The Happy Mothers Theorem

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Abstract: We revisit a 25 years old approach of the twin primes conjecture, and after a simple adjustment, push it forward by means of ordinary sieves to an important conclusion: the twin prime conjecture is true.

Keywords: twin primes, sieves

The twin prime conjecture is one of the most famous unsolved problems in mathematics. Recent developments[3][4] have incrementally taken the bar down to a bounded gap of only 246. It is unclear though, whether the same techniques will allow the mathematical community to go all the way down to just 2, and most mathematicians doubt that it will.

Over the centuries, different (and generally less advanced) methods and techniques have been used to attempt to prove the conjecture. In some cases[2], authors came close to what we describe below. The inspiration for the attempt presented here comes essentially from an article by Gold & Tucker (1991)[1]. We believe that the roadblock they hit was caused by the usage of their $G(n)$ function. And thus, we will start where they left off, with their final Theorem 2, simply transposing it to a much more natural formulation.

Theorem 1: *Any twin prime pair greater than* $(3;5)$ *is of the form* $[6z-1;6z+1]$ *where* $z \in \mathbb{N}^*$ *and satisfies all inequalities in (1), for all* $(x,y) \in \mathbb{N}^2$*; and conversely, for any such integer* z*, the pair* $[6z-1;6z+1]$ *is a twin prime pair.*

(1)

$$\begin{aligned} 6xy + 5x + 5y + 4 &\neq z \\ 6xy + 7x + 5y + 6 &\neq z \\ 6xy + 7x + 7y + 8 &\neq z \end{aligned}$$

Proof: see Gold & Tucker (1991)[1]. Briefly, we observe that their mention of " $[G(n),\ G(n+1)]$, n being odd" is equivalent to $[3(2k+1)+2,\ 3(2k+2)+1]$ which we can write $[6(k+1)-1,\ 6(k+1)+1]$. We simply pose $z = k+1$.

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Since there is a one-to-one correspondence between the twin prime pairs and their z "seeds" or "happy mothers" (as we heard them called online), we will aim to count and measure the density over the integers of such numbers z.

A tabulated view will help visualize the effect of the above 3 inequations, seen as sieves:

Table 1: $6xy + 5x + 5y + 4$

	0	1	2	3	4	5	6	7	8	9	10
0	4	9	14	19	24	29	34	39	44	49	54
1	9	20	31	42	53	64	75	86	97	108	119
2	14	31	48	65	82	99	116	133	150	167	184
3	19	42	65	88	111	134	157	180	203	226	249
4	24	53	82	111	140	169	198	227	256	285	314
5	29	64	99	134	169	204	239	274	309	344	379
6	34	75	116	157	198	239	280	321	362	403	444
7	39	86	133	180	227	274	321	368	415	462	509
8	44	97	150	203	256	309	362	415	468	521	574

Table 2: $6xy + 7x + 5y + 6$

	0	1	2	3	4	5	6	7	8	9	10
0	6	11	16	21	26	31	36	41	46	51	56
1	13	24	35	46	57	68	79	90	101	112	123
2	20	37	54	71	88	105	122	139	156	173	190
3	27	50	73	96	119	142	165	188	211	234	257
4	34	63	92	121	150	179	208	237	266	295	324
5	41	76	111	146	181	216	251	286	321	356	391
6	48	89	130	171	212	253	294	335	376	417	458
7	55	102	149	196	243	290	337	384	431	478	525
8	62	115	168	221	274	327	380	433	486	539	592

Table 3: $6xy + 7x + 7y + 8$

	0	1	2	3	4	5	6	7	8	9	10
0	8	15	22	29	36	43	50	57	64	71	78
1	15	28	41	54	67	80	93	106	119	132	145
2	22	41	60	79	98	117	136	155	174	193	212
3	29	54	79	104	129	154	179	204	229	254	279
4	36	67	98	129	160	191	222	253	284	315	346
5	43	80	117	154	191	228	265	302	339	376	413
6	50	93	136	179	222	265	308	351	394	437	480
7	57	106	155	204	253	302	351	400	449	498	547
8	64	119	174	229	284	339	394	449	504	559	614

Sidenote: one can verify that the first missing elements from these 3 tables, {1,2,3,5,7,10,12,17,18,...} match exactly the http://oeis.org/A002822 list.

We begin by defining the infinite sets $A_k \subset \mathbb{N}$, $B_k \subset \mathbb{N}$ and $C_k \subset \mathbb{N}$ of the form

$$
\begin{aligned}
A_k \; &= \; \{\, m \equiv \; 5k - 1 \quad (mod\; 6k - 1)\,\} \\
&\cup \{\, \{m \equiv \; k \qquad (mod\; 6k - 1)\,\} \setminus \{k\}\} \\
B_k \; &= \; \{\, m \equiv \; 5k + 1 \quad (mod\; 6k + 1)\,\} \\
&\cup \{\, \{m \equiv \; k \qquad (mod\; 6k + 1)\,\} \setminus \{k\}\} \\
C_k \; &= \; A_k \cup B_k
\end{aligned}
\tag{2}
$$

with m and k nonzero positive integers.

For instance, with $k = 1$ we have

$$
\begin{aligned}
A_1 &= \{m \equiv 4\; (mod\; 5)\} \;\cup\; \{\{m \equiv \; 1\; (mod\; 5)\} \setminus \{1\}\} \\
B_1 &= \{m \equiv 6\; (mod\; 7)\} \;\cup\; \{\{m \equiv \; 1\; (mod\; 7)\} \setminus \{1\}\} \\
C_1 &= \; A_1 \cup B_1
\end{aligned}
$$

And with $k = 2$, we get

$$
\begin{aligned}
A_2 &= \{m \equiv 9\; (mod\; 11)\} \;\cup\; \{\{m \equiv \; 2\; (mod\; 11)\} \setminus \{2\}\} \\
B_2 &= \{m \equiv 11\; (mod\; 13)\} \;\cup\; \{\{m \equiv \; 2\; (mod\; 13)\} \setminus \{2\}\} \\
C_2 &= \; A_2 \cup B_2
\end{aligned}
$$

etc.

How these sets act as sieves is presented below in Table 4. The rightmost column presents the result (in orange, the sifted-out elements, in white, the remaining ones) and each A_k and B_k sets are shown by their congruences.

C_1		C_2		C_3		C_4		C_5		C_6		C_7				
A_1 (5)	B_1 (7)	A_2 (11)	B_2 (13)	A_3 (17)	B_3 (19)	A_4 (23)	B_4 (25)	A_5 (29)	B_5 (31)	A_6 (35)	B_6 (37)	A_7 (41)	B_7 (43)	…		
															1	**5,7**
2	2														**2**	**11,13**
3	3	3	3												**3**	**17,19**
4	4	4	4	4	4										4	
0	5	5	5	5	5	5	5								**5**	**29,31**
1	6	6	6	6	6	6	6	6	6						6	
2	0	7	7	7	7	7	7	7	7	7	7				**7**	**41,43**
3	1	8	8	8	8	8	8	8	8	8	8	8	8		8	
4	2	9	9	9	9	9	9	9	9	9	9	9	9		9	
0	3	10	10	10	10	10	10	10	10	10	10	10	10		**10**	**59,61**
1	4	0	11	11	11	11	11	11	11	11	11	11	11		11	
2	5	1	12	12	12	12	12	12	12	12	12	12	12		**12**	**71,73**
3	6	2	0	13	13	13	13	13	13	13	13	13	13		13	
4	0	3	1	14	14	14	14	14	14	14	14	14	14		14	
0	1	4	2	15	15	15	15	15	15	15	15	15	15		15	
1	2	5	3	16	16	16	16	16	16	16	16	16	16		16	
2	3	6	4	0	17	17	17	17	17	17	17	17	17		**17**	**101,103**
3	4	7	5	1	18	18	18	18	18	18	18	18	18		**18**	**107,109**
4	5	8	6	2	0	19	19	19	19	19	19	19	19		19	
0	6	9	7	3	1	20	20	20	20	20	20	20	20		20	
1	0	10	8	4	2	21	21	21	21	21	21	21	21		21	
2	1	0	9	5	3	22	22	22	22	22	22	22	22		22	
3	2	1	10	6	4	0	23	23	23	23	23	23	23		**23**	**137,139**
4	3	2	11	7	5	1	24	24	24	24	24	24	24		24	
0	4	3	12	8	6	2	0	25	25	25	25	25	25		**25**	**149,151**
1	5	4	0	9	7	3	1	26	26	26	26	26	26		26	
2	6	5	1	10	8	4	2	27	27	27	27	27	27		27	
3	0	6	2	11	9	5	3	28	28	28	28	28	28		28	
4	1	7	3	12	10	6	4	0	29	29	29	29	29		29	
0	2	8	4	13	11	7	5	1	30	30	30	30	30		**30**	**179,181**
1	3	9	5	14	12	8	6	2	0	31	31	31	31		31	
2	4	10	6	15	13	9	7	3	1	32	32	32	32		**32**	**191,193**
3	5	0	7	16	14	10	8	4	2	33	33	33	33		**33**	**197,199**
4	6	1	8	0	15	11	9	5	3	34	34	34	34		34	
0	0	2	9	1	16	12	10	6	4	0	35	35	35		35	
1	1	3	10	2	17	13	11	7	5	1	36	36	36		36	
2	2	4	11	3	18	14	12	8	6	2	0	37	37		37	
3	3	5	12	4	0	15	13	9	7	3	1	38	38		**38**	**227,229**
4	4	6	0	5	1	16	14	10	8	4	2	39	39		39	
0	5	7	1	6	2	17	15	11	9	5	3	40	40		**40**	**239,241**
1	6	8	2	7	3	18	16	12	10	6	4	0	41		41	
2	0	9	3	8	4	19	17	13	11	7	5	1	42		42	

Table 4

The question is: how dense is the cumulative $\bigcup_{k=1}^{\infty} C_k$ sieve? And does it leave infinitely many "holes", otherwise known as "happy mothers"?

Let's start with A_1 . Its arithmetic density is (obviously) exactly:

$$\delta(A_1) = \frac{2}{5}$$

We know that $\delta(C_1)$ will satisfy

$$\delta(C_1) = \delta(A_1 \cup B_1) = \delta(A_1) + \delta(B_1) - \delta(A_1 \cap B_1)$$

Since A_1 and B_1 are composed of arithmetic progressions with differences (moduli 5 and 7) that are coprimes, we deduce that

$$\delta(A_1 \cap B_1) = \delta(A_1) \cdot \delta(B_1)$$

and we get

$$\delta(C_1) = \frac{2}{5} + \frac{2}{7} - \frac{2}{5}\frac{2}{7} = \frac{14 + 10 - 4}{35} = \frac{20}{35} \quad = \frac{2}{5} + \frac{2}{5}\frac{3}{7}$$

Similarly, for $\delta(C_1 \cup A_2)$ we obtain

$$\delta(C_1 \cup A_2) = \frac{20}{35} + \frac{2}{11} - \frac{20}{35}\frac{2}{11} = \frac{250}{385} \quad = \frac{2}{5} + \frac{2}{5}\frac{3}{7} + \frac{2}{5}\frac{3}{7}\frac{5}{11}$$

And for $\delta(C_1 \cup C_2)$ we have

$$\delta(C_1 \cup C_2) = \delta(C_1 \cup A_2 \cup B_2) \quad = \frac{2}{5} + \frac{2}{5}\frac{3}{7} + \frac{2}{5}\frac{3}{7}\frac{5}{11} + \frac{2}{5}\frac{3}{7}\frac{5}{11}\frac{9}{13}$$

One can observe[5] that

$$\delta(C_1) = \frac{2}{5}\left(1 + \frac{3}{7}\right) = \frac{2}{5}\frac{10}{7} \quad = \frac{4}{7}$$

and

$$\delta(C_1 \cup C_2) - \delta(C_1) = \frac{2}{5}\frac{3}{7}\frac{5}{11}\left(1 + \frac{9}{13}\right) = \frac{2}{5}\frac{3}{7}\frac{5}{11}\frac{22}{13} \quad = \frac{4}{7}\frac{3}{13}$$

Similarly, the natural density increment for C_3 simplifies to

$$\delta(C_1 \cup C_2 \cup C_3) - \delta(C_1 \cup C_2) = \frac{4}{7}\frac{3}{13}\frac{9}{19}$$

If the moduli of the A_k and B_k sets would all be relatively prime, the following generic formula would be exact:

$$\delta\left(\bigcup_{k=1}^{n} C_k\right) = \frac{4}{7}\sum_{k=0}^{n-1}\frac{(\frac{3}{6})_k}{(\frac{13}{6})_k} \tag{2}$$

where $(\alpha)_n = \prod_{k=0}^{n-1}(\alpha + k)$ is the Pochhammer symbol (rising factorial).

Though, a potential issue seems to appear for A_k and B_k sets with composite moduli, like B_4 (mod 25), A_6 (mod 35), etc. As can be seen in Table 4, the B_4 and A_6 sieves are "cancelled" by A_1 and B_1. It is shown in Annex A that such sets always fully "collide" with one or more previous sets. For instance, if $6k - 1$ is composite, then an integer $m < k$ can be found such that either $A_k \subset A_m$ or $A_k \subset B_m$, and therefore

$$\delta\left(\bigcup_{j=1}^{k-1} C_j \cup A_k\right) = \delta\left(\bigcup_{j=1}^{k-1} C_j\right) \tag{3}$$

the density of the global sieve stays unaffected by these sets.

The pattern identified above (2) and the favorable cases of the composite moduli (3) allow us to define the $\Delta : \mathbb{N}^* \to [0,1]$ function as an estimate and (admittedly mediocre) upper-bound for the arithmetic density of all C_k sieves up to n:

$$\delta\left(\bigcup_{k=1}^{n} C_k\right) \leq \Delta(n) = \frac{4}{7}\sum_{k=0}^{n}\frac{(\frac{3}{6})_k}{(\frac{13}{6})_k} \tag{4}$$

While following observation doesn't help much for our purpose, it is interesting to note that

$$\lim_{n\to\infty}\Delta(n) = \lim_{n\to\infty}\frac{4}{7}\sum_{k=0}^{n}\frac{(\frac{3}{6})_k}{(\frac{13}{6})_k} = \frac{4}{7}\ {}_2F_1\left(1, \tfrac{3}{6}; \tfrac{13}{6}; 1\right) = 1 \tag{5}$$

Proof: see math.SE discussion[5]

A corollary for the above result (5) is that the density of the "happy mother" numbers over the whole set of integers, which is lower-bounded by $1 - \lim_{n\to\infty}\Delta(n)$, can't be negative, which of course isn't very helpful. But it would have been very problematic to find $\lim_{n\to\infty}\Delta(n) < 1$, since it would have meant that the set of "happy mother" numbers is substantial, in absolute contradiction with the well-known density of the prime numbers.

Since $5n-1$ is the smallest integer possibly affected by the $\bigcup_{k=n}^{\infty} C_k$ sieves, we can define the function $U : \mathbb{N}^* \to \mathbb{R}$

$$U(n) = (5n+3)\cdot(1-\Delta(n))$$

as a mediocre lower-bound estimate for the total number of unsifted elements (the "happy mothers") lower than $5n+3$. Or alternatively, and for sufficiently large n ($n \geq 8$), the function $V : \mathbb{N}^* \to \mathbb{R}$

$$V(n) = n\cdot(1-\Delta(\lfloor \tfrac{n-3}{5} \rfloor))$$

can be defined as a lower-bound estimate for the total number of unsifted elements up to n.

Result: We find

$$\lim_{n\to\infty} U(n) = \lim_{n\to\infty} (5n+3)\cdot(1-\Delta(n)) = \infty \tag{6}$$

Proof: simple application of the Stolz-Cezàro theorem

$$\lim_{n\to\infty} (5n+3)\cdot\left(1-\frac{4}{7}\sum_{k=0}^{n}\frac{(\frac{3}{6})_k}{(\frac{13}{6})_k}\right) = \lim_{n\to\infty} \frac{1-\frac{4}{7}\sum_{k=0}^{n}\frac{(\frac{3}{6})_k}{(\frac{13}{6})_k}}{\frac{1}{5n+3}} = \lim_{n\to\infty} \frac{-\frac{(\frac{3}{6})_{n+1}}{(\frac{13}{6})_{n+1}}}{\frac{1}{5n+8}-\frac{1}{5n+3}}$$

We therefore conclude that there are infinitely many "happy mothers", and consequently, per Theorem 1, infinitely many twin prime pairs as well.

dedicated to Bernard Capelle

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References:

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Annex A

The set of congruence classes $\{\overline{1}_6, \overline{5}_6\}$ is closed under multiplication. Therefore, all non-prime integers of the form $6k+1$ can be written as either

$$6k+1 = (6i-1)(6j-1)$$

or

$$6k+1 = (6i+1)(6j+1)$$

And similarly, all non-prime integers of the form $6k-1$ can be written as

$$6k-1 = (6i-1)(6j+1)$$

Case by case study:

- $6k+1 = (6i-1)(6j-1) \implies k = 6ij - i - j$

  If $X \equiv k \ (mod\ 6k+1)$, then $\exists\ n$ such that

$$\begin{aligned} & X = (6k+1)n + k \\ \Leftrightarrow\ & X = (6i-1)(6j-1)n + 6ij - i - j \\ \Leftrightarrow\ & X = (6i-1)\,[(6j-1)n + j] - i \\ \Leftrightarrow\ & X = (6i-1)\,[(6j-1)n + j] - 6i + 5i - 1 + 1 \\ \Leftrightarrow\ & X = (6i-1)\,[(6j-1)n + j] - (6i-1) + 5i - 1 \\ \Leftrightarrow\ & X = (6i-1)\,[(6j-1)n + j - 1] + (5i-1) \end{aligned}$$

$$\implies X \equiv 5i - 1 \ \ (mod\ 6i-1)$$

  If $Y \equiv 5k+1 \ (mod\ 6k+1)$, then $\exists\ n$ such that

$$\begin{aligned} & Y = (6k+1)n + 5k + 1 \\ \Leftrightarrow\ & Y = (6i-1)(6j-1)n + 5(6ij - i - j) + 1 \\ \Leftrightarrow\ & Y = (6i-1)\,[(6j-1)n + 5j] - 5i + 1 \\ \Leftrightarrow\ & Y = (6i-1)\,[(6j-1)n + 5j] - (6i-1) + i \\ \Leftrightarrow\ & Y = (6i-1)\,[(6j-1)n + 5j - 1] + i \end{aligned}$$

$$\implies Y \equiv i \ \ (mod\ 6i-1)$$

- $6k+1 = (6i+1)(6j+1) \implies k = 6ij + i + j$

  ...

... *(tbd)*