^2}\right)=1+4\gamma\pi^2+4\pi^2\ln 2\pi-4\pi^2\int_0^{2\pi}\frac{1-\cos t}{t}dt;$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\ln\binom{2n}{n}-2n\ln n\right)=\frac{2}{\pi}\int_0^{2\pi}\frac{\sin t}{t}dt-\frac{1}{\pi}\int_0^{4\pi}\frac{\sin t}{t}dt$$

Now the formula with Barnes G-function is left to derive. We will use the well known formula

$$\ln G(1+x)=\frac{x(1-x)}{2}+\frac{x}{2}\ln 2\pi+x\ln\Gamma(x)-\int_0^x\ln\Gamma(t)dt$$

and full Stirling's formula to derive

$$\ln G(1+x)=\frac{x^2}{2}\ln x-\frac{3}{4}x^2+\frac{x}{2}\ln 2\pi-\frac{1}{12}\ln x+\int_0^{\infty}\frac{2t\ln t}{e^{2\pi t}-1}dt-\int_0^{\infty}\frac{t\ln(1+\frac{t^2}{x^2})}{e^{2\pi t}-1}dt$$

or, in terms of Glaisher-Kinkelin constant A, given by the equality $\frac{1}{24}-\frac{1}{2}\ln A=\int\limits_0^\infty \frac{t\ln t}{e^{2\pi t}-1}dt$:

$$\ln G(1+x)=\frac{x^2}{2}\ln x-\frac{3}{4}x^2+\frac{x}{2}\ln 2\pi-\frac{1}{12}\ln x+\frac{1}{12}-\ln A-\int\limits_0^\infty \frac{t\ln(1+\frac{t^2}{x^2})}{e^{2\pi t}-1}dt$$

Differentiating $\Gamma\zeta$ integral we get the equality

$$\zeta'(2)=\frac{\pi^2}{6}(\gamma+\ln 2\pi-12\ln A)$$

so

$$\sum_{n=1}^{\infty}\frac{\mu(n)\ln n}{n^2}=-\left.\frac{d}{ds}\frac{1}{\zeta(s)}\right|_{s=2}=\frac{6}{\pi^2}(\gamma+\ln 2\pi-12\ln A)$$

Using this formula with the fact that $\sum\limits_{n=1}^{\infty}\frac{\mu(n)}{n^2}=\frac{6}{\pi^2}$ we get

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\left(\ln G(1+nx)-\frac{(nx)^2}{2}\ln(nx)+\frac{3}{4}(nx)^2\right)=-\frac{1}{2\pi^2}(\gamma+\ln 2\pi x-1)-$$

$$-\int\limits_0^\infty t\ln(1+t^2)e^{-2\pi tx}dt=(\gamma+\ln 2\pi x)\left(\frac{x\sin 2\pi x}{\pi}+\frac{\cos 2\pi x}{2\pi^2}-\frac{1}{2\pi^2}\right)+$$

$$+\frac{x\cos 2\pi x}{2}-\frac{\sin 2\pi x}{4\pi}+\left(\frac{\sin 2\pi x}{2\pi^2}-\frac{x\cos 2\pi x}{\pi}\right)\int\limits_0^{2\pi x}\frac{\sin t}{t}dt-$$

$$-\left(\frac{\cos 2\pi x}{2\pi^2}+\frac{x\sin 2\pi x}{\pi}\right)\int\limits_0^{2\pi x}\frac{1-\cos t}{t}dt$$

Setting $x=1$ we obtain the desired result. Similarly using Abel-Plana formula for Hurwitz zeta function we can obtain this type formulas in terms of incomplete Γ-function.
But the Möbius μ-function is not the only non-trivial arithmetic function which have the closed form of Lambert series. Another example is the Euler's totient function φ. This function is appropriate to use it in an analogue of

Möbius inversion formula for Γ-function $\left(\sum\limits_{n=1}^{\infty} \mu(n) \ln\Gamma\left(1+\frac{x}{n}\right) = x - \ln(1+x)\right)$:

$$\sum_{n=1}^{\infty} \varphi(n)\left(\ln\Gamma\left(1+\frac{x}{n}\right)+\frac{\gamma x}{n}-\frac{\pi^2 x^2}{12n^2}\right)=\frac{x}{2}\ln 2\pi-\frac{x+(1+\gamma)x^2}{2}-\ln G(1+x)$$

So let's try to use this function in analogue to the sums with full Stirling's expansion, but firstly deformating it for convergence:

$$\ln x! = x\ln x - x + \frac{1}{2}\ln 2\pi x + \frac{1}{12x} + \int_0^{\infty} \frac{2x(\arctan t - t)}{e^{2\pi t x}-1}dt$$

Now using the Lambert series formula of the totient function $\sum\limits_{n=1}^{\infty} \frac{\varphi(n)}{e^{ny}-1} = \frac{e^y}{(e^y-1)^2}$ $(y>0)$ we have:

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n}\left(\ln\Gamma(1+nx)-(nx)\ln(nx)+nx-\frac{1}{2}\ln 2\pi nx-\frac{1}{12nx}\right)=$$

$$=-\frac{1}{\pi}\int\limits_0^{\infty}\frac{t^2}{1+t^2}\frac{dt}{e^{2\pi t x}-1}$$

But because it's $\frac{t^2}{1+t^2}$, not $\frac{t}{1+t^2}$, this result is not representable in terms of generalized Harmonic number. That's why it is not so interesting.

Derivation of stated results related to Lambert W-function

Lagrange inversion theorem implies the formula for coefficients of Lambert function Taylor's expansion, using which it can be derived that

$$\forall x \in [0;1]: \quad \sum_{n=1}^{\infty} \frac{n^{n-1}x^n e^{-nx}}{n!} = x$$

Then

$$\int\limits_0^1 x\,dx = \sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}\int\limits_0^1 x^n e^{-nx}dx$$

Integrating by parts it is easy to prove that
$e^{-x}\left(1+x+...+\frac{x^n}{n!}\right)=1-\frac{1}{n!}\int\limits_0^x t^n e^{-t}dt$

Then

$$\forall x\in[0,1]:\quad \int\limits_0^x tdt=\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}\frac{1}{n^{n+1}}\int\limits_0^{nx} t^n e^{-t}dt=$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-e^{-nx}\left(1+(nx)+...+\frac{(nx)^n}{n!}\right)\right)$$

So

$$\forall x\in[0,1]:\quad \sum_{n=1}^{\infty}\frac{e^{-nx}}{n^2}\left(1+nx+\frac{(nx)^2}{2!}+...+\frac{(nx)^n}{n!}\right)=\frac{\pi^2}{6}-\frac{x^2}{2}$$

Let's define the sequence of polynomials

$$P_1(x)=x,\ P_{n+1}(x)=\int_0^x\frac{P_n(t)(1-t)}{t}dt$$

Then

$$\forall x\in[0;1]:\quad \sum_{n=1}^{\infty}\frac{n^{n-k}x^n e^{-nx}}{n!}=P_k(x)$$

So $\forall k\in\mathbb{N}_{k\geqslant 2},\forall x\in[0;1]$:

$$\sum_{n=1}^{\infty}\frac{e^{-nx}}{n^k}\left(1+(nx)+\frac{(nx)^2}{2!}+...+\frac{(nx)^n}{n!}\right)=\zeta(k)-\int_0^x P_{k-1}(t)dt$$

But we can continue this sequence of polynomials at least to all real variable $\geqslant 1$ (but these functions of course won't be polynomials anymore).

$$\sum_{n=1}^{\infty}\frac{n^{n-y}x^n e^{-nx}}{n!}=\frac{1}{\Gamma(y-1)}\sum_{n=1}^{\infty}\frac{n^{n-1}x^n e^{-nx}}{n!}\int\limits_0^{\infty}t^{y-2}e^{-nt}dt=$$

$$=-\frac{1}{\Gamma(y-1)}\int\limits_0^{\infty}t^{y-2}W_0(-xe^{-x-t})dt$$

But $x \in [0;1]$, so we can change the variable $t = xz - x - \ln z$ to obtain:

$$P_y(x) = \frac{1}{\Gamma(y-1)} \int_0^1 (xz - x - \ln z)^{y-2}(x - x^2 z)dz = \frac{x}{\Gamma(y)} \int_0^1 (xz - x - \ln z)^{y-1} dz$$

(at least for all $x \in [0;1],\ y \in [1;+\infty)$) But now we are interested only in case when $y \in \mathbb{N}$.

$$P_y(x) = \frac{x}{(y-1)!} \int_0^1 \sum_{m=0}^{y-1} \binom{y-1}{m} (x(t-1))^m (-\ln t)^{y-1-m} dt =$$

$$= \frac{x}{(y-1)!} \sum_{m=0}^{y-1} \binom{y-1}{m} x^m (-1)^{y-1} \frac{d^{y-1-m}}{ds^{y-1-m}} \frac{\Gamma(s+1)\Gamma(m+1)}{\Gamma(s+m+2)}\Bigg|_{s=0} =$$

$$= \sum_{m=1}^{y} \frac{x^m}{m!} \sum_{j=1}^{m} \frac{(-1)^{m+j}}{j^{y-m}} \binom{m}{j}$$

So

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^k} \left(1 + n + \frac{n^2}{2} + ... + \frac{n^n}{n!}\right) = \zeta(k) - \sum_{m=1}^{k-1} \sum_{j=1}^{m} \frac{(-1)^{m+j}}{(m+1)! j^{k-1-m}} \binom{m}{j}$$

or using the fact that $\sum\limits_{m=1}^{n-1} \sum\limits_{k=1}^{m} f(k,m) = \sum\limits_{m=1}^{n-1} \sum\limits_{k=1}^{m} f(k, n-m+k-1)$:

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^k} \left(1 + n + \frac{n^2}{2} + ... + \frac{n^n}{n!}\right) = \zeta(k) + \sum_{m=1}^{k-1} \frac{(-1)^{k-m}}{\Gamma(k-m)} \sum_{j=1}^{m} \frac{j^{j-m}}{(k-m+j)j!}$$

Now let's investigate the behavior of this Dirichlet series at $k = 1$. It is well known that

$$\lim_{n \to \infty} e^{-n} \left(1 + n + \frac{n^2}{2} + ... + \frac{n^n}{n!}\right) = \frac{1}{2}$$

so we can try to find the closed-form expression of the sum

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n} \left(1 + n + \frac{n^2}{2} + ... + \frac{n^n}{n!}\right) - \frac{1}{2n} = M$$

To regularize the $W'(x)$ series let's use the expansion of the function $f(x) = (1-x)^{-\frac{1}{2}}$. Then

$$\frac{x}{1-x} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2-2xe^{1-x}}} = \sum_{n=1}^{\infty} \frac{n^n x^n e^{-nx}}{n!} - \frac{\binom{2n}{n} x^n e^{n-nx}}{2^{2n}\sqrt{2}}$$

There's no limit for $x \to 1^+$, but to apply Abel's theorem we need only the limit for $x \to 1^-$, which is $\frac{1}{\sqrt{2}} - \frac{2}{3}$. And so we can integrate this formula.

$$\int_0^1 \frac{x}{1-x} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2-2xe^{1-x}}} dx =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - e^{-n}\left(1 + n + \frac{n^2}{2} + ... + \frac{n^n}{n!}\right)\right) - \frac{\binom{2n}{n}}{2^{2n}\sqrt{2}} \int_0^1 x^n e^{n-nx} dx =$$

$$= -M + \sum_{n=1}^{\infty} \frac{1}{2n} - \frac{\binom{2n}{n}}{2^{2n}\sqrt{2}} \int_0^1 x^n e^{n-nx} dx$$

Γ-function duplication theorem and change of variable implies that

$$\frac{1}{n\binom{2n}{n}} = \frac{e^n}{4^n} \int_0^1 t^{n-1} e^{-nt} \frac{1-t}{\sqrt{1-te^{1-t}}} dt$$

so

$$\int_0^1 \frac{x}{1-x} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2-2xe^{1-x}}} dx =$$

$$= -M + \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}\sqrt{2}} \int_0^1 x^n e^{n-nx} \left(\frac{1-x}{x\sqrt{2-2xe^{1-x}}} - 1\right) dx =$$

$$= -M + \int_0^1 \left(\frac{1}{\sqrt{2-2xe^{1-x}}} - \frac{1}{\sqrt{2}}\right)\left(\frac{1-x}{x\sqrt{2-2xe^{1-x}}} - 1\right) dx$$

That's why

$$M = \int_0^1 \frac{1-x}{2x\sqrt{1-xe^{1-x}}}\left(\frac{1}{\sqrt{1-xe^{1-x}}} - 1\right) - \frac{x}{1-x} dx$$

Using the substitution $t = xe^{1-x}$ and reverse we can find the antiderivative of this function:

$$\int \frac{1-x}{2x\sqrt{1-xe^{1-x}}}\left(\frac{1}{\sqrt{1-xe^{1-x}}}-1\right)-\frac{x}{1-x}dx =$$

$$= x+\ln(1-x)+\frac{1}{2}\ln(xe^{1-x})-\frac{1}{2}\ln(1-xe^{1-x})-\frac{1}{2}\ln(1-\sqrt{1-xe^{1-x}})+$$

$$+\frac{1}{2}\ln(1+\sqrt{1-xe^{1-x}})+C$$

counting the limits of this function for $x \to 0$ and $x \to 1^-$ we finally have:

$$\boxed{\sum_{n=1}^{\infty}\frac{e^{-n}}{n}\left(1+n+\frac{n^2}{2}+...+\frac{n^n}{n!}\right)-\frac{1}{2n}=1-\frac{1}{2}\ln 2}$$

Conclusion

At first, we gave some new infinite series, which have closed-form expressions in terms of Trigonometric Integral functions, using the formal approach to the derivation of Stirling's formula. Then we derived some results for new Dirichlet series. Defining

$$M(s)=\sum_{n=1}^{\infty}\frac{e^{-n}}{n^s}\left(1+n+\frac{n^2}{2}+...+\frac{n^n}{n!}\right)$$

we obtained less obvious results than

$$\lim_{s\to\infty}M(s)=\frac{2}{e}$$

We gave the formulas for $M(2)$, $M(3)$ and general closed-form formula for $M(k)$ for all natural k except 1 and integral formula for real $k>1$. We also derived that

$$\lim_{s\to 1}M(s)-\frac{\zeta(s)}{2}=1-\frac{1}{2}\ln 2$$

or

$$\lim_{s\to 1}M(s)-\frac{1}{2s-2}=1+\frac{\gamma}{2}-\frac{1}{2}\ln 2$$

An interesting problem is to find analytic continuation of $M(s)$ and find it's values at negative points.

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